DIFFERENTIAL GEOMETRY

MATH 136

2. Homework

This is the second homework. It is due Friday September 19th.

Surfaces

Problem 2.1: We have mentioned in class twice the fundamental theorem of linear algebra tells that for any $n \times m$ matrix,

$$\ker(A^T) = \mathrm{im}(A)^\perp .$$

a) Prove this formally and also illustrate your proof with an example.
b) Why does it imply the **rank-nullity theorem**: $\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$ holds for any $n \times m$ matrix. Use the same example to illustrate your proof.
c) If A is a $n \times m$ matrix and $b \in \mathbb{R}^n$. For which k, n, m is it possible that $\{x, Ax - b = 0\}$ is a k-dimensional manifold?

Problem 2.2: a) Verify that the parametrization

$$r(\theta, \psi, \phi) = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \cos(\psi)\sin(\phi) \\ \sin(\psi)\sin(\phi) \end{bmatrix}$$

parametrizes the 3-sphere $S : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ in $\mathbb{R}^4$. It has been used first by Heinz Hopf. Here $\theta \in [0, 2\pi], \psi \in [0, \pi]$ and $\phi \in [-\pi/2, \pi/2]$. (To verify that this is the right choice of parametrization just check that $\theta \to \theta, \psi \to \psi + \pi, \phi \to -\phi$ produces the same point.)
b) Verify that the set $SU(2)$ of complex 2×2 matrices of the form

$$A = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$$

which have determinant 1 also represent the 3-sphere. Verify that there is a $1 - 1$ correspondence between $SU(2)$ and S.
c) Conclude that there is a multiplicative structure $*$ on the 3-sphere. We can define $x * y$ are points in S and get a new point. This multiplication is associative and each element has a unique inverse. Which point on the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ represents the 1-element?

Problem 3.3: a) Verify that the 2 dimensional surface parametrized as

$$r(u,v) = \begin{bmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \cos(u)\sin(v) \\ \sin(u)\sin(v) \end{bmatrix}$$

is a subset of the 3-sphere. Check that it is a regular surface and so a 2-manifold by checking that dr has rank 2 everywhere.
b) What kind of surface is it? Compute its surface area with the formula given in the notes. Note that u, v both go from 0 to 2π.

Curves

Problem 2.4: a) Verify that for every non-zero integers a, b the curve

$$r(t) = \begin{bmatrix} \cos(at)\cos(bt) \\ \sin(at)\cos(bt) \\ \cos(at)\sin(bt) \\ \sin(at)\sin(bt) \end{bmatrix}$$

t goes from 0 to 2π is contained in the 3-sphere.
b) Find the arc length using the formula you know. The answer depends on a, b.
c) Give an explicit parametrization of a closed curve in the 3-sphere for which the arc length is larger than 1000.

Problem 2.5: a) Given two points A, B in $\mathbb{R}^n$. Prove the **theorem of Archimedes** telling that the straight line gives the shortest connection between A and B. That is, among all smooth curves connecting A with B. We will later call the shortest connection a **geodesic**.
b) Prove that any smooth regular curve $r : [a, b] \to \mathbb{R}^n$ can be parametrized by arc length s: there is a new parametrization $f(s)$ such that the velocity is $|f'(s)| = 1$ at all points. We will need this result next week, when we prove the **fundamental theorem of curves**.