

# DIFFERENTIAL GEOMETRY

MATH 136

## Homework 8

This is the eighth homework. It is due Friday, November 7st:

**Problem 13.1:** To warm up to Greens theorem. Solve the following problems: a) Use Green to compute the area of the region  $|x|^{2/3}/a^2 + |y|^{2/3}/b^2 \leq 1$ .  
b) Assume  $F$  is a smooth 1-form on a 2-torus  $M$ , what can you say about  $\iint_M dF$  ?  
c) What can you say about  $\iint_M K dA$  if  $M$  is a 2-manifold with a regular parametrization  $r$ .

**Problem 13.2:** a) Green's theorem tells that if  $R \subset \mathbb{R}^2$  is a region and  $X = [P, Q]$  is a vector field in the plane, then  $\iint_R \text{curl} X dudv = \int_{\delta R} X(r(t)) \cdot r'(t) dt$  where  $\delta R$  is the boundary. Look up and write down a proof of this.  
b) Look up the discrete Green theorem and give a proof

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**Problem 13.3:** Verify here that Stokes theorem on  $S = r(R)$  can be reduced to Green on  $R$ :

$$\iint_R \text{curl}(F) r_u \times r_v dudv = \iint_R \text{curl}(X) dudv$$

Assume  $F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$  is a vector field in space. Prove the important formula

$$\text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u .$$

As we have seen in class, this implies that the 2D field  $X = [F \cdot r_u, F \cdot r_v]$  satisfies  $\text{curl}(X) = F_u \cdot r_v - F_v \cdot r_u$ .

**Problem 13.4:** We have seen half of the proof that the form  $X$  is intrinsic. Verify that also the second component of  $X = [z \cdot w_u, z \cdot w_v]$  can be expressed from  $I$  alone.

<sup>1</sup><https://people.math.harvard.edu/~knill/teaching/math22b2022/handouts/lecture33.pdf>

**Problem 13.5:** Below you see Gauss's original statement of the theorem Egregium translated into English. Explain what he means with "developing a surface upon any other surface" and why it is not possible for example to find a map of the earth in  $\mathbb{R}^3$  which preserves distances.

Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates  $x, y, z$ , will correspond a definite point of the latter surface, whose coordinates are  $x', y', z'$ . Evidently  $x', y', z'$  can also be regarded as functions of the indeterminates  $p, q$ , and therefore for the element  $\sqrt{(dx'^2 + dy'^2 + dz'^2)}$  we shall have an expression of the form

$$\sqrt{(E' dp^2 + 2 F' dp \cdot dq + G' dq^2)}$$

where  $E', F', G'$  also denote functions of  $p, q$ . But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically

$$E = E', \quad F = F', \quad G = G'.$$

Thus the formula of the preceding article leads of itself to the remarkable

**THEOREM.** *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

FIGURE 1. Gauss original statement translated into English. Source: Wikipedia