DIFFERENTIAL GEOMETRY

MATH 136

Lecture 1: What is differential geometry?

Introduction

1.1. Differential geometry deals with geometric objects called **manifolds**. We later will define manifolds intrinsically. It makes more sense however to look first at manifolds embedded in a Euclidean space $\mathbb{R}^n$ and in particular in $\mathbb{R}^3$ and even give concrete parametrizations for them. One-dimensional manifolds are known as **curves** or two dimensional manifolds as **surfaces**. The level sets of smooth function $f : \mathbb{R}^n \to \mathbb{R}$ can produce hypersurfaces of dimension $n-1$. The 3-sphere $x_1^2+x_1^2+x_3^2+x_4^2 = 1$ for example is a 3-dimensional manifold embedded in 4-dimensional space. Since we live in a 3-dimensional space, we are at first mostly interested in one or two dimensional manifolds in $\mathbb{R}^3$. These are **curves** or **surfaces** in a traditional sense.

1.2. A **curve** is given as a **parametrization** $r(t) = [x(t), y(t), z(t)]$, where $t \in [a, b]$ is the parameter domain and $x(t), y(t), z(t)$ are functions of one variable. In two dimensions, some curves can also be written implicitly as a **level curve**. An example is the ellipse $x^2/4 + y^2/9 = 1$. A **surface** is given as a parametrization $r(u, v) = [x(u, v), y(u, v), z(u, v)]$, where (u, v) is in some domain $R \subset \mathbb{R}^2$ in the uv-plane. Some surfaces can be given as **level surfaces** $f(x, y, z) = 0$. The intersection of level surfaces $f(x, y, z) = 0, g(x, y, z) = 0$ is often a **curve**. If f, g are polynomials it is often a one-dimensional **variety**.

1.3. We are interested in **global quantities** like **arc length** $\int_a^b |r'(t)|\ dt$ or **surface area** $\iint_R |r_u \times r_v|\ dudv$, where $\times$ is the **cross product**. We also use **local quantities** like **curvature** $\kappa(t) = |r'(t)\times r''(t)|/|r'(t)|^3$ or **torsion** $\tau(t) = \det[r'(t), r''(t), r'''(t)]/|r'\times r''|^2$ for curves. For a surface, the **curvature** of a point can be defined as $K(p) = \lim_{r\to 0} 3\frac{2\pi r-|S_r(p)|)}{\pi r^3}$, where $|S_r(p)|$ is the length of the **wave front** $S_r(p)$ of points on the surface in distance r from p which is a circle for small r. This intrinsic definition does not making use of the embedding of the surface in an ambient space. It even makes sense on non-smooth surfaces, like polyhedra.

1.4. Curvature plays an important role in differential geometry. We will define it differently later in the course and verify that it is independent of the embedding in space. This is the **Theorema egregium**, the "great theorem" of Gauss from 1827. Riemannian geometry, the idea of doing geometry on a manifold without having to embed it into an ambient emerged in an inaugural lecture of Riemann in 1854. The theory is used heavily in Einstein's 1915 theory of general relativity. Schwarzschild found a black hole solution in 1916. A 100 years later, gravitational waves from black hole mergers have been observed.

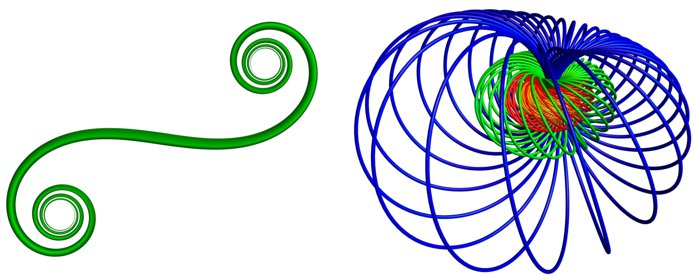

Figure 1. To the left we see a curve called Euler curve, to the right, we see a visualization of the 3 dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. It is foliated by 2 dimensional flat tori. We can not embed flat tori in $\mathbb{R}^3$ but we can in $\mathbb{R}^4$.

1.5. The focus of differential geometry is to investigate relations between **local quantities** and **global properties**. An important example to observe what happens if curvature is integrated up. We are especially interested in quantities that do not depend on the metric, like the **Euler characteristic**. Here are examples which will appear early in this course: for a planar closed curve in $\mathbb{R}^2$ one can define the **signed curvature** as $\kappa(t) = (r'(t) \times r''(t))/|r'(t)|^3$. We will then see the **Hopf Umlaufsatz** $\int_C \kappa(t)\ dr(t) = 2\pi$. For a two-dimensional surface with g holes of Euler characteristic $\chi(G) = 2 - 2g$, one has the **Gauss-Bonnet theorem** $\iint_R K(x)\ dV(x) = 2\pi\chi(G)$. We also want to understand **geodesics**, curves that locally minimize length. One can start geodesics into any direction $v/|v|$ and let it run for a distance $|v|$. This produces the **exponential map** $\exp_p$, a map from the tangent space T_pM of a point p to the manifold. If S_r is the sphere of radius r in $\mathbb{R}^2$, then the image $W_r(t) = \exp_p(S_r)$ is called the **wave front** at p. These waves can become complicated for large r. This can also be studied on polyhedra. We expect wave fronts to become dense in the manifold, except for very special cases like the round sphere.

1.6. Differential geometry then extends curve and surface theory to arbitrary dimensions. One study then so called **Riemannian manifolds** or **pseudo-Riemannian manifolds** which appear in physics. There is an intrinsic geometry but also interest when manifolds M are embedded in larger manifolds M'. In general relativity for example, space is a 3-dimensional manifold embedded in a four dimensional **space-time manifold** M'. The above formulation of curves or surfaces dealt with embeddings of one or two dimensional manifolds in Euclidean 3-manifold M'. One can use the exponential map to define **sectional curvature** and to use it to define a **curvature tensor** or **scalar curvature**. The extrema of the functional that gives the total scalar curvature are the Einstein equations. **General relativity** studies solutions of these equations as they tell how matter bends space. The geodesic equations then tell, how matter moves in this space.