DIFFERENTIAL GEOMETRY

MATH 136

Homework 4

This is the fourth homework. It is due Friday, October 3rd.

Problem 1: There is a **discrete Hopf Umlaufsatz** for polygons.
a) Assume first we have a simple convex polygon with n vertices. Define the curvature at the vertex v_k to be κ_k which is the outer angle $\pi - \alpha_k$ where α_k is the angle you have defined in third grade for polygons. The discrete Hopf Umlaufsatz tells $\sum_{k=1}^{n} \kappa_k = 2\pi$. Prove this.
b) Now formulate the general (not necessarily convex) case. Define suitable curvatures (which are no more positive now in general) such that the result works.

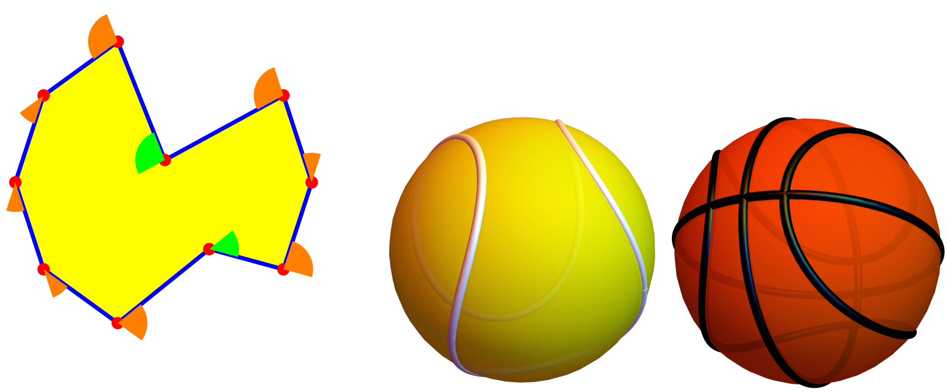

FIGURE 1. In the Umlaufsatz for polygons, curvatures can get positive or negative. The tennis ball and basket ball curve.

Problem 2: Check the four vertex theorem in the example $r(t) = 4[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)]$ The expressions for $\kappa(t)$ and $\kappa'(t)$ are not that bad. Plot the function $\kappa(t)$ and find the critical points, the roots of κ'.

Problem 3: For parameters a, define $c = 2\sqrt{a}$ and the curve $r(t) = [a\cos(t)+\cos(3t), a\sin(t)-\sin(3t), c\sin(2t)]$. For $a = 2$ one has the **tennis ball curve**, for $a = 1/2$ the **base ball curve** and for $a = 1.8$, the **basket ball curve** (Basketballs have two additional grand circles).
a) Verify that these curves are located on a sphere.
b) Look up the tennis-ball theorem, state its content, then write down the main idea on how the theorem is proven.

Problem 4: Let us reformulate the Frenet equations in the plane using complex coordinates. Assume $\kappa(t)$ is an arbitrary continuous function. We have seen that if $K(t) = \begin{bmatrix} 0 & \kappa(t) \\ -\kappa(t) & 0 \end{bmatrix} \in so(2)$, then $Q'(t) = K(t)Q(t)$ defines a path $\begin{bmatrix} \cos(\alpha(t)) & \sin(\alpha(t)) \\ -\sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$ of rotation matrices in $SO(2)$ such that $T(t) = [\cos(t), \sin(t)]$ is the velocity $r'(t)$ and $N(t) = [-\sin(t), \cos(t)]$ is the normal vector. The matrix $Q(t)$ gave us the Frenet frame and so the curve $r(t) = \int_0^t r'(s)\, ds$.
a) Explain why we can identity $K(t)$ with $i\kappa(t)$ and $Q(t)$ with a complex number of length 1. In other words, you explain why $so(2)$ is isomorphic to $u(1)$ and $SO(2)$ is isomorphic to $U(1)$.
b) Now show that we have an explicit formula

$$r(t) = \int_0^t e^{i\int_0^s \kappa(u)\, du}\, ds$$

for the curve.

Problem 5: This is a continuation of Problem 4.4.
a) Lets look at the example $\kappa(t) = 1 + 3\sin(3t)$. We get to the explicit formula assuming $r(0) = 0$ and $r'(0) = 1$

$$r(t) = \int_0^t e^{i\int_0^s (1+3\sin(3u))du}\, ds\ .$$

Draw this curve by integration. Verify that this is a closed bounded curve by checking numerically that $r(2\pi) = 0$.
(If you want to have fun, try something like $\kappa(t) = 1 + 30\sin(5t)$).
b) Now look at $\kappa(t) = 1 + \sin(t)$. Draw this curve by integration. Verify that this produces an unbounded curve because $r(2\pi) \neq 0$.
c) Apropos Hopf theorem. What is the rotation index of the closed curve in a)?
d) Apropos Scheitelsatz: How many vertices does the curve in a) have. How about the curve in b) on the interval from 0 to 2π?

P.S. If you want to have some fun, check out Oliver's recent stability result that $\kappa(t) = k + a\sin(mt)$ for integers k, m and $a \neq 0$ produces a bounded curve if and only if k is not a multiple of m. It appears to be a new result.

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