

DIFFERENTIAL GEOMETRY

MATH 136

Homework 3

This is due Friday, September 26.

Problem 1: We have seen in class proofs of the formulas

$$\kappa = |r' \times r''|/|r'|^3, \quad \tau = (r' \times r'') \cdot r''' / |r' \times r''|^2.$$

Prove the easier reverse: if $r(t)$ is parametrized by arc length, then these formulas agree with the formulas for curvature and torsion you have seen in class when we have arc length parametrization and defined in general $\kappa_j = e'_j \cdot e_{j+1}$. So, show that the above formulas simplify to $\kappa = T' \cdot N = e'_1 \cdot e_2$ and τ simplifies to $\tau = N' \cdot B = e'_2 \cdot e_3$.

Problem 2: a) Look up and write down a proof that if $F(t, x)$ is a differentiable function from $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$, then there exists an open interval $(-a, a)$ and a unique path $x(t) : (-a, a) \rightarrow \mathbb{R}^n$ such that $x' = F(t, x)$ and $x(0) = x_0$. We want you to write down the proof in the differentiable case which is a bit more special than the usual assumption assuming a Lipschitz property for F .

b) Justify that if $x(t)$ stays bounded meaning that there is constant such that $|x(t)| \leq M$ for all t , then the solution exists for all t . We call this a global solution. c) Now conclude that if $Q' = K(t)Q$ is a differential equation for a matrix $Q(t)$ with skew symmetric $K(t)$, then there is a global solution $Q(t)$ in $SO(n)$

Problem 3: a) Determine from each of the spaces $SO(n)$, $so(n)$, $SU(n)$, $su(n)$ whether they are linear spaces or not.

b) Check that if $x(t)$ is a differentiable curve in $SO(n)$, then $x(t)$ satisfies the differential equation $x'(t) = A(t)x(t)$, where $A(t) \in so(n)$, the space of skew-symmetric matrices.

c) Show that $A(t) = A$ is a constant skew symmetric matrix, then the **matrix exponential** $Q(t) = e^{At}$ is an orthogonal matrix. What is this matrix $Q(t)$ in the case $n = 2$?

Problem 4: a) First verify that the helix $r(t) = [\cos(at), \sin(at), bt]$ has constant curvature and torsion. What are the values?
 b) Now prove that if a curve has constant curvature and torsion, it must be a helix.

Problem 5: a) There is an interval of c values for which the curve

$$r(t) = [\cos(t), \sin(t) + c \sin(3t)]$$

a simple closed curve. What is the interval?

b) Verify the Hopf Umlaufsatz in the case $c = 1/10$ for which the curve is simple. A computer algebra system gives you

$$f(t) = \kappa(t)|r'(t)| = \frac{6c \cos(2t) - 3c \cos(4t) + 1}{(3c \cos(3t) + \cos(t))^2 + \sin^2(t)}.$$

Verify this using a computer algebra system. Research then which integration method can be used to solve $\int_0^{2\pi} f(t) dt = 2\pi$. No need to actually do the integral by hand, just describe what you would have to do.

c) Look at the Hopf Umlaufsatz in the example of an asteroid

$$r(t) = [\cos^3(t), \sin^3(t)].$$

First verify that $|r'(t)| = |3 \cos(t) \sin(t)|$ and $\kappa(t) = -(2/3)/|\sin(2t)|$ then compute the rotation index $\int_0^{2\pi} \kappa(t)|r'(t)| dt / (2\pi)$. While your result complies with the Hopf Umlaufsatz, there is something strange going on given how you rotate counterclockwise around the region. Figure it out!

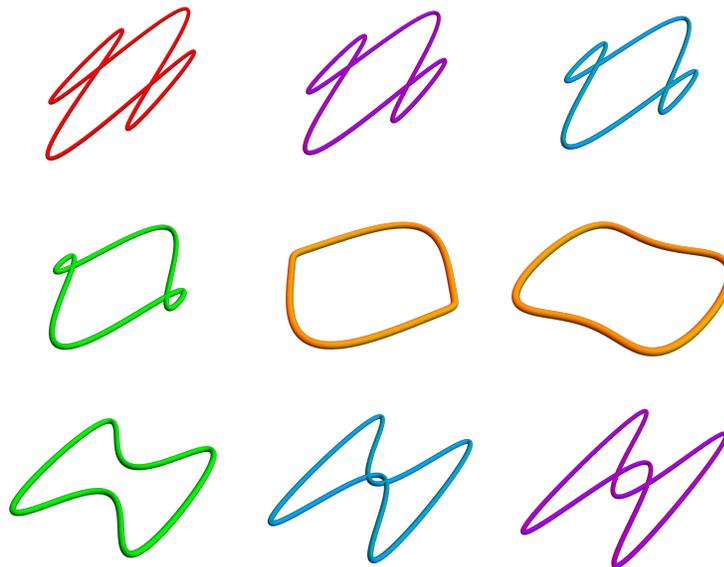


FIGURE 1. A few cycloid curves described in problem 5. There are cases for which the curve is simple and curves for which it is not.