

DIFFERENTIAL GEOMETRY

MATH 136

1. Homework

This first homework deals with two ways to describe manifolds: implicit equations or parametrization. The core mathematical topic is the implicit function theorem. This homework is a guide from multi-variable calculus to differential topology. This problem set is due Friday September 12.

DUPIN CYCLIDE

Problem 1.1: The level surface $S \subset \mathbb{R}^3$ given by $(x^2 + y^2 + z^2 + 7)^2 - 4(3x - 1)^2 - 32y^2 = c$ is an example of a **Dupin cyclide**.

- Draw $\{f = c\}$ for $c = -10, c = 0, c = 10, c = 100$.
- There are three parameters c for which S is singular meaning that there are points on S for which the gradient $df^T = \nabla f = \vec{0}$. Find these parameters.
- Re-derive the multi-variable calculus fact that if the gradient of f is not zero at a point, then the surface can near this point be written as a graph. E.g. $z = g(x, y)$ with $g_x = -f_x/f_z, g_y = -f_y/f_z$.
- Conclude from c) that if c is not one of the three critical values, the surface has a tangent plane at each point and so is a manifold.
- A surface is called compact if it is closed and bounded. Prove that S is bounded for any c .
- Look up and state the Sard theorem about the **critical values**, the set of c is for which $f = c$ is a manifold.

Problem 1.2: Intersecting the surface with $z = 0$ gives the algebraic curve $(x^2 + y^2 + 7)^2 - 4(3x - 1)^2 - 32y^2 = c$. It is the solution set of a polynomial equation in two variables.

- Draw the curves for the parameters $c = -10, c = 0, c = 10, c = 100$.
- Again verify that there are three parameter values for which the curve is singular.
- Again verify that in the regular case the curve looks locally like a graph. E.g. $y = g(x)$ with slope $g'(x) = -f_x/f_y$.
- Conclude that if c is not one of the critical parameters, then we deal with a manifold, a curve that near every point can be written as a graph.
- Look up the proof of the classification of one dimensional manifolds (without boundary) and give the argument for a proof.

Problem 1.3: a) Verify that for $c = 0$ the surface $f = c$ defined in a) can be parametrized as $\vec{r}(u, v) = [x, y, z]^T$ with

$$\begin{aligned}x &= R(1 + 8 \cos(u) - 3 \cos(u) \cos(v)) \\y &= R\sqrt{8}(3 - \cos(v)) \sin(u) \\z &= R\sqrt{8}(\cos(u) - 1) \sin(v),\end{aligned}$$

where $R = 1/(3 - \cos(u) \cos(v))$.

b) Compute the first fundamental form $g = dr^T dr$.

c) Check that $|r_u \times r_v|^2/R^4 = 256(3 - \cos(v))^2 \sin^4(u/2)$. For which (u, v) values is this zero?

Problem 1.4: Define $f_1(x, y, z) = f(x, y, z)$ from problem 1.1 and define $f_2(x, y, z) = z$. Let's look at the map from \mathbb{R}^3 to \mathbb{R}^2 given by

$$f(x, y, z) = \begin{bmatrix} f_1(x, y, z) \\ f_2(x, y, z) \end{bmatrix}.$$

The Cassini oval can be written as a system of non-linear equations $f(x, y, z) = C$ with $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The Cassini ovals are intersection of two polynomial equations.

a) Compute the **Jacobian matrix**

$$df(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 & \frac{\partial}{\partial z} f_1 \\ \frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 & \frac{\partial}{\partial z} f_2 \end{bmatrix}.$$

b) Look up the implicit function theorem in this case. When does df have not maximal rank?

c) We have seen in problem 1.1 that there are three critical c values. See whether you get the same points.

Problem 1.5: This problem should now more routine after battling problems 1.1-1.4.

a) By using a good picture or coordinates, explain why $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$ is a torus.

b) Find a parametrization for the surface similarly as 1.3 did for the cyclid.

c) Use the implicit description $f = 0$ to show that the manifold is compact.

d) Using the parametrization $S = r(R)$, check the maximal rank condition, again verifying that we have a manifold in the sense that every neighborhood of a point on the surface can be parametrized $\phi(U)$ where U is an open region in \mathbb{R}^2 .