

DIFFERENTIAL GEOMETRY

MATH 136

Final Exam Paper

ASSIGNMENT: DUE DECEMBER 14, 2025

Write an expository paper about a topic in differential geometry of your choice.

RULES

- The paper must be written by yourself. No human or AI assistance is allowed in writing.
- Acknowledge any references used: books, papers, web, discussions, the use of computer algebra.
- Aim for 4 or more pages with references. There can be illustrations. Also for illustrations, either do them yourself or give credit.

GRADING CRITERIA

- (1) Mathematical correctness
- (2) Clarity, readability and elegance
- (3) References and sources
- (4) Adaptation to the course, notation
- (5) Originality or depth or surprise

The paper is 60 points, the in class exam is 40 points. In total: 100 points.

TOPIC SUGGESTIONS

Pick or modify one of the following topics. If you prefer an other topic, check with Oliver. The topic needs to be related to this course - of course!

- (1) "**Positive Curvature Manifolds**". Examples of even dimensional manifolds for which all sectional curvatures are positive.
- (2) "**Clifton-Pohl torus**" A pseudo Riemannian manifold on which the geodesic flow is not complete.
- (3) "**Caustics in Riemannian geometry?**"
- (4) "**What is a Jacobi field?**"
- (5) "**Differential geometry of evolutes.**"
- (6) "**Principal curvatures and umbilic points**"
- (7) "**The symmetries of the Riemann tensor**" Like Bianchi identities.
- (8) "**Minimal surfaces**"
- (9) "**Chaotic geodesic flows**" (like on surfaces of constant negative curvature).

- (10) **"My favorite Riemannian manifold"**. i.e. projective plane, the Klein bottle, the hyperbolic plane, the projective 3-space $SO(3)$ (a 3 manifold) or $SU(3)$ (a 8 dimensional manifold).
- (11) **"The Differentiable Jordan Curve theorem"**. Easier than the topological Jordan Curve theorem. Prove it. See Do Carmo page 400.
- (12) **"The Hopf-Rynov Theorem"**: Two points on a complete surface can be joined by a minimal geodesic. See Do Carmo page 338.
- (13) **"The formulas of Codazzi-Meinardi"**. See section 4.C in Kuehnel.
- (14) **"The world of discrete manifolds"**. Explore small examples of discrete manifolds, I.e. Klein bottles, projective planes, or higher genus Klein bottles.
- (15) **"The Brachistochrone problem"** An example of a variational problem.
- (16) **"The Hilbert Action"** Outline the proof that critical points of the Hilbert action are Einstein manifolds.
- (17) **"What are Minimal Surfaces?"**. See chapter 3D in Kuehnel.
- (18) **"Complex manifolds" or Kähler manifolds.**
- (19) **"The complex projective plane $\mathbb{C}P^2$.**
- (20) **"What is a Kalabi-Yau manifold?"**
- (21) **"How can Morse theory help to understand Riemannian geometry?"**
- (22) **"What are Ruled surfaces?"** Section 3C in Kuehnel.
- (23) **"The rigidity of the sphere"** a compact connected regular surface of constant curvature must be a sphere. There is a section in Do Carmo.
- (24) **"Clairaut's relation for geodesics"** A conserved quantity for geodesics in surfaces of revolution".
- (25) **"The isoperimetric inequality"**. There is an exposition in Do Carmo.
- (26) **"Geodesic rigidity of Gauss"**: if p, q are two points on a manifold and if $\gamma_s(t)$ is a smooth family of geodesics connecting p, q , then all these geodesics must have the same length.
- (27) **Gauss Lemma in Riemannian geometry**: why are geodesics are perpendicular to wave fronts.
- (28) **"Exotic spheres?"** What are spheres with non-standard differential structure.
- (29) **"What is Willmore energy?"** The boy surface is known as "Oberwolfach surface". A theorem of Bryant-Kusner tells that the Boy surface minimizes the Willmore energy.
- (30) **"Geometry of hyperbolic space"**. i.e. geodesics, in a 2 dimensional manifold of constant curvature -1. Gauss-Bonnet for polygons.
- (31) **"Fenchel's theorem"** or the Fary-Milnor theorem about total curvature of curves. There is a section in Kuehnel.
- (32) **"The fundamental theorem of Riemannian geometry"** a connection satisfying three axioms must be the Christoffel connection.
- (33) **"Classifying 2-manifolds"**. Describe the classification of 2-manifold as a connected sum of tori or projective planes.
- (34) **"Spherical space forms"**. See for example Theorem 7.30 in Kuehnel.
- (35) **"What are lense spaces?"** What are spherical 3-manifolds?
- (36) **"Syngé's theorem"**: an even dimensional orientable positive curvature manifold is simply connected.
- (37) **"The Poincare conjecture"**. How was the Poincare conjecture proven. Especially define the Ricci flow using notation we have used.

- (38) **"Surgery of 3-manifolds"**. Explain how one can use knots to build 3-manifolds. Especially Dehn surgery and the Lickorish-Wallace theorem.
- (39) **"The Kerr Metric"**. A solution of the Einstein equations. The Kerr metric models a rotating non-charged black hole. The Kerr-Newman metric a rotating charged black hole.
- (40) **"Bertrand-Puiseux formulas"**. Prove $|S_r(p)| = 2\pi r - \pi K \epsilon^3/3 + \dots$
- (41) **"Chaotic Billiards"**. The geodesic flow in manifolds with boundary. is called a billiard. This involves curvature.
- (42) **"Riemannian metrics on Lie groups"**. Like on $SO(3)$ or $SU(2)$. How does one get a Riemannian metric on such spaces?
- (43) **"The Willmore conjecture"**. What is the Willmore energy? Explain the conjecture and how it was solved.
- (44) **"The tensor algebra"**. Especially the exterior algebra of skew symmetric tensors.
- (45) **"Counting Periodic geodesics."**
- (46) **"The isometric embedding problem R^3 ."**
- (47) **Open questions in Riemannian geometry**
- (48) **Look at the inverse Frenet problem in the case of piecewise linear periodic curvatures.**
- (49) **"Curvature of convex polyhedra in four dimensions."**
- (50) **"Curvature in a conformal metric."**