9th Homework

This is the ninth' homework. It is due Friday, November 14th:

Problem 1: a) Prove the **Euler Handshake lemma** $\sum_{i=1}^V d_i = 2E$ in graph theory.
b) Assume M is a discrete 2-torus with F faces. How many vertices does it have?
c) In the handout of lecture 17, you see a picture of a discrete torus with $V = 64$ vertices. Determine E and F in that case.

Problem 2: We parametrize a paraboloid M as $r(u,v) = (u, v, u^2 + v^2)$. for $R = \{u^2 + v^2 \leq 1\}$. This is a **2-manifold with boundary**.
a) Compute the curvature K.
b) Compute $|r_u \times r_v| = \sqrt{\det(g)}$.
c) Compute $\iint_R K \, dV$.

Problem 3: We continue with the same paraboloid as before.
a) Compute the curvature of the boundary curve $x(t)$ (parametrized by arc length).
b) Compute the normal curvature $\kappa_n(t) = n(t) \cdot \ddot{x}(t)$ as well as the geodesic curvature $\kappa_g(t) = (n \times \dot{x}) \cdot \ddot{x}$.
c) Verify the local Gauss-Bonnet result. That is show that

$$\iint_R K \, dV + \int_0^L \kappa_g(t) \, dt = 2\pi \, .$$

Problem 4: Use a computer algebra system to verify that

$$\iint_M K \, dV = 4\pi$$

if $M = \{x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ for $a = 2, b = 3, c = 5$. What is the maximal and what is the minimal curvature of this ellipsoid?

Problem 5: The "angular defect" $K(p)$ at a vertex of a convex polyhedron M is the angle needed to add to complete the angle to 2π. For a cube for example, it $2\pi - 3\pi/2 = \pi/2$ at every corner.
a) Descartes theorem states that the total defect of a convex polyhedron is 4π so that the angular defect is a curvature. This is a polyhedral Gauss-Bonnet theorem. Verify this for an icosahedron to see what is going on.
b) Modify the local to global proof of Gauss-Bonnet to see that for a general polyhedron, the value $K(p) = 2\pi - \sum_i \alpha_i$ gives a curvature that adds up to 2π times the Euler characteristic of the surface. Here, α_i are the angle interior angles and the result you want to show is $\sum_p K(p) = 2\pi\chi(M)$. It is a version of Gauss-Bonnet.
c) Illustrate your theorem with the Escher stair polyhedron built in **mine craft** or **Lego**. Compute all the angular defects and add them up. The total curvature should be the Euler characteristic of the stair.

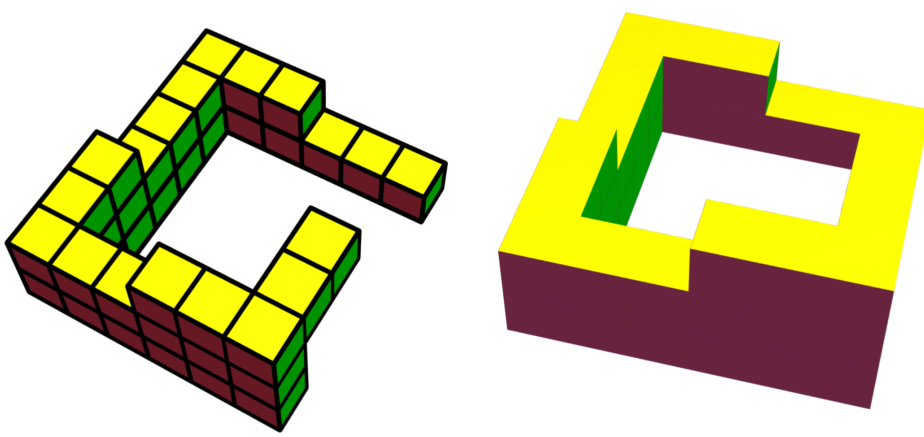

FIGURE 1. The Escher Stairs built in mine craft. If you look at it from the right angle and do glue the bricks nicely, you see an impossible stair, which always goes down or up depending on whether you are a "wineglass half empty" or "wine glass half full" type of person.

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 136, FALL, 2025