DIFFERENTIAL GEOMETRY

MATH 136

Homework 5

This is the fifth homework. It is due Friday, October 10th.

Problem 1: a) Compute the first and second fundamental form for the surface

$$r(u,v) = \begin{bmatrix} u \\ u^2 - v^2 \\ v \end{bmatrix} .$$

In this problem, we do not want you to use a computer algebra system. You need to write down especially all the matrices dr, dn and A.
b) Use a) to compute the curvature and especially the curvature at $(0,0,0)$.

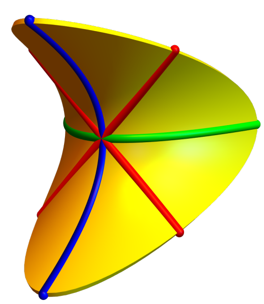

FIGURE 1. A surface with negative curvature.

Problem 2: Compute the first and second and third fundamental form for the torus $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$

$$r(u,v) = \left[\ (5+\cos(v))\cos(u), (5+\cos(v))\sin(u), \sin(v) \ \right] ,$$

where $u, v \in [0, 2\pi)$. We want you to write down explicit expressions for the matrices dr, dn. This problem again can be solved by hand, but you are allowed here to use a computer algebra system to assist you.

Problem 3: a) Compute the matrix A for the shape operator for the same torus.
b) Compute the Gauss curvature K and the mean curvature H.

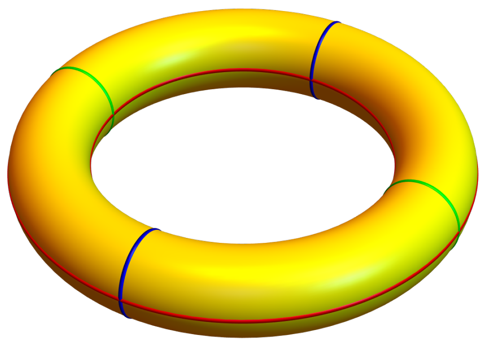

FIGURE 2. The torus for problem 3-4.

Problem 4: a) Again continuing with the torus, integrate $\int_0^{2\pi} \int_0^{2\pi} K dV$. Even if you should use a computer algebra system to integrate, you need to find out what integration method can be used to solve the integral at hand.
b) Your result will be an integer but it will not be compatible with the Gauss Bonnet result that you have seen in unit 9. What was going wrong?

Problem 5: a) Verify that the shape operator matrix A is symmetric in the inner product $\langle v, w \rangle = v^T I w$ so that we can parametrize the surface with a basis such that A is diagonal with respect to the inner product given by I and so that $n_u = -\lambda r_u$ and $n_v = -\mu r_v$. (You assume that I is already diagonal).
b) Use a) to prove the matrix identity $III - 2HII + KI = 0$, where $H = (\lambda + \mu)/2$ and $K = \lambda\mu$ and λ, μ are the eigenvalues of A.

P.S. Here is Mathematica sample code to compute. If you chose to use a computer algebra system, we ask you in problem 1) and 2) to comment what each of the commands you enter does.

```
r={Sin[v] Cos[u],Sin[v] Sin[u],Cos[v]};
ru=D[r,u]; rv=D[r,v];
n=Cross[ru,rv]; n=n/Sqrt[n.n];
nu=D[n,u]; nv=D[n,v];
drt={ru,rv};dr=Transpose[drt];
dnt={nu,nv};dn=Transpose[dnt];
g=drt.dr; h=-dnt.dr; e=dnt.dn;
A=Inverse[g].h; H=Tr[A]/2; K=Det[A];
```

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