DIFFERENTIAL GEOMETRY

MATH 136

10th Homework

This is the 10th homework. It is due Friday, November 21th:

Problem 1: Tensor field or not?
a) The calculus gradient field $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$.
b) The Jacobian of a function $df = [f_x, f_y, f_z]$
c) The second fundamental form II
d) The shape operator A like $(Av)^k = \sum_j A^k_j v^j$.
e) The inverse I^{-1} of the first fundamental form g^{ij}.
f) The second derivative $r_{u^i u^j}$.
g) The Christoffel symbols $\Gamma_{ijk} = r_{u^i u^j} r_{u^k}$.

Problem 2: Prove that $SU(2)$ is a manifold by explicitly giving the charts.

Problem 3: The following code allows you to experiment with level sets in discrete manifolds. The host manifold is a discrete 4-manifold, the join of a 2-sphere and a 1-sphere. Running the code builds a random function from the vertex list to $\{1, 2, 3\}$. It defines a co-dimension 2 manifold.
a) Run the code as it is, report the numbers V,E,F as well as the Euler characteristic of M_f.
b) Change the code and see what happens if the function takes 4 values rather than 3.
c) Build a 5 manifold as the join of two 2-manifolds and build a 3-manifold by taking a function taking 3 random values.
d) Report the curvature values of your 3-manifold.
e) Report the curvature vales of a 4 manifold by taking a function taking 2 values on the 5 manifold in c).

```
Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]]; w[x_]:=-(-1)^k;
R[G_,k_]:=Module[{},R[x_]:=x->RandomChoice[Range[k]]; Map[R,Union[Flatten[G]]]];
F[G_]:=Delete[BinCounts[Map[Length,G]],1]; Euler[G_]:=F[G].Table[w[k],{k,Length[F[G]]}];
Surface[G_,g_]:=Select[G,SubsetQ[#/.g,Union[Flatten[G] /. g]] &];
S[s_,v_]:=VertexDelete[NeighborhoodGraph[s,v],v];         Sf[s_,v_]:=F[Whitney[S[s,v]]];
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Curvature[s_,v_]:=Module[{f=Sf[s,v]},1+f.Table[(-1)^k/(k+1),{k,Length[f]}]];
Curvatures[s_]:=Module[{V=VertexList[s]},Table[Curvature[s,V[[k]]],{k,Length[V]}]];
J[G_,H_]:=Union[G,H+Max[G]+1,Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1}],0]]]];
ToGraph[G_]:=UndirectedGraph[n=Length[G];Graph[Range[n],
  Select[Flatten[Table[k->l,{k,n},{l,k+1,n}],1],(SubsetQ[G[[#[[2]]]],G[[#[[1]]]]])&]]];
Barycentric[s_]:=ToGraph[Whitney[s]];

G=J[Whitney[Barycentric[CompleteGraph[{2,2,2}]]],Whitney[CycleGraph[7]]];   (* J=Join *)
g=R[G,3]; H=Surface[G,g];     (* A codimension 2 manifold in the 4-sphere G=Oct * C_7  *)
Print["EulerChi= ",Euler[H]];  Print["Fvector: ",F[H]]; s=ToGraph[H]; GraphPlot3D[s]
Print["Gauss-Bonnet Check: "];  Print[Total[Curvatures[s]]==Euler[H]];
Print["Curvature Values:  "];   Print[Union[Curvatures[s]]];
```

Problem 4: The Reissner-Nordstroem metric

$$g = \begin{bmatrix} -\frac{e^2}{r^2}+\frac{2M}{r}-1 & 0 & 0 & 0 \\ 0 & \frac{1}{\frac{e^2}{r^2}-\frac{2M}{r}+1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\phi) \end{bmatrix}$$

is a static solution of the Einstein-Maxwell equations

$$R - \frac{1}{2}Sg = T \, .$$

But it is not a vacuum. It describes the field of a charged, non-rotating body of mass M and charge e. An example is a charged black hole. What is the entry T_{44} for $e=1, r=1, \phi=\pi/3$?

Problem 5: a) Verify that the metric on $SU(2)=S^3$ given by the parametrization

$$r = [\cos(u)\cos(w), \sin(u)\cos(w), \cos(v)\sin(w), \sin(v)\sin(w)]$$

satisfies equation $R-2g=0$. What is S? b) Finally check that the pseudo sphere given by the parametrization

$$r = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)+\log\left(\tan(\frac{v}{2})\right)]$$

satisfies the vacuum Einstein equations. What is the curvature K?

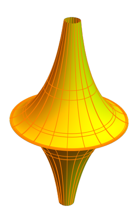

FIGURE 1. The Pseudo sphere

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