DIFFERENTIAL GEOMETRY

MATH 136

Homework 7

This is the seventh homework. It is due Friday, October 31st:

Problem 1: Assume you make a trip and your effort is $F(x, \dot{x}) = x^2 \dot{x}^2$ rather than the kinetic energy $\dot{x}^2$ because things are easy initially but get harder later on. What is the best strategy to reach from $x(0) = 0$ to $x(1) = 1$? Go slow first or go fast first? In order to find the best way, solve the Euler-Lagrange problem to minimize the action

$$E(x) = \int_0^1 F(x, \dot{x}) \, dt$$

for $F(x, \dot{x}) = x^2 \dot{x}^2$ with $x(0) = 0, x(1) = 1$.

Problem 2: Look at the paraboloid $r(u, v) = [u, v, u^2 + v^2]$.
a) Compute all the Christoffel symbols Γ_{ijk}. These are 8 terms. (Do this by hand). b) Now compute all the Christoffel symbols Γ_{ij}^k. (Also here, do this by hand. There is Mathematica code which can be used for the last problem).

Problem 3: A geodesic $x(t)$ is called **closed**, if there exists L such that $x(L) = x(0)$ and $\dot{x}(L) = \dot{x}(0)$. It is a closed curve on M for which also initial and end velocities agree. Note that the curve x does not have to be simple. Two closed geodesics $x_1(t), x_2(t)$ are called **homotopic**, if one can deform them to each other on the manifold. Formally this means to parametrize both on $[a, b] = [0, 1]$ and then give a continuous $F(t, s)$ of two variables, such that $F(t, 0) = x_1(t)$ and $F(t, 1) = x_2(t)$ and $t \to x_s(t) = F(t, s)$ is on the manifold. We verify here that on any torus, there are infinitely many geodesics that are pairwise not homotopic to each other.
a) Invent and then define a "winding vector" $(n, m) \in \mathbb{Z}^2$.
b) Verify that this number is the same for two closed curves that are homotopic.
c) Draw a torus and a geodesic for winding number $(4, 5)$.
d) Verify that there each homotopy class is not empty by telling how to write one.
e) Argue why there is at least one geodesic in each homotopy class.

Problem 4: We draw some wave fronts $W_r(p)$ on the flat Clifford torus $\mathbb{R}^2/\mathbb{Z}^2$ (Pac-Man square). This manifold can be realized as all point (x, y) in $\mathbb{R}^2$, where points $(x, y), (x + n, x + m)$ identified if $n, m \in \mathbb{Z}$. Work with the point $p = (1/2, 1/2)$.
a) Draw the wave front $W_{1/2}(p)$.
b) Draw the wave front $W_1(p)$.
c) Draw the wave front $W_5(p)$.
You are welcome to become physical part c and use scissor, ruler and compass to do that).

Problem 5: The torus
$$r(u, v) = ((a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), b\sin(v)) ,$$
has the metric
$$\begin{aligned} g_{11} &= (a + b\cos(v))^2 \\ g_{22} &= b^2 \\ g_{12} &= g_{21} = 0 \end{aligned}$$
Use the following example code (done for the sphere) to compute all the Christoffel symbols Γ_{ij}^k. Make sure to simplify.

```
r={Sin[v] Cos[u],Sin[v] Sin[u],Cos[v]};
ru=D[r,u];        rv=D[r,v];
```

[1]See a project done with Emily Kang during Summer 2024

```
n=Cross[ru,rv];  n=n/Sqrt[n.n];
nu=D[n,u];        nv=D[n,v];
drT={ru,rv};      dr=Transpose[drT];
g=drT.dr;         gi=Inverse[g];
dnT={nu,nv};      dn=Transpose[dnT];
h=-dnT.dr;        e=dnT.dn;
K=Det[h]/Det[g];
X={u,v};  d=2;
c[i_,j_,k_]:=(D[g[[j,k]],X[[i]]]
              +D[g[[k,i]],X[[j]]]
              -D[g[[i,j]],X[[k]]])/2;
Christoffel[i_,j_,k_]:=Sum[gi[[k,l]]*c[i,j,l],{l,d}];
S=Table[Simplify[Christoffel[i,j,k]],{i,d},{j,d},{k,d}];
TableForm[S]
```