DIFFERENTIAL GEOMETRY

MATH 136

Unit 0: Introduction

0.1. This course is an introduction to the Riemannian geometry of curves, surfaces and manifolds. It also develops less technical discrete differential geometry. We see how it is applied to general relativity. Most differential geometry texts go too far for a 21 hour long course in which the goal is to prove Gauss-Bonnet in full detail using a multi-variable calculus and linear algebra background only.

0.2. Some texts in Riemannian geometry try to develop the theory "coordinate free". This is more elegant but also more abstract. It can come with frustrations. How can one "see" or "work" with an axiomatically defined connection for example? We opted for a concrete approach in which everything can be computed explicitly, either by hand or with computer algebra. A handful of lines of code suffice to compute the Einstein tensor of a manifold. Geodesics are directly introduced using the Euler-Lagrange variational principle. One can then generate geodesics on an arbitrary manifold.

0.3. Low dimensional Riemannian geometry is an active area of mathematics. There are many open problems and applications. There are close relations to computer science, computer graphics or data analysis; this again motivates is to sticking to coordinates. This is also how Einstein worked. A coordinate free approach is more elegant once one understands the nuts and bolts, but it can feel like dream walking without intuition. Especially for a beginner, it is good to be able to see everything in a concrete way at first. Going abstract is easier afterwards. Abstraction is also often used to hide ignorance with buzz words.

0.4. Riemannian geometry is not only a prototype mathematical theory, it also helps to inspire topology or combinatorics. Topological statements like that any simply connected 3-manifold is topologically equivalent to a sphere was solved by adding a Riemannian metric structure on the manifold first. Combinatorial notions of manifolds have led to nice combinatorial questions. We see also here, how easy the discrete set-up is. A precise frame work for discrete manifolds including all differential geometry is possible in a few paragraphs. Gauss-Bonnet for general networks almost feels like a joke, compared to the difficulties encountered in the continuum. Also a Sard works well in the discrete.

0.5. From the application side, the subject of differential geometry is a marvel. The geometry of Riemannian manifolds is the lingua franca of gravity. Riemannian manifolds are also an inspiration for the arts. The actual metric implementation of a manifold is important in aesthetics. Both very smooth or polyhedral implementations can matter. When looking at the shape of objects like cars, houses, cloths or tools, one can see fashion bounce between smooth and edgy. Geometric considerations also matter in data analysis. Artificial intelligence frame work embed knowledge in higher dimensional spaces and use distances to build large language models.

0.6. The subject is also saturated with unsolved problems. Simple sounding questions are unresolved like whether a positive curvature even dimensional manifold has positive Euler characteristic or whether there is a positive curvature metric on the product of two 2-spheres. We have even in the case of ellipsoids no answer yet about Jacobi's last problem estimating the number of cusps of caustics of wave fronts. How many closed geodesics are there for a given manifold? What are the manifolds in higher dimensions for which all geodesics are closed? For which type of manifolds do spectral properties determine the manifold? Are two manifolds with isomorphic geodesic flows and equal diameter automatically isometric?

0.7. In the 21 lectures allocated, we reach four mountain peaks:

(1) the Frenet-Serret theorem
(2) the Gauss-Bonnet theorem
(3) the Theorema egregium
(4) the Einstein equations

Unlike other first course in Riemannian geometry, we do not restrict to curves and surfaces. But Gauss-Bonnet and the Theorema egregiums is only proven for 2-manifolds. The higher dimensional Gauss-Bonnet-Chern theorem would be harder to prove. We reuse much of the lecture notes from last year. Lectures cover the notes with short-cuts and uses also time for discussion and examples not written down here.

0.8. The literature list is same as in 2024:

- W. Kühnel, Differential Geometry: Curves - Surfaces - Manifolds", 3. Edition.
- M. P. Do Carmo, Differential Geometry of Curves and Surfaces", 2. Edition. [1]
- M. Berger, "A panoramic view of Riemannian geometry". A marvel.
- S. Gudmundsson, "An introduction to Gaussian Geometry". (Nice 2023 notes).
- I.A. Taimanov, "Lectures on Differential geometry". (Refreshingly short).
- H.L. Cycon, R.G. Froese, W.Kirsch, B.Simon: "Schrödinger operators". (Ch.12).
- J.A. Thorpe, "Elementary Topics in Differential geometry".
- A. Pressley, "Elementary Differential geometry". (2. edition).
- M. Lipschultz, "Schaum Outline: differential Geometry".
- C.W. Misner, K.S. Thorne, and J.A. Wheeler "Gravitation".
- Y.Choquet-Bruhat: "General relativity and the Einstein equations".
- J.A. Wheeler: "Journey into gravity and space-time".
- Y. Choquet Bruhat and C. DeWitt-Morette: "Analysis, Manifolds and Physics".
- J. Marsden and T. Ratiu: "Manifolds, Tensor Analysis".
- O. Knill, "Introduction to geometry and geometric analysis". (1995 notes).

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[1]Both texts have been used traditionally in the last 20 years here at Harvard

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MATH 136

Lecture 1: What is differential geometry?

Introduction

1.1. Differential geometry deals with geometric objects called **manifolds**. We later will define manifolds intrinsically. It makes more sense however to look first at manifolds embedded in a Euclidean space $\mathbb{R}^n$ and in particular in $\mathbb{R}^3$ and even give concrete parametrizations for them. One-dimensional manifolds are known as **curves** or two dimensional manifolds as **surfaces**. The level sets of smooth function $f : \mathbb{R}^n \to \mathbb{R}$ can produce hypersurfaces of dimension $n-1$. The 3-sphere $x_1^2+x_1^2+x_3^2+x_4^2 = 1$ for example is a 3-dimensional manifold embedded in 4-dimensional space. Since we live in a 3-dimensional space, we are at first mostly interested in one or two dimensional manifolds in $\mathbb{R}^3$. These are **curves** or **surfaces** in a traditional sense.

1.2. A **curve** is given as a **parametrization** $r(t) = [x(t), y(t), z(t)]$, where $t \in [a, b]$ is the parameter domain and $x(t), y(t), z(t)$ are functions of one variable. In two dimensions, some curves can also be written implicitly as a **level curve**. An example is the ellipse $x^2/4 + y^2/9 = 1$. A **surface** is given as a parametrization $r(u, v) = [x(u, v), y(u, v), z(u, v)]$, where (u, v) is in some domain $R \subset \mathbb{R}^2$ in the uv-plane. Some surfaces can be given as **level surfaces** $f(x, y, z) = 0$. The intersection of level surfaces $f(x, y, z) = 0, g(x, y, z) = 0$ is often a **curve**. If f, g are polynomials it is often a one-dimensional **variety**.

1.3. We are interested in **global quantities** like **arc length** $\int_a^b |r'(t)| \, dt$ or **surface area** $\iint_R |r_u \times r_v| \, dudv$, where $\times$ is the **cross product**. We also use **local quantities** like **curvature** $\kappa(t) = |r'(t)\times r''(t)|/|r'(t)|^3$ or **torsion** $\tau(t) = \det[r'(t), r''(t), r'''(t)]/|r'\times r''|^2$ for curves. For a surface, the **curvature** of a point can be defined as $K(p) = \lim_{r\to 0} 3\frac{2\pi r-|S_r(p)|)}{\pi r^3}$, where $|S_r(p)|$ is the length of the **wave front** $S_r(p)$ of points on the surface in distance r from p which is a circle for small r. This intrinsic definition does not making use of the embedding of the surface in an ambient space. It even makes sense on non-smooth surfaces, like polyhedra.

1.4. Curvature plays an important role in differential geometry. We will define it differently later in the course and verify that it is independent of the embedding in space. This is the **Theorema egregium**, the "great theorem" of Gauss from 1827. Riemannian geometry, the idea of doing geometry on a manifold without having to embed it into an ambient emerged in an inaugural lecture of Riemann in 1854. The theory is used heavily in Einstein's 1915 theory of general relativity. Schwarzschild found a black hole solution in 1916. A 100 years later, gravitational waves from black hole mergers have been observed.

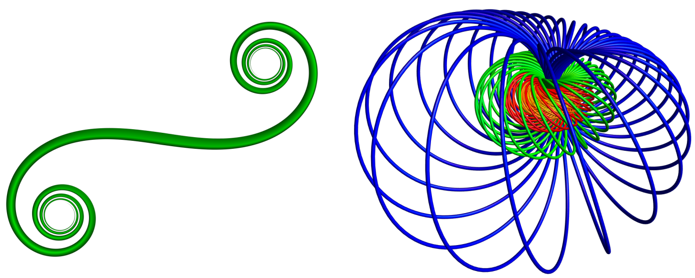

FIGURE 1. To the left we see a curve called Euler curve, to the right, we see a visualization of the 3 dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. It is foliated by 2 dimensional flat tori. We can not embed flat tori in $\mathbb{R}^3$ but we can in $\mathbb{R}^4$.

1.5. The focus of differential geometry is to investigate relations between **local quantities** and **global properties**. An important example to observe what happens if curvature is integrated up. We are especially interested in quantities that do not depend on the metric, like the **Euler characteristic**. Here are examples which will appear early in this course: for a planar closed curve in $\mathbb{R}^2$ one can define the **signed curvature** as $\kappa(t) = (r'(t) \times r''(t))/|r'(t)|^3$. We will then see the **Hopf Umlaufsatz** $\int_C \kappa(t)\ dr(t) = 2\pi$. For a two-dimensional surface with g holes of Euler characteristic $\chi(G) = 2 - 2g$, one has the **Gauss-Bonnet theorem** $\int\int_R K(x)\ dV(x) = 2\pi\chi(G)$. We also want to understand **geodesics**, curves that locally minimize length. One can start geodesics into any direction $v/|v|$ and let it run for a distance $|v|$. This produces the **exponential map** $\exp_p$, a map from the tangent space T_pM of a point p to the manifold. If S_r is the sphere of radius r in $\mathbb{R}^2$, then the image $W_r(t) = \exp_p(S_r)$ is called the **wave front** at p. These waves can become complicated for large r. This can also be studied on polyhedra. We expect wave fronts to become dense in the manifold, except for very special cases like the round sphere.

1.6. Differential geometry then extends curve and surface theory to arbitrary dimensions. One study then so called **Riemannian manifolds** or **pseudo-Riemannian manifolds** which appear in physics. There is an intrinsic geometry but also interest when manifolds M are embedded in larger manifolds M'. In general relativity for example, space is a 3-dimensional manifold embedded in a four dimensional **space-time manifold** M'. The above formulation of curves or surfaces dealt with embeddings of one or two dimensional manifolds in Euclidean 3-manifold M'. One can use the exponential map to define **sectional curvature** and to use it to define a **curvature tensor** or **scalar curvature**. The extrema of the functional that gives the total scalar curvature are the Einstein equations. **General relativity** studies solutions of these equations as they tell how matter bends space. The geodesic equations then tell, how matter moves in this space.

Lecture 2: Preliminaries

2.1. A map

$$f : \mathbb{R}^m \to \mathbb{R}^n, \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} \mapsto \begin{bmatrix} f_1(x_1, \dots x_m) \\ \dots \\ f_n(x_1, \dots, x_m) \end{bmatrix}$$

is called **differentiable** or C^1, if all derivatives $\frac{\partial}{\partial x^j} f_i$ are continuous. For such a map, define the $n \times m$ **Jacobean matrix**

$$df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_1(x_1, \dots, x_m) \\ & \dots & \\ \frac{\partial}{\partial x_1} f_n(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_n(x_1, \dots, x_m) \end{bmatrix} .$$

2.2. a) If $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a curve, then $dr(t) = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ is the **velocity**.
b) If $f(x, y, z)$ is a function of 3 variables, then $df(x, y, z) = [f_x, f_y, f_z]$ is a 1×3 matrix. It is the transpose of the gradient.
c) For a vector field $f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$, we have $df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$. It is used also when analyzing systems of differential equations $x' = f(x)$.
d) If $f(x) = Ax$ is a linear map, then $df(x) = A$.

2.3. We say df has **maximal rank** if its rank is the minimum of m, n. If $m < n$, then the image of f is in general a m-dimensional set in $\mathbb{R}^n$. If $n < m$, then we can look at the roots $f(x) = 0$ which is a $n - m$ dimensional set. If $m \leq n$, we can say that f is **manifold like** near $f(x)$ if $df(x)$ has maximal rank. If $n \leq m$, then $M = \{f - c = 0\}$ is **manifold like** near x, if $df(x)$ has maximal rank. In a homework you review the fundamental theorem of linear algebra $\ker(A^T) = \text{im}(A)^\perp$.

2.4. Example: $m = 2, n = 1$ and $f(x, y) = x^3y^2 - xy^3 = -4$. What happens near $(x_0, y_0) = (1, 2)$. To investigate this, we look at the Jacobean matrix $df(x, y) = [3x^2y^2 - y^3, 2x^3y - 3xy^2]$ which is at $(1, 2)$ equal to $[4, -8]$. The line $4x - 8y = -12$ is tangent to the curve. If we write the curve as $y = g(x)$ near $x = 1$, then $f(x, g(x)) = -4$ and differentiating gives $f_x + f_y g' = 0$ so that $g' = -f_x/f_y$. The next theorem assures that $g(x)$ exists.

Theorem 1 (Implicit function theorem)**.** *If $f(x, y) = f(x_0, y_0) = c$ and $f_y(x_0, y_0) \neq 0$, then $f(x, y) = c$ can near (x_0, y_0) be written as $y = g(x)$ for some C^1 function $g(x)$.*

Proof. Take a small neighborhood $U = I \times J$ of (x_0, y_0), where $|f_y(x,y)| \geq c$ and $|f_x(x,y)| \leq d$. Given $(x,y) \in U$ and $f(x_0,y_0) = 0$ and $y \to f_y(x,y)$ is bounded away from 0, we have $f(x, y_0 - t)f(x, y_0 + t) < 0$ and by the **intermediate value theorem**, there exists for x close to x_0 a $t = t(x)$ such that $f(x, y_0 + t(x)) = 0$. This gives us a function $g(x) = y_0 + t(x)$ and $f(x,y) = c$ agrees with $y = g(x)$ in U. By the chain rule $\frac{\partial}{\partial x} f(x, g(x)) = f_x \cdot 1 + f_y \cdot g' = 0$, we see that g is differentiable with $g' = -f_x/f_y$ and $|g'(x)| \leq d/c$ is bounded in U. □

2.5. It follows that if a C^1 function f is manifold like near (x_0, y_0) then the level set $f(x,y) = c$ is near (x_0, y_0) the graph of a function: Proof: if $f_y \neq 0$ use the implicit function theorem giving $y = g(x)$. If $f_y = 0$, we have $f_x \neq 0$ and $f(x,y) = c$ is the graph of $x = g(y)$ for some C^1 function g.

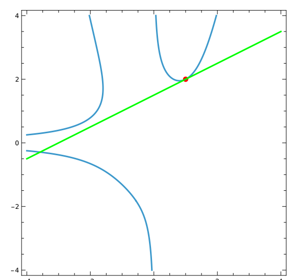

FIGURE 1. $f(x,y) = c$ can near (x_0, y_0) be written as a graph $y = g(x)$.

2.6. If $m < n$, then f has maximal rank if df has rank m. We look at this case more next week and look at $g = df^T df$. Lets look at a C^1 curve C defined by the parametrization $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Its velocity if $dr = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$. If the velocity is not zero at $t = t_0$ then the curve is manifold like. The curve is then close to the line $l(s) = r(t_0) + sr'(t_0)$. If $r'(t_0) \neq 0$, then C is a graph $(s, g(s))$ close to the line $(s, l(s)$.

2.7. In the case $m = n$, the map $f : \mathbb{R}^n \to \mathbb{R}^n$, the image of f is manifold like near x if $\det(df) \neq 0$. In that case f is invertible near x:

Theorem 2 (Inverse function theorem)**.** *If $h(y) \in C^1$ and $h'(y_0) \neq 0$, then h is invertible near y_0 and $y = g(x)$ near x_0. Also, g is C^1 with $dg(x_0) = dh(y_0)^{-1}$.*

Proof. Define $f(x,y) = x - h(y) = 0$. As $f_y = -h'(y)$ is non-zero, the theorem applies: there is $g(x)$ with $y = g(x)$ near x_0. It is the inverse: $x = h(y)$ and $y = g(x)$. □

2.8. All proofs can be generalized to maps $f : \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto f(x,y) \in \mathbb{R}^n$. Now f_x is a $n \times k$ matrix and f_y is a $n \times n$ matrix. If f_y is invertible, then df has maximal rank and $f(x,y) = c$ can be written as $y = g(x)$ and $n \times k$ matrix $dg = -f_y^{-1} f_x$. This is the implicit function theorem. If $k = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, then $df(x)$ invertible implies f is invertible near x.

2.9. A point $x \in \mathbb{R}^m$ is a critical point of f if $df(x)$ does not have maximal rank. The value $f(x)$ is then called a **critical value**. In the homework you look up the Sard theorem assuring that for smooth f almost all values are critical values.

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Lecture 3: Surfaces

3.1. Geometric objects can be given as **level sets**, kernels $\{f = 0\}$ of smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$ with $n < m$ or **parametrizations**, images of smooth maps f from a subset R of $\mathbb{R}^m \to \mathbb{R}^n$ with $m < n$. If in the level set case, df has maximal rank n everywhere, we get a **manifold**. [1] The same happens in the parametrization case, if f is injective and df has maximal rank m everywhere.

3.2. An example of **level surface** $\{f = 0\}$ of a function $f : \mathbb{R}^3 \to \mathbb{R}$ with $df \neq 0$ everywhere is the sphere $x^2 + y^2 + z^2 - 1 = 0$. An other example is a curve, the image of an interval $[a, b]$ to $\mathbb{R}^n$. The duality between kernel and image manifests already in linear algebra. The **kernel** $\ker(A)$ of a matrix A is the linear space $\{Ax = 0\}$ The **image** $\mathrm{im}(A)$ is the linear space $\{Ax\}$. The **fundamental theorem of linear algebra** is the wonderful duality $\boxed{\mathrm{im}(A^T) = \ker(A)^\perp}$.

Theorem: The image of A^T is perpendicular to the kernel of A.

Contour surfaces

3.3. If $f : \mathbb{R}^n \to \mathbb{R}$ is given, then the solutions of $f(x_1, \cdots, x_n) = d$ is called **hyper surface** or simply **surface** if $n = 3$. If the Jacobian matrix df (or equivalently the gradient $\nabla f = df^T$ is non-zero (meaning has maximal rank at every point), then $f = d$ is an example of a manifold. We will give more definitions later.

3.4. The case $f(x) = Ax$ is a hyperplane. **Quadratic manifolds** are $f(x) = x \cdot Bx + Ax = d$, where B is a symmetric matrix, A is a row vector and $d \in \mathbb{R}$ and df has maximal rank. Write $\mathrm{Diag}(a_1, \dots, a_n)$ for diagonal and 1 for the identity matrix.

3.5. Examples: For $B = 1$ and $A = 0$ and $d = 1$ we get the **sphere** $|x|^2 = 1$. For $B = \mathrm{Diag}(1/a^2, 1/b^2, 1/c^2)$ is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ we get ellipsoids. For $B = \mathrm{Diag}(1, 1, -1)$ and $d = 1$, we get a **one-sheeted hyperboloid** $x^2 + y^2 - z^2 = 1$. For $B = \mathrm{Diag}(1, 1, -1)$ and $d = -1$, we get a **two-sheeted hyperboloid** $x^2 + y^2 - z^2 = -1$. For $B = \mathrm{Diag}(1, 1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **paraboloid** $x^2 + y^2 = z$, for $B = \mathbf{Diag}(1, -1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **hyperbolic paraboloid** $x^2 - y^2 = z$. We can recognize paraboloids by intersecting with $x = 0$ or $y = 0$ to see parabola. If $B = \mathrm{Diag}(1, 1, -1)$ and $d = 0$, we get a **cone** $x^2 + y^2 - z^2 = 0$. For $B = \mathrm{Diag}(1, 1, 0)$ and $d = 1$ we get the **cylinder** $x^2 + y^2 = 1$.

[1] A theorem of Nash assures that every m-manifold can be embedded in some $\mathbb{R}^n$.

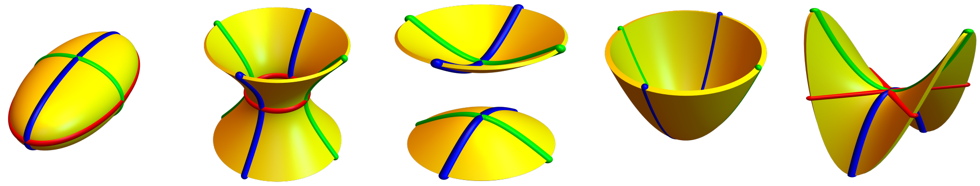

FIGURE 1. Ellipsoid, hyperboloids and paraboloids.

PARAMETRIZATIONS

3.6. A map $r : \mathbb{R}^m \to \mathbb{R}^n$ is called a **parametrization**. It is custom to use the letter r here, rather than f. We take the case $m < n$ and especially $m = 2, n = 3$. A map r from $\mathbb{R}$ to $\mathbb{R}^n$ is a **curve**. The image of a map $r : R \subset \mathbb{R}^m \to \mathbb{R}^n$ is then a **m-dimensional surface** in $\mathbb{R}^n$.

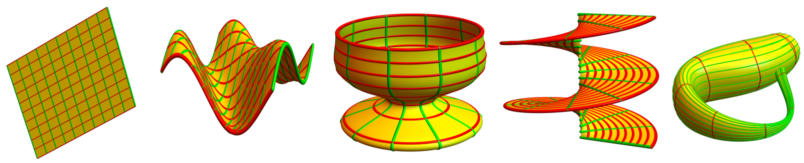

FIGURE 2. A plane, graph, surface of revolution, helicoid and Klein bottle

3.7. The parametrization $r(\phi, \theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ produces the **sphere** $x^2 + y^2 + z^2 = 1$. The full sphere uses $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$. By modifying the coordinates, we get an **ellipsoid** $r(\phi, \theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$ satisfying $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By allowing a, b, c to be functions of ϕ, θ we get "bumpy spheres" like $r(\phi, \theta) = (3 + \cos(3\phi)\sin(4\theta))[\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$.

3.8. If $r : \mathbb{R}^m \to \mathbb{R}^n$ with $m < n$ is a parametrization, then **Jacobian matrix** $dr(x)$ produces the $m \times m$ matrix with $\boxed{g = dr^T dr}$. It is the **first fundamental form**. For a parametrization $R : \mathbb{R}^2 \to \mathbb{R}^3$, the 3×2 matrix $dr(u, v)$ contains the vectors $\partial_u r = r_u, \partial_v r = r_v$ as columns and g is a 2×2 matrix.

3.9. The number $\boxed{|dr| = \sqrt{\det(dr^T dr)}}$ is called the **volume distortion factor**. The integral $\boxed{\int_R |dr(x)|\, dx}$ is the m-dimensional volume of the images $r(R) \subset \mathbb{R}^n$.

3.10. For a surface in $\mathbb{R}^3$, the surface area is $\boxed{\int\int_R |r_u \times r_v|\, dudv}$ because

Theorem: $\det(dr^T dr) = |r_u \times r_v|^2$ for $r : \mathbb{R}^2 \to \mathbb{R}^3$.

Proof. As $dr^T dr = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$, the identity is the **Cauchy-Binet identity** $|r_u \times r_v|^2 = |r_u|^2 |r_v|^2 - |r_u \cdot r_v|^2$ which boils down to $\sin^2(\theta) = 1 - \cos^2(\theta)$, where θ is the angle between the tangent vectors r_u and r_v. □

Lecture 4: Curves

4.1. **Curves** in $\mathbb{R}^n$ can be be either given as images of smooth maps $r : \mathbb{R} \to \mathbb{R}^n$ or as solutions $f = 0$ to $(n-1)$ equations $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$. In the first homework, you have seen the intersection of $n - 1 = 2$ surfaces $f_1 = c$ and $f_2 = 0$ in $\mathbb{R}^3$ which gave sections of the cyclide. Looking at **solution sets of equations** is more like a **algebraic geometry** thing. Here, in differential geometry, we primarily look at **parametrizations** $[a, b] \to \mathbb{R}^n$. An example of a curve is the **helix** $r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$ in $\mathbb{R}^3$. [1]

4.2. The **Jacobean matrix** of a curve $r(t)$ is $\boxed{dr(t)}$

$$r(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}, dr(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \cdots \\ x_n'(t) \end{bmatrix} = r'(t) \; .$$

It is also known as the **velocity** and just abbreviated $\boxed{r'(t)}$. The **first fundamental form** is

$$g = dr^T dr = |r'(t)|^2 \; .$$

It is the square of the speed. The **arc length** of the curve is defined as

$$L = \int_a^b |r'(t)| \; dt \; .$$

Related to arc length is the **action**

$$I = \int_a^b |r'(t)|^2 \; dt \; .$$

which has the advantage that it can be computed better and produces equivalent variational problems. Minimizing the arc-length is equivalent to minimize the action and leads to geodesics. Here we are in flat Euclidean space and geodesics are straight lines. We will say more about this in class. You show in the homework:

Theorem 1 (Archimedes)**.** *The straight line is the shortest path connecting $A, B \in \mathbb{R}^n$.*

[1] We will often write also just $[\cos(t), \sin(t), t]^T$ or simply $[\cos(t), \sin(t), t]$ without the transpose for typographic reasons.

4.3. A curve is called **simple** if r is it does not have self intersections. It is called **regular** if the first fundamental form is nowhere zero. Equivalently, this means that the velocity is nowhere zero. A simple closed curve in space is called a **knot**. An example is the **figure 8 knot**

$$r(t) = [(2 + \cos(2t))\cos(3t), (2 + \cos(2t))\sin(3t), \sin(4t)]^T$$

parametrized on $[0, 2\pi]$. We talk more about this in class like that it lives on a torus and why you can not tie knots in $\mathbb{R}^n$ for $n > 3$. r is **simple** can be rephrased that the map $r : [0, 2\pi) \to \mathbb{R}^3$ is **injective**.

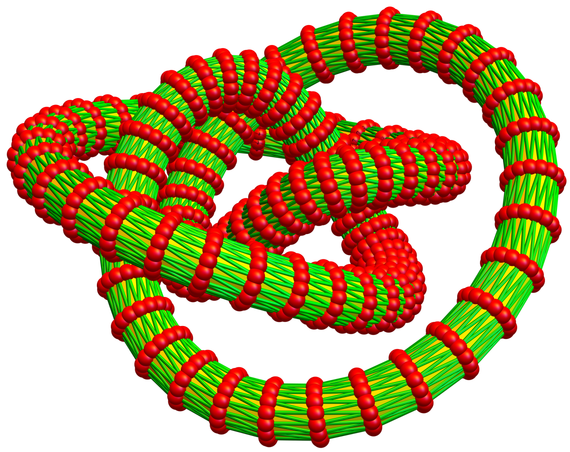

FIGURE 1. A picture of a knot. Drawing the curve in such a fancy way, needs concepts like curvature and torsion, which we will learn next week. The first fundamental form of the figure 8 knot is $r'(t)^2 = 101/2 + 36\cos(2t) + (5/2)\cos(4t) + 8\cos(8t)$. The action is $I = 101\pi$, the arc length involves elliptic integrals. Numerically it is $L = 42.966\dots$. It is typical that we can explicitly give the action but not the length.

4.4. A curve is **parametrized by arc length** if $|\vec{r}\,'(t)| = 1$ for all t. You will prove in homework the following important result:

Theorem 2. *Every smooth regular curve in $\mathbb{R}^n$ can be parametrized by arc-length.*

4.5. It is custom to write $r(s)$ to indicate that we have an arc length parametrization. For the helix above, the arc length parametrization is $r(s) = [\cos(s/\sqrt{2}), \sin(s/\sqrt{2}, 1/\sqrt{2}]$. In general we do not bother to actually compute the arc length parametrization. Already in simple cases like the ellipse it would get nasty. We can use the theorem however to build theory and prove stuff about curves.

DIFFERENTIAL GEOMETRY

MATH 136

Lecture 5: Frenet Equations

5.1. Today, we work at smooth curves $r : [a, b] \to \mathbb{R}^3$. [1] Define the **unit tangent vector** $T(t) = r'(t)/|r'(t)|$, the **normal vector** $B(t) = T'(t)/|T'(t)|$ and the **binormal vector** $B(t) = T(t) \times N(t)$. The three vectors are defined, as long as as r' and T' are both non-zero. One calls it a **Frenet frame** (T, N, B). A smooth curve is called a **Frenet curve** if r', r'' are linearly independent at every t. This is equivalent to the statement $\boxed{r' \times r'' \neq 0}$ for every t.

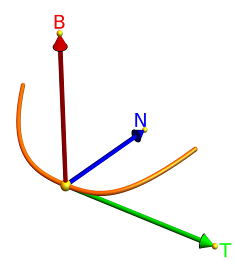

Figure 1. Direction T, normal N and binormal direction B.

Theorem 1. *For a Frenet curve, the Frenet frame TNB is orthonormal at every point.*

Proof. By assumption, r' and r'' are both not zero. If the parametrization is arc length, then $T = r'$ and $N = T'/|T'| = r''/|r''|$. Since $T \cdot T = 1$, we have by Leibniz product rule, $2T' \cdot T = 0$ so that N is perpendicular to T. The cross product $T \times B$ now also has length 1 and is perpendicular to both T and N. □

5.2. Every Frenet curve $r(t)$ can be parametrized by arc length as you work out in the homework. The **curvature** κ is then defined as $\boxed{\kappa = T' \cdot N}$ which is $|T'|$. The curvature measures the deviation of the curve from being linear. The **torsion** τ is defined as $\boxed{\tau = N' \cdot B}$. It measures the deviation from the curve of being planar. We can encode the three vectors T, N, B by turning them into row vectors of an **orthogonal** 3×3 **matrix** $Q(t) = \begin{bmatrix} T & N & B \end{bmatrix}^T$. We get now $Q'(t) = K(t)Q(t)$, where $K(t)$ is skew-symmetric:

[1] In $\mathbb{R}^3$, one requires the map $r : [a, b] \to \mathbb{R}^3$ to be at least C^3.

5.3.

Theorem 2 (Frenet equations). $\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$

Proof. Because $N \cdot T = 0$, we have $N' \cdot T = -T' \cdot N = -\kappa$. The relation $N \cdot N = 1$ implies $N' \cdot N = 0$. The relation $B \cdot B$ implies $B' \cdot B = 0$. Now expand with respect to the basis $\{T, N, B\}$ and use $\kappa = |T'|$ and $\tau = N' \cdot B$:

$$\begin{aligned} T' &= (T' \cdot T)T \ + (T' \cdot N)N + (T' \cdot B)B = 0 \ + \ 0 \ + \ \kappa N \\ N' &= (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B = -\kappa T + 0 \ + \ \tau B \\ B' &= (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B = 0 \ - \tau N + \ 0 \ . \end{aligned}$$

□

5.4. In the two dimensional case, we only have to consider T and N. We can reduce to the planar case if τ is constant 0. The Frenet equations can then be written as $\begin{bmatrix} T \\ N \end{bmatrix}' = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}$.

5.5. The **fundamental theory of curves** in $\mathbb{R}^3$ tells that curvature and torsion determines a curve up to Euclidean congruences given by rotations or translations.

"The shape of a Frenet curve is determined by curvature and torsion".

Lemma 1. *For any differentiable curvature and torsion functions $\kappa(t) \geq 0$ and $\tau(t)$, there exists up to translation and rotation a* **unique** *curve $r(t)$ parametrized by arc length that has the given curvature and torsion.*

Proof. The initial $r(0)$ and $(T(0), N(0), B(0))$ fixes the initial point and location. Now "build the curve": the functions $\kappa(t), \tau(t)$ define a **skew symmetric** matrix $K(t) = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$. We define orthogonal matrix $Q(t)$ satisfying the differential equation $Q'(t) = K(t)Q(t)$. In the homework you will verify that a differential equation $x' = F(t, x)$ with a C^1 function F has locally a unique solution and that if $x(t)$ stays bounded, the solutions exist for all times. You also will check that if $Q(t)$ is a curve of orthogonal matrices, then $Q' = KQ$ with skew symmetric K. This can be reversed: if $Q(0)$ is orthogonal and $K(t)$ is skew symmetric, then the solution $Q(t)$ of the differential equation is orthogonal. Having $Q(t)$, it produces $r'(s) = Q(s)r'(0)$ and so $r(t) = r(0) + \int_0^t r'(s) \, ds$, where $r'(t) = r'(0) + \int_0^t Q^T(s) r'(0) \, ds$. □

5.6. We have mentioned in the warm-up class expressions for curvature and torsion for a curve $r(t)$. These formulas worked if the curve was not necessarily arc-length parametrized. In the Frenet case, meaning that $r' \times r'' \neq 0$, we will prove them in class:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}, \quad \tau(t) = \frac{\det[r'(t), r''(t), r'''(t)]}{|r' \times r''|^2} \ .$$

In class we will look at examples where we can compute the curvatures and torsion. A case where κ is arc length and a case where $\kappa = \tau$.

Lecture 6: Fundamental theorem of curves

6.1. A **Frenet curve** is given by a C^n map $r : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ for which $r', r'', \dots, r^{(n)}$ are linearly independent at every point. In the case $n = 3$ we had seen this as $r' \times r'' \neq 0$. Let $e_1, e_2, \dots, e_n$ denote the orthonormal frame obtained by Gram-Schmidt. One can get this as follows: build the matrix R with $r', r'' \dots, r^{(n)}$ as rows and perform the QR decomposition to get an orthonormal matrix Q in which the vectors $e_1, e_2, \dots, e_n$ are the rows. [1] Define the curvatures $\boxed{\kappa_j = e_j' \cdot e_{j+1}}$. It is positive for $1 \leq j \leq n - 2$. The largest κ_{n-1} is also called the **torsion** and is not necessarily positive. A natural generalization of the Frenet formulas to arbitrary dimensions is

Theorem 1 (Frenet-Serret formulas)**.**

$$\begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ \cdots \\ e_{n-1} \\ e_n \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \cdots & \cdots \\ 0 & -\kappa_2 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & \kappa_{n-1} \\ 0 & \cdots & \cdots & 0 & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ \cdots \\ e_{n-1} \\ e_n \end{bmatrix} .$$

Proof. To get the entries of K expand e_j' in terms of the $e_1, \dots, e_n$.

$$e_j' = \sum_{i=1}^{n} (e_j' \cdot e_i) e_i \ .$$

This means $Q' = KQ$ with skew symmetric K. Especially, the diagonal entries of K are zero. The skew symmetry can be seen from $e_j \cdot e_k = 0$ for all $j \neq k$ implying $e_j' \cdot e_k = -e_j \cdot e_k'$. For every $j \leq n-1$, the e_j by definition are in the subspace generated by $r', r'', \dots, r^{(j)}$ which is the subspace generated by $e_1, \dots, e_j$ and e_j' therefore generated by $e_1, \dots, e_j$. This implies $e_j' \cdot e_{j+2} = e_j' \cdot e_{j+3} = \cdots = e_j' \cdot e_n = 0$. The only entry in the upper triangular part is $(e_j' \cdot e_{j+1}) = \kappa_j$. $\square$

6.2. One can check the skew symmetry of K abstractly starting with $Q^T Q = 1$. A fancy way to restate is that in the Lie group $SO(n)$, the tangent space is the Lie algebra $so(n)$.

[1] Frenet and Serret have discovered the $n = 3$ dimensional case independently. The higher dimensional case has appeared only in the 20th century.

Lemma 1. *If $Q(t)$ is a curve of orthogonal matrices, then $Q'(t) = K(t)Q(t)$ with skew symmetric K.*

6.3. Given curvatures $\kappa_1(t) > 0, \dots, \kappa_{n-2}(t) > 0, \kappa_{n-1}(t)$ which are all continuous, we get a continuous path $K(t)$ of skew symmetric matrices.

Theorem 2 (Fundamental theorem of curves)**.** *Given curvatures κ_j, there is up to translation and rotation a unique Frenet curve which has these curvatures.*

Proof. The curvatures define a curve $K(t)$ of skew symmetric matrices. The differential equation $Q' = K(t)Q = F(t, Q)$ is linear in Q and so smooth. Since the solution of this differential equation gives orthogonal matrices $Q(t)$ (check it!) the solution exists for all times. Proceed as in the 3 dimensional case by writing $r(t) = r(0) + \int_0^t r'(s)\, ds$ where $r'(s) = Q(s)_1$ is the first row of $Q(s)$. □

6.4. Examples.
1) If K is constant, then e^{Kt} solves $Q' = KQ$.
2) If K is constant and $n = 3$, then the curve is a spiral if $\tau \neq 0$ and a circle if $\tau = 0$.
3) In $\mathbb{R}^3$, the torsion is constant zero if and only if the curve is contained in a plane.
4) In $\mathbb{R}^n$ the torsion is constant zero if and only if the curve is contained in a $(n-1)$ dimensional hyperplane.
5) A line is not a Frenet curve and the above does not apply.
6) For non-Frenet curves, lots of things can go wrong. Assume for example, you have a curve which contains some part which is a line. While traveling along that line, we can turn around and lose track of the Frenet frame.

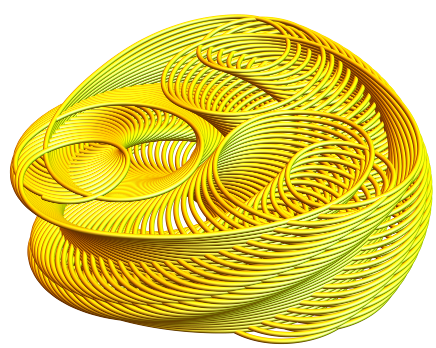

FIGURE 1. It can be fun to watch curve with simple $\kappa(t) = a + b\cos(ct), \tau(t) = d\sin(et)$

6.5. A famous example is the Euler curve. It is a plane curve for which $\kappa(t) = t$ is fixed. Here is a possible open research problem. Given periodic functions $\kappa(t) > 0, \tau(t)$ not necessarily with the same period. Under which conditions is the resulting curve r bounded when looking at $t \in \mathbb{R}$?

Lecture 7: Hopf Umlaufsatz

7.1. The theorem of today deals with **signed curvature** $\kappa = \frac{r' \times r''}{|r'|^3}$ using the cross product in 2 dimensions. [1]. We do not assume today that the curve is Frenet. The curvature is allowed to become zero. We assume however that the curve is regular, meaning that $dr = r'$ is never zero, as well as closed. We have seen that there is then an arc length parametrization of the curve and $|\kappa| = |r''|$ because $r' \cdot r' = 1$ implies r'' is perpendicular to r'. We deal with the last curvature and so torsion which always can have signs.

7.2. If the curve is parametrized on $[a, b]$, the **rotation index** is defined as $\frac{1}{2\pi} \int_a^b \kappa(t)\, dt$. In general, if the closed curve is not arc length parametrized, the rotation index is defined as $\boxed{\int_a^b \kappa(t)|r'(t)|\, dt}$.

Theorem 1. *The rotation index of a closed C^2 curve is in $\mathbb{Z}$.*

Proof. Using arc length parametrization, write

$$r'(t) = [\cos(\alpha(t)), \sin(\alpha(t)]$$

then $\kappa = \alpha'$ (we did that more detailed in class). Since the curve is closed, we have $\alpha(b) - \alpha(a) = 2\pi n$, where n is an integer. □

7.3. The case $r(t) = [\cos(nt), \sin(nt)]$ with $t \in [0, 2\pi]$ shows that the rotation index can take any integer value n. It is intuitively clear that if a curve has no self intersections, then the index must be either 1 or -1. This is not so obvious however. We do not want for example to refer to the Jordan curve theorem, telling that a continuous simple closed curve in the plane divides the plane into an inside and outside. Heinz Hopf found a nice argument which proves this "Umlaufsatz" (rotation angle theorem) in an elegant way using a deformation picture:

Theorem 2 (Hopf Umlaufsatz)**.** *A simple closed regular C^2 curve has rotation index 1 or -1.*

Proof. We assume that $r(t)$ is parametrized on $[0, 1]$ and parametrized by arc-length. Define on the square $Q = [0, 1] \times [0, 1]$ the function $f : Q \to \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\}$ given by $f(t, s) = \arg((r(t) - r(s))/|r(t) - r(s)|)$ for $t \neq s$ and $\alpha(t) = f(t, t) = \arg(r'(t)/|r'(t)|)$ for $t = s$. Because $r \in C^1$, the function f is continuous on $Q = [0, 1] \times [0, 1]$. Now comes

[1]The cross product in n dimensions has $\binom{n}{2} = n(n-1)/2$ components. For $n = 2$ it is a scalar

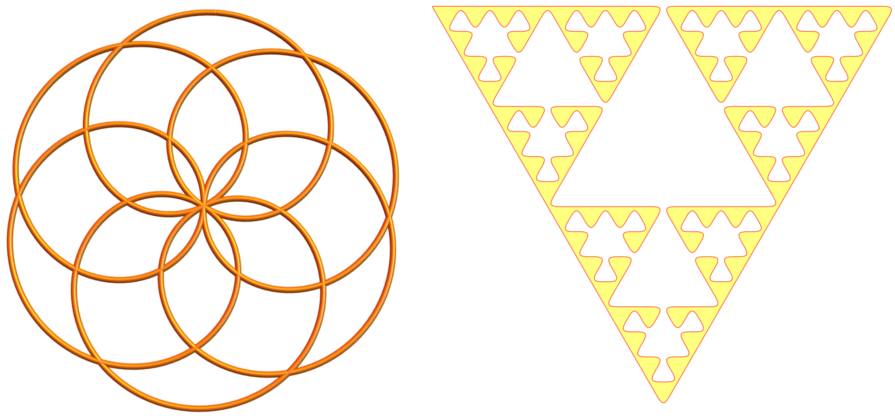

FIGURE 1. To the left, we see $r(t) = [\cos(t)+\cos(7t), \sin(t)+\sin(7t)], t \in [0, 2\pi]$ which looks like the flower of life. Its rotation number is 7. We can compute $\kappa(t)|r'(t)| = 4 + 72/(25 + 8\cos(6t))$ which integrates on $[0, 2\pi]$ up to 14π. To the right, a simple closed smooth curve in the plane. What is its rotation number?

a **homotopy argument**. The index is $[f(1,1) - f(0,0)]/(2\pi)$ and so an integer. If we move along the diagonal and look at $\alpha(t) = f(t,t)$ we see a continuous curve which moves on the circle $\mathbb{T}$. If we deform the curve, the total change remains the same. We can continuously deform the parameter curve so that we first deform from $(0,0)$ straight to $(0,1)$ and then straight from $(0,1)$ to $(1,1)$. Choose a coordinate system so that is in $y \geq 0$ just touching the x-axes. If $r'(0) = [a, 0]$ with positive a then $f(t,s) \in [0,\pi]$ with $f(0,0) = 0$ and $f(0,1) = \pi$ and then $f(1,1) = 2\pi$. If $a < 0$, then $f(t,s) \in [-\pi, 0]$ with $f(0,0) = \pi$ and $f(0,1) = 0$ and then $f(1,1) = -\pi$. In the former case, $i = 1$ in the later $i = -1$. $\square$

7.4. Remarks:
1) This is a Gauss-Bonnet type result for a flat 2-manifold with boundary.
2) The proof shows that this even works for C^1 curves as $f(t,t) - f(s,s)$ is just the angle change of the tangent. This works even if the curvature is not defined. In the homework you will push it to polygons. Most texts assume C^2.
3) Closed C^0 curves are trickier: Jordan curve theorem, open peg problem of Toeplitz.

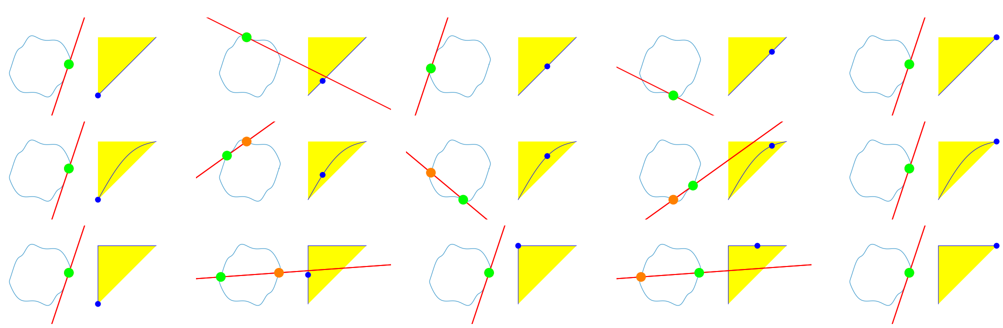

FIGURE 2. The deformation argument.

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Lecture 8: Four vertex theorem

8.1. A **vertex** of a C^2 planar curve is a point $r(t)$ where the curvature function $\kappa(t)$ has a maximum or minimum. In the case of a non-circular ellipse, there are two maxima and two minima, and so 4 vertices in total. What happens in general? A curve is called **convex** if it bounds a convex region R. A region $R \subset \mathbb{R}^2$ is called **convex** if the line segment between any two points $A, B \in R$ is part of the region.

Theorem 1. *A simple closed regular convex C^3 plane curve has at least 4 vertices.*

Proof. We can assume κ is not constant. As a continuous function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has by the extremal value theorem at least one maximum a and one minimum b. We first show that we have more than just 2 extrema. So, assume, $r(a)$ and $r(b)$ were the only critical points. Choose the coordinate system so that these two points $r(a), r(b)$ are on the x-axis so that $r(t) = [x(t), y(t)]$ and $y(a) = y(b) = 0$. Convexity assures that there is no other root $y(t) = 0$. So, k' changes sign at $s = a$ and $s = b$ and nowhere else and $k'(s)y(s)$ does not change sign at all. The Frenet equations tell $e_1 = T = [x', y'], e_2 = N = [-y', x'], [x'', y''] = e_1' = \kappa e_2 = \kappa[-y', x']$ and imply that $x'' = -\kappa y'$. Now, we integrate over the entire curve $\int_0^L \kappa'(s)y(s)\ ds = \kappa y|_0^L - \int_0^L \kappa(s)y'(s)\ ds = \int_0^L x''(s)\ ds = x'(L) - x'(0) = 0$ (because C is closed). But note that $\kappa'(s)y(s)$ does not change sign since both κ' and y only change sign at two points. There is therefore an other maximum or minimum. The number of local max and local min are the same for a periodic function (they must alternate as two successive maxima have a minimum between) so that there must be 4 vertices. $\square$

8.2. Examples.
1) The ellipse $r(t) = [2\cos(t), \sin(t)]$ has the curvature $\kappa(t) = 2(\cos^2(t) + 4\sin^2(t))^{-3/2}$ which has maxima at $t = 0, \pi$ and minima at $t = \pm\pi/2$.
2) The curve $r(t) = 5[\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature $r'(t) \times r''(t)/|r'(t)|^{3/2}$ that has minima at $0, \pi$ and maxima at $\pm \arccos(-2/5)$.
3) The **limacon** $r(t) = [\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature with only 2 vertices! Why is this not a counter example?

8.3. Remarks.
1) Convexity is not really needed. Osserman showed that for any simple closed C^2 curve C there are $2n$ vertices if the smallest circle enclosing C intersects it in at least n connected components. Also, a general simple closed C^2 curve has at least 4 vertices.
2) V. Arnold conjectured that the result holds for any curve that can be obtained from

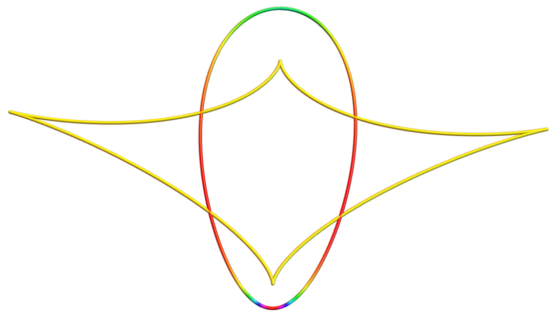

Figure 1. A simple closed curve for which curvature is coded as color. The 4 vertices are visualized by plotting the **evolute** $e(t) = r(t) + n(t)/\kappa(t)$, where $n(t)$ is the normal vector pointing inside. The vertices of the curve correspond to cusps of the evolute.

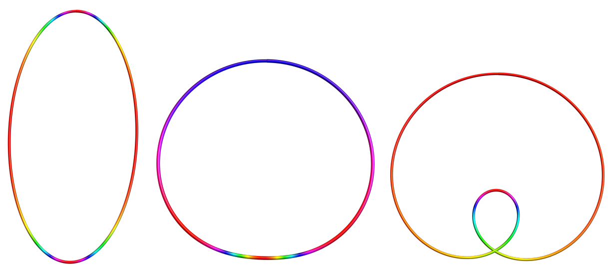

Figure 2. The ellipse, a deformation of an ellipse and a limacon.

a circle by suitable deformations.
3) Converse result: If a continuous real valued, periodic function has at least two local maxima and two local minima, it is the curvature function of a simple closed curve.

8.4. History.

1) The theorem was proven in 1909 by Syamadas Mukhopadhyaya for convex curves. 2) The general case was published in 1912 by Adolph Kneser. 3) The proof given here in the convex case is due to G. Herglotz in 1930. 4) Robert Osserman in 1985 generalized the result to the "four or more vertex theorem". 5) The converse result started with H. Gluck in 1971 and was proven in 1997 by Bjoern Dahlberg.

8.5. Related.

1) The **evolute** of a plane curve is defined as $e(t) = r(t) + n(t)/\kappa(t)$. It is the caustic of the normal map. At points, where $\kappa'(t)$ is zero, the evolute has **cusps**. A caustic of a simple closed curve therefore has at least 4 cusps. For the ellipse the evolute is called the Lamé curve.
2) The **tennis ball theorem** states that a C^2 curve on the sphere that divides the sphere into regions of equal area have at least 4 inflection points.
3) The open **last geometrical problem of Jacobi** asks whether a caustic on an ellipse has at least 4 cusps.

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Lecture 9: Fundamental Forms

9.1. A surface M in $\mathbb{R}^3$ is defined by a C^2 map $r : R \to r(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ from a planar domain R to space $\mathbb{R}^3$. The partial derivatives r_u, r_v are tangent to the **grid curves** $u \to r(u,v)$ and $v \to r(u,v)$ and so tangent to M. If r is regular, the **unit normal vector** $n = r_u \times r_v / |r_u \times r_v|$ is defined and perpendicular to the surface.

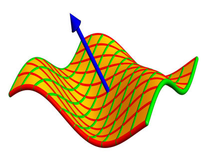

FIGURE 1. A parametrized surface $r(u,v)$ with unit normal vector $n(u,v)$. When seen as a map from M to S^2 it is known as the **Gauss map**.

9.2. We have already seen that the **first fundamental form** $I = g = dr^T dr$ satisfies $\det(I) = |r_u \times r_v|^2$.

Theorem: First fundamental form:

$$I = g = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$$

is a symmetric positive definite bilinear form.

Proof. The matrix dr is a 3×2 matrix and dr^T is a 2×3 matrix: $dr^T = \begin{bmatrix} - & r_u & - \\ - & r_v & - \end{bmatrix}$, $dr = \begin{bmatrix} | & | \\ r_u & r_v \\ | & | \end{bmatrix}$. The product is a 2×2 matrix. Now $g = dr^T dr$. We have seen already that $\det(g) = |r_u \times r_v|^2$. The trace of g is $\text{tr}(g) = |r_u|^2 + |r_v|^2$. Having positive trace and positive determinant assures that we have a positive definite matrix. We call g a **bilinear form** because it maps two vectors X, Y to a number $\langle X, Y\rangle = X^T g Y$. It defines us a scalar product on the surface. $\square$

9.3. Examples:

1) In the case of a graph of a function $r(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$ we have $g = \begin{bmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{bmatrix}$.

2) In the sphere case $r(u,v) = \begin{bmatrix} \sin(v)\cos(u) \\ \sin(v)\sin(u) \\ \cos(v) \end{bmatrix}$ we have $g = \begin{bmatrix} \sin^2(v) & 0 \\ 0 & 1 \end{bmatrix}$. Note that at $v=0$ and $v=\pi$ this is not regular.

9.4. If $r : R \to \mathbb{R}^3$ is a regular C^2 parametrization of a surface M, define

$$n(u,v) = \frac{r_u \times r_v}{|r_u \times r_v|} .$$

It is continuously differentiable because r was assumed to be C^2. The Jacobean derivative dn is the 3×2 matrix $dn = \begin{bmatrix} | & | \\ n_u & n_v \\ | & | \end{bmatrix}$. We can combine it with dr^T and define the **second fundamental form** $h = -dr^T dn$. It agrees with $(d^2 r)^T n$.

> **Theorem:** Second fundamental form
> $$II = h = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} -r_u \cdot n_u & -r_u \cdot n_v \\ -r_v \cdot n_u & -r_v \cdot n_v \end{bmatrix} = \begin{bmatrix} n \cdot r_{uu} & n \cdot r_{uv} \\ n \cdot r_{vu} & n \cdot r_{vv} \end{bmatrix}$$
> is a symmetric bilinear form.

Proof. From $r_u \cdot n = 0$ we get $r_{uu} \cdot n = -r_u \cdot n_u$ and similarly get $r_{uv} \cdot n = -r_u \cdot n_v$. Now, $II = -dr^T dn$ is symmetric because Clairaut applies. Clear is $r_u \cdot n_v = n_v \cdot r_u$. □

9.5. The **third fundamental form** is $III = e = dn^T dn$ is the first fundamental form of the sphere map n.

> **Theorem:** Third fundamental form:
> $$III = e = \begin{bmatrix} n_u \cdot n_u & n_u \cdot n_v \\ n_v \cdot n_u & n_v \cdot n_v \end{bmatrix} .$$
> is a symmetric bilinear form and $|n_u \times n_v|^2 = \det(III)$.

Proof. $III = dn^T dn$ is symmetric as the dot product is commutative. The proof of $|n_u \times n_v|^2 = \det(III)$ is word by word identical what we have done in the second class for $r_u \times r_v$. □

9.6. The third fundamental form is not independent from the other two fundamental forms. In homework: with $H = \mathrm{tr}(A)/2$ and $K = \det(A)$ are trace and determinant of $A = I^{-1} II$:

> **Theorem:** Compatibility: $III - 2H II + K I = 0$

DIFFERENTIAL GEOMETRY

MATH 136

Lecture 10: Curvature

10.1. The **shape operator** S or **Weingarten map** encodes how the surface curves in $\mathbb{R}^3$. S maps the tangent space T_pM to the tangent space T_pS^2. It maps r_u to $-n_u$ and r_v to $-n_v$, meaning $Sdr^T = -dn^T$. By identifying T_pM with T_pS^2 it is a self-map of T_pM. In the basis $\{r_u, r_v\}$, it becomes a 2×2 matrix $A = S^T$ satisfying $\boxed{dn = -drA}$. Take this matrix equation for 3×2 matrices as the relation defining the shape operator. [1]

Theorem 1 (Shape operator)**.** *The shape operator matrix is* $\boxed{A = I^{-1}II}$.

Proof. Using the 3×2 matrices dr, dn we have defined A as $dn = -drA$. The second fundamental form is $II = -dr^T dn = dr^T drA = IA$. Since I is invertible, we can solve for A and get $A = I^{-1}II$. □

While A is not necessarily symmetric, it is symmetric with respect to the inner product $\langle v, w\rangle = v^T gw$. Proof $\langle Av, w\rangle = (Av)^T Iw = v^T A^T I = v^T II^T (I^{-1})^T Iw = v^T II^T w = v^T IIw = w^T IIv$ because II was symmetric. Having been able to switch v,w shows $\langle Aw, v\rangle = \langle w, Av\rangle$.

10.2. Define the **Gaussian curvature** as $\boxed{K = \det(A)}$. Written out, the curvature is

$$K = \det(A) = \frac{\det(II)}{\det(I)} = \frac{LN - M^2}{EG - F^2} = \lambda\mu \ .$$

From linear algebra, we know it is the product of the eigenvalues λ, μ of A. The **mean curvature** H is defined as the average of eigenvalues λ, μ of A. It is

$$H = \frac{\mathrm{tr}(A)}{2} = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{\lambda + \mu}{2} \ .$$

Theorem 2.

$$K = \frac{\det(II)}{\det(I)}$$

is independent of the basis.

Proof. We immediately have from the product determinant formula and the fact that I is invertible that $\det(K) = \det(A) = \det(II)/\det(I)$. Since determinants are independent of the basis, also the curvature is. □

[1] Einstein: $I = g_{ij}$ and $I^{-1} = g^{ij}$ and $II = h_{ij}$ and $A_i^k = g^{kj} h_{ji}$. The shape operator is a "linear transformation" $A_k^i v^k = w^i$ on vectors. I, II are quadratic forms **"(0,2) tensor fields"** while A is a transformation at every point, a **"(1,1) tensor field"**. $dn = -drA$ are called Weingarten equations.

10.3. We write $\iint_M f dV$ for the integral $\iint_R f(u,v)|r_u \times r_v| dudv$. For $f = 1$, this is the **surface area** $|M| = \iint_R |r_u \times r_v| \, dudv$. Since $n(u,v)$ parametrizes the unit sphere, we have $\iint_R |n_u \times n_v| \, dudv = 4\pi$. For convex surfaces, we can use the same parameter domain $R = [0, 2\pi) \times [0, \pi)$ and see that the total curvature is the same than the total curvature of a sphere. This requires that K is positive. The area of the image of S is called the **total curvature**. We have now already a cool version of Gauss-Bonnet: The general version will work for any surface, not only for convex (and so positive curvature) surfaces.

Theorem 3 (Gauss-Bonnet for convex closed surfaces)**.** *$\iint_M K dV = 4\pi$.*

Lemma 1. $\boxed{III = II^T A}$ *and so* $\det(III) = \det(A)^2 \det(I)$.

Proof. Start with the definition $dn = -drA$. Multiply with dn^T from the left to get $III = dn^T dn = -dn^T drA = -(dr^T dn)^T A = II^T A$. Taking determinants gives $\det(III) = \det(II)\det(A) = \det(I)\det(A)\det(A)$. □

10.4. The two identities $II = IA$ and $III = IIA$ can be used for a proof of the identity $\boxed{III - 2HII + KI = 0}$ without using the inner product defined by I. [2] Now to the proof of the Gauss-Bonnet result:

Proof. Take square roots of the lemma gives $\sqrt{\det(III)} = K\sqrt{\det(I)}$. This step has required K to be non-negative. Therefore,

$$\begin{aligned} 4\pi &= \iint_R |n_u \times n_v| dudv = \iint_R \sqrt{\det(III)} \, dudv \\ &= \iint_R K\sqrt{\det(I)} \, dudv = \iint_R K|r_u \times r_v| dudv = \iint K \, dV \, . \end{aligned}$$

□

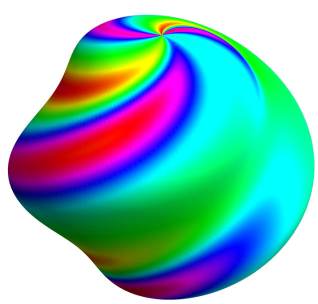

FIGURE 1. We see a convex surface colored with the curvature function K. Gauss-Bonnet establishes that the total curvature is 4π.

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[2] Thanks to some students of the course to point this out.

DIFFERENTIAL GEOMETRY

MATH 136

Lecture 11: Discrete Gauss-Bonnet

11.1. To see the Gauss-Bonnet result for a general surface M, we need to define **Euler characteristic** $\chi(M)$. It is defined by a **triangulation** of M. This can well be done in the language **graph theory**, initiated by Leonhard Euler.

11.2. A **graph** $G = (V, E)$ is a finite set V of **vertices** or **nodes** and a finite set E of different **edges** or **connections** (a, b) with $a \neq b$. Every subset V' of V **generates** a subgraph (V', E'), where $E' = \{(a, b) \in E, a \in V', b \in V'\}$. We can so associate a subset V' of V the subgraph it generates. A pair of adjacent vertices for example generates a K_2 subgraph. A pair of non-adjacent vertices generates $S^0 = \overline{K_2}$ the graph with two points and no vertices, which is also known as the **0-sphere**.

11.3. A circular graph C_n with $n \geq 4$ vertices is called a **circle** of length n. The **unit sphere** $S(v)$ of a vertex v is the subgraph generated by all immediate neighbors of v. A **2-manifold** is a graph for which every unit sphere is a circle. A 2-manifold graph G embedded as a subset $|G| \subset M$ defines a **triangulation** of M; $v \in V$ is realized as a point in M, an edge $e \in E$ is realized as a simple curve in M parametrized by an interval, a connected component in the complement of $|G|$ is regularly parametrized by a triangle $R \subset \mathbb{R}^2$.

11.4. A complete subgraph K_3 of G is also called a **triangle** or a **face** in G. The **Euler characteristic** of a 2-manifold is defined as $\chi(G) = |V| - |E| + |F|$, where $|X|$ is the **cardinality** of X. The **curvature** of a 2-manifold is defined as $\boxed{K(v) = 1 - |S(v)|/6}$. The following theorem goes back to **Victor Eberhard**.

Theorem 1 (Gauss-Bonnet). *For a 2-manifold,* $\sum_{v \in V} K(v) = \chi(G)$.

Proof. Define the function $\omega(x)$ on $X = V \cup E \cup F$ as $\omega(x) = (-1)^{\dim(x)}$ where $\dim(x) = |x| - 1$ is the dimension one less than the number $|x|$ of vertices in x. So, $\chi(G) = |V| - |E| + |F| = \sum_{|x|=1} (-1)^0 + \sum_{|x|=2} (-1)^1 + \sum_{|x|=3} (-1)^2 = \sum_{x \in X} \omega(x)$. If all values -1 from an edge (a, b) are distributed equally to (a, b) and all the values 1 from a face (a, b, c) are distributed equally to the vertices a, b, c, we end up with a function K that is only non-zero on vertices v and equal there to $K(v) = 1 - S_0(v)/2 + S_1(v)/3$, where $S_0(v), S_1(v)$ are the number of vertices and edges in $S(v)$ for $v \in V$. In the case of a circular $S(v)$ we know $S_0(v) = S_1(v) = |S(v)|$ so that $K(v) = 1 - |S(v)|(1/3 - 1/2) = 1 - |S(v)|/6$. $\square$

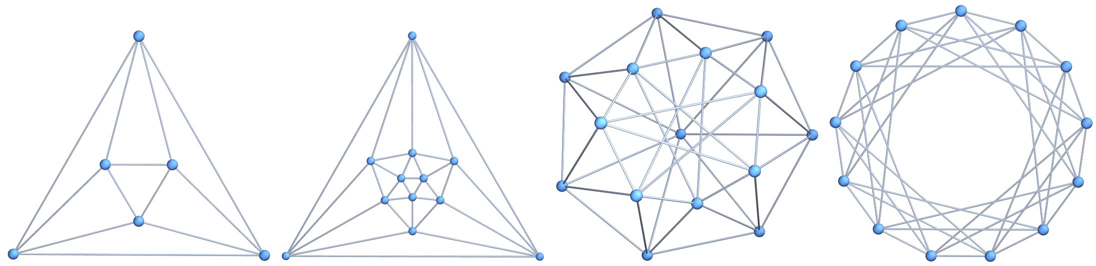

FIGURE 1. The **octahedron** has $K(v) = 1/3$ for all v. The **icosahedron** has $K(v) = 1/6$ for all v. A **projective plane** has curvatures in $\{0, 1/6, -1/6\}$. The flat torus = Clifford torus has constant 0 curvature.

11.5. An **edge collapse** $G \to G'$ takes $(a, b) \in E$ and identifies a with b. It removes 2 faces, 3 edges and 1 vertex so that $\chi(G) = \chi(G')$. A 2-manifold G is a **2-sphere** S^2, if $\chi(G) = 2$. The **connected sum** $G\#H$ of two 2-manifolds G, H is obtained by removing an edge in both manifolds and identifying the C_4 boundaries of the holes. If $v \in V(G)$ and $w \in V(H)$, with $|S(v)| = |S(w)|$, one can also remove v from G and w from H and glue boundaries to get a $G\#H$ with $\chi(G\#H) = \chi(G) + \chi(H) - 2$. A **2-ball** is a graph obtained from a S^2 by removing a vertex v. A **2-cylinder** or **handle** is a 2-sphere in which two vertices in distance > 2 removed. A **2-torus** is a 2-manifold obtained from a 2-cylinder by gluing the boundaries, matching orientation. A **Moebius strip** is a projective plane with one vertex removed. When glued into a hole of a sphere it is a **cap**. The **Klein bottle** is a S^2 with two caps. The projective plane is a sphere with a cap. The **boundary** of a G is $\{w \in V | S(v) \text{ is not a circle }\}$. The boundary of a ball or a Moebius strip is a circle. 2-manifolds have no boundary.

11.6. A **topological deformation** of a 2-manifold G takes a 2-ball in G and replaces it with an other 2-ball with the same boundary. In other words, a topological deformation is the process $G \to G' = G\#S^2$ implying $\chi(G) = \chi(G')$. Two 2-manifolds G, H are **topologically equivalent** if they can be deformed into each other by a finite set of topological deformations. An example of a topological deformation is to take out an edge and fill in the opposite diagonal edge. This **diagonal flip** is known as **Pachner transformation**. The following theorem is a milestone of 19'th century mathematics:

Theorem 2 (Classification of 2-manifolds). *Every connected 2-manifold is equivalent to a 2-sphere S^2 or a connected g-sum of either $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ or $\mathbb{P}^2 \# \cdots \# \mathbb{P}^2$.*

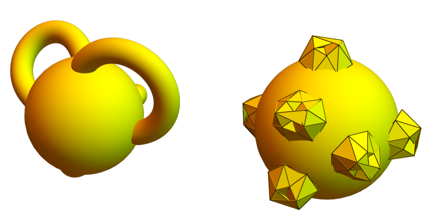

FIGURE 2. A 2-manifold is either S^2 with g handles (orientable) and $\chi = 2 - 2g$ or a S^2 with g cross caps and $\chi = 2 - g$ (non-orientable).

DIFFERENTIAL GEOMETRY

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Lecture 12: Geodesics

12.1. If $M = r(R)$ is a regular manifold, define the space X of regular paths $x(t)$ that start at $x(a) \in R$ and end at $x(b) \in R$. If $F(x, \dot{x})$ is a function of position x and velocity $\dot{x}$, we can minimize $E(x) = \int_a^b F(x, \dot{x}) \, dt$ by looking for paths $x(t)$ at which the variation is zero. [1]

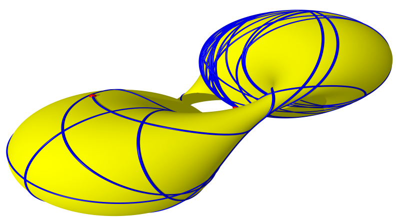

Figure 1. This geodesic on a torus was computed in Povray with the Schield's ladder method: evolve freely in $\mathbb{R}^3$ but stay glued to the surface.

Theorem 1 (Euler-Lagrange). *If x minimizes E, then* $\boxed{F_x(x, \dot{x}) = \frac{d}{dt} F_{\dot{x}}(x, \dot{x})}$.

Proof. For a minimum, the change $E(x + \xi) - E(x)$ of a displacement $x + \xi$ of x satisfies $\int_a^b F(x + \xi, \dot{x} + \dot{\xi}) - F(x, \dot{x}) \, dt \geq 0$. As Fermat knew, we better have $dE\xi = \lim_{h \to 0}(E(x + h\xi) - E(x))/h = 0$ because a non-zero limit would make $E(x + h\xi)$ larger or smaller than $E(x)$ for small h. By the chain rule, $dE\xi = \int_a^b F_x(x, \dot{x})\xi + F_{\dot{x}}(x, \dot{x})\dot{\xi} \, dt$. Integration by parts, using $\xi(a) = \xi(b) = 0$, gives $dE\xi = \int_a^b [F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x}]\xi(t) \, dt$. In order that this is zero for all ξ, we better have $[F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x})] = 0$ for all $t \in [a, b]$. Proof. If $\neq 0$ at some point $t \in [a, b]$, it would be non-zero in a neighborhood U of t, allowing to find a smooth function ξ that is positive in U and 0 else, producing a nonzero change $dE\xi$. □

12.2. To understand minima if $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle = g(x)(\dot{x}, \dot{x}) = \sum_{i,j} g_{ij}(x)\dot{x}^i \dot{x}^j$, we need notation. M was defined as a regular map $r : R \to \mathbb{R}^n$ giving points $r(u^1, \dots, u^m) \in \mathbb{R}^n$. Define the **Christoffel symbols** $\boxed{\Gamma_{ijk} = r_{u^i u^j} \cdot r_{u^k}}$. The product rule gives

$$\partial_{u^k} g_{ij} = r_{u^i u^k} \cdot r_{u^j} + r_{u^i} \cdot r_{u^j u^k} = \Gamma_{ikj} + \Gamma_{jki} \; ,$$

[1] $x(t) = (x^1(t), \dots, x^m(t)) = (u^1(t), \dots u^m(t))$ as most texts use this notation. For typographical reasons, write $\dot{x}^k$ rather than x'^k. If $r(u, v)$ parametrizes M, paths $x(t) = (u(t), v(t)) \in R$ define curves $r(x(t)) \in M$. "Variation" instead of "derivative" avoids confusion with $\dot{x}$. Variations are directional derivatives in an infinite dimensional space X of paths between two fixed points.

$$\partial_{u^i} g_{jk} = r_{u^j u^i} \cdot r_{u^k} + r_{u^j} \cdot r_{u^k u^i} = \Gamma_{jik} + \Gamma_{kij} \ ,$$
$$\partial_{u^j} g_{ki} = r_{u^k u^j} \cdot r_{u^i} + r_{u^k} \cdot r_{u^i u^j} = \Gamma_{kji} + \Gamma_{ijk} \ .$$

Adding the second and third and subtracting the first, using Clairaut $\Gamma_{ijk} = \Gamma_{jik}$, gives $2\Gamma_{ijk}$ on the right hand side. So:

Lemma 1. $\boxed{\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}] \ .}$

Using the notation $g^{ij} = (g^{-1})_{ij}$ and $\Gamma^k_{ij} = \sum_{l=1}^m g^{kl}\Gamma_{ijl}$, we get to the main point: [2]

Theorem 2 (Geodesics)**.** *Minima of the action functional $E(x) = \int_a^b \langle \dot{x}, \dot{x}\rangle \ dt$ satisfy*

$$\boxed{\ddot{x}^k + \sum_{i,j=1}^m \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0.}$$

Proof. To show that Euler-Lagrange for $F(x,\dot{x}) = \sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j$ is $2\sum_j g_{jk}\ddot{x}^j + 2\sum_{i,j}\Gamma_{ijk}\dot{x}^i\dot{x}^j = 0$: use notation $\partial_{x^k} g_{ij} = g_{ij,k}$. First get $F_{\dot{x}^k} = \sum_j g_{kj}\dot{x}^j + \sum_j g_{jk}\dot{x}^j$. Then $\frac{d}{dt}F_{\dot{x}^k} - F_{x^k} = \sum_{j,i} g_{kj,i}\dot{x}^i\dot{x}^j + \sum_j g_{kj}\ddot{x}^j + \sum_{j,i} g_{jk,i}\dot{x}^i\dot{x}^j + \sum_j g_{jk}\ddot{x}^j - \sum_{i,j} g_{ij,k}\dot{x}^i\dot{x}^j$. The 1st, 3rd and 5rd terms add up to $2\sum_{i,j}\Gamma_{ijk}\dot{x}^i\dot{x}^j$. The 2nd and 4th give $2\sum_j g_{jk}\ddot{x}^j$. □

We see that the acceleration of a particle moving on a geodesic is determined by the velocity and "gravitational force" terms Γ which involves changes in the metric. Einstein would interpret these changes in metric as "mass".

12.3. With $G(x,\dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} = \sqrt{F}$, we get the **arc length functional**

$$I(r) = \int_a^b ||\dot{x}|| \ dt = \int_a^b \sqrt{\langle \dot{x}(t), \dot{x}(t)\rangle} \ dt = \int_a^b \sqrt{\sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j} \ dt \ .$$

Theorem 3 (Maupertius)**.** *Action and length functionals have the same extrema.*

Proof. Because $\frac{d}{dt}G_{\dot{x}} = \frac{F_{\dot{x}}}{2\sqrt{F}}$ and $G_x = \frac{F_x}{2\sqrt{F}}$, the Euler-Lagrange equations $\frac{d}{dt}G_{\dot{x}} = G_x$ are equivalent to the Euler-Lagrange equations $\frac{d}{dt}F_{\dot{x}}/\sqrt{F} = F_x/\sqrt{F}$. We have used that x was regular, so that $F(x,\dot{x})$ is never zero. □

12.4. If $x(t)$ is an arc length parametrized curve on M, the **normal curvature** is defined as $\boxed{\kappa_n = \ddot{x} \cdot n}$. It is the scalar projection acceleration $\ddot{x}$ onto n. By Cauchy-Schwarz, it is smaller or equal than $\kappa = |\ddot{x}|$. Define the **geodesic curvature** $\kappa_g = \ddot{x} \cdot (n \times \dot{x})$. Pythagoras gives $\kappa_n^2 + \kappa_g^2 = \kappa^2$. Note that both κ_n and κ_g can be signed.

Theorem 4 (Schield's ladder)**.** *Geodesics have zero geodesic curvature.*

Proof. Geodesics minimize arc length $L = \int_t^{t+2h} ||\dot{x}|| \ dt$ between two close points $x(t), x(t+2h)$. If $\kappa_g = \ddot{x} \cdot (\dot{x} \times n) \neq 0$ at some point $x(t)$, there would be a shorter connection between $x(t)$ and $x(t+2h)$ than $x(t), x(t+h), x(t+2h)$ violating minimality. □

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[2]It's musical! ∂_{u^i} is co-variant and u^i is contra-variant. Einstein would write $g_{ij}\dot{x}^i\dot{x}^j$ for $\langle \dot{x}, \dot{x}\rangle$.

DIFFERENTIAL GEOMETRY

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Lecture 13: The exponential map

13.1. The geodesic differential equation $\boxed{\ddot{x}^k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0}$ can be written as a first order system $\frac{d}{dt}[x, \dot{x}] = [\dot{x}, f(x, \dot{x})]$ if the first fundamental form g is twice differentiable. This ordinary differential equation therefore has **local solutions** for some time $t \in (-a, a)$ by the Piccard existence theorem. But the solutions exist for all time. No "blow up" is possible if the surface is smooth, regular and closed. The reason is the following:

Lemma 1. *If $x(t)$ is geodesic, then $\langle \dot{x}, \dot{x} \rangle = \sum_{i,j} g_{ij} \dot{x}^i \dot{x}^j$ is preserved.*

Proof. The Hamiltonian $H = -F + \sum_j \dot{x}^j F_{\dot{x}^j}$ is also called the Legendre transform. We get $\frac{d}{dt} H = -\sum_j F_{x^j} \dot{x}^j - \sum_j F_{\dot{x}^j} \ddot{x}^j + \sum_j \ddot{x}^j F_{\dot{x}^j} + \sum_j \dot{x}^j \frac{d}{dt} F_{\dot{x}^j} = 0$. Use Euler-Lagrange to replace the last term with $\sum_j \dot{x}^j F_{x^j}$. For $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle$, we have $H = -F + 2\sum_j \dot{x}^j \dot{x}^j = -F + 2F = F$. We see $H = F = \langle \dot{x}, \dot{x} \rangle$ is preserved energy. □

Theorem 1 (Hopf-Rynov)**.** *For regular, compact, smooth M, geodesics exist globally.*

Proof. If M is C^4 then Γ is differentiable. The Piccard existence theorem gives local solutions in the unit tangent bundle $(p, v) \in M \times S^{m-1}$. A regular compact manifold is complete in the sense that all Cauchy sequences have limits. The only way that a solution path could not be continued is that $\dot{x}(t)$ blows up. Otherwise, we could restart the differential equation at the end point a of a maximal interval $(-a, a)$ of existence. By the lemma, a blow up of $\dot{x}(t)$ is not possible. □

13.2. Remarks: **a)** The regularity is necessary. On a piece-wise smooth manifold like a cube, a geodesic hitting a corner can not be continued continuously. **b)** There are compact Lorentzian manifolds like the Clifton-Pohl torus that are not complete. **c)** The lemma is important. The proof shows that each variational problem gets with a **Legendre transform** $H = -F + \sum_j \dot{x}^j F_{x^j}$ to an "energy" H that is preserved. [1]

13.3. The **exponential map** $\exp_p : T_pM \to M$ is obtained by defining $\exp_p(0) = p$ and for $v \neq 0$, define $\exp_p(v)$ by taking $v/|v|$ as initial direction of the geodesic flow and evolving it for time $|v|$. The image $\exp(S_r(0)) = W_r(p)$ is called the **wave front**. It is the set of all points which can be reached from p by running from it a geodesic of length p. Wave fronts are **geodesic circles** for small t but in general become very complicated.

[1]For more, see J. Moser, Selected Topics in the Calculus of Variations. (Notes by O. Knill) 2002

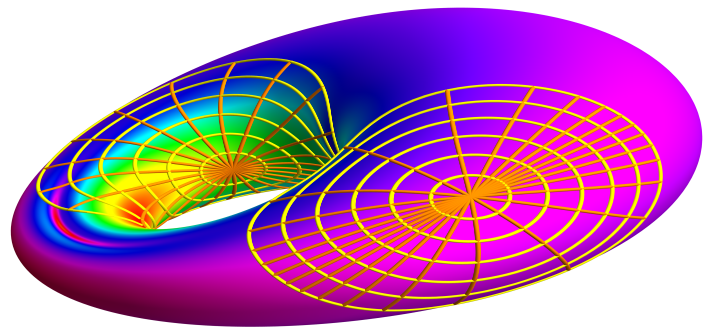

FIGURE 1. The exponential map evolves all possible geodesics from p. If all these geodesics are stopped at time t, we get a wave front $W_t(p)$.

13.4. The **radius of injectivity** of M is the smallest r such that the exponential map $B_r \subset T_pM \to M$ is injective.

Lemma 2. *For a compact manifold M, the radius of injectivity is positive.*

Proof. When fixing a point p, there is a $B_r(0) \subset \mathbb{R}^m$ such that that $\exp_p$ is invertible. This follows from the **inverse function theorem** and the fact that $d\exp_p = 1$ (identity matrix) at p because $\exp_p(v) - v = O(|v|^2)$ by definition. Let $r(p)$ be maximal radius on which $\exp_p(B_r(0))$ is differentiable. This function $r(p)$ is continuous in p and positive. By compactness of M and the **extremal value theorem**, there is a minimum, a lower bound. □

13.5. For fixed p, critical values of $\exp_p$ form the **caustic** of p. If r is the radius of injectivity, the open set $U = \exp_p(B_r(0)) \subset M$ is called the **normal neighborhood** of p. Lets look at the two dimensional case:

Lemma 3. *On U there are coordinates (ρ, θ) such that $g = I = \begin{bmatrix} 1 & 0 \\ 0 & G \end{bmatrix}$ satisfying $\lim_{\rho \to 0} G(\rho, \theta) = 1$.*

Proof. These are called **geodesic polar coordinates** because they come from the exponential map. Since velocity is preserved, the radial direction does not expand. □

13.6. This implies:

Theorem 2 (Gauss Lemma)**.** *For every unit vector v, the radial geodesics $\{\exp_p(sv), s \leq t\}$ is normal to the wave front $W_t(p)$.*

Proof. Within $U = \exp_p(B_r(0))$ this is clear by the coordinates. □

13.7. Remarks. 1) Geodesic coordinates with $I = g = \text{diag}(1, g_{22} \dots, g_{mm})$ exist on any m-manifolds. **2)** For 2-manifolds, linearising the geodesic flow affects only the vector perpendicular to the geodesic $x(t)$. This is called a **Jacobi field**. For surfaces, for fixed p and v, we get a **Jacobi differential equation** $\boxed{z'' = -K(x(t))z}$, where $z(t) = G(x(t))$ in the normal patch. The roots of $z(t)$ belong to caustic points $\exp_t(p)$.

Lecture 14: Curvature is a Curl

14.1. The proof of the **Gauss-Bonnet theorem** will invoke **Green's theorem** from calculus. Also the **Theorema egregium** will boil down to the fact that curvature form KdV is the curl dX of a 1-form X, that only depends on the first fundamental form I. **Differential geometry** so builds heavily on **multi-variable calculus**.

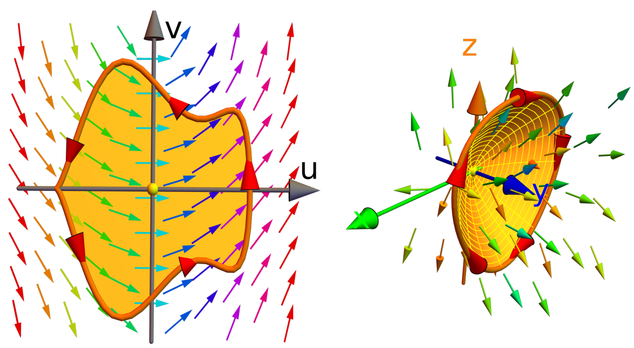

FIGURE 1. **Green's theorem** integrates the 2D curl dX over a planar region R. **Stokes theorem** integrates the 3D curl dF over a surface M. If $M = r(R)$, one can **pull back** the 1-form F in $\mathbb{R}^3$ to a 1-form X in $\mathbb{R}^2$ and so get Stokes from Green: $dF(r_u, r_v) = \text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u = \text{curl}(X)$ for $X = [F \cdot r_u, F \cdot r_v]$ (see homework). In differential geometry, a particular X will lead to Gauss-Bonnet.

14.2. Green's theorem is usually written for planar vector fields $X^T = \begin{bmatrix} P \\ Q \end{bmatrix}$: the double integral of the curl dX of X in a R agrees with the line integral of X along the boundary δR. If we change to row vectors, we have a **1-form** $X = [P, Q]$. 'Power=force times velocity" $\begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}$ is then the **matrix product** $X\dot{x}$ rather than $X^T \cdot \dot{x}$. [1]

14.3. Assume $X = [P, Q]$ is a 1-form and assume $x(t) = \left[\ u(t), v(t)\ \right]^T$ is the parameterization of a **closed curve** δU with $t \in [0, L]$ bounding the region $U \subset R$. The **curl** of X is defined as $dX = \text{curl}(X) = Q_u - P_v$. The **1-form** X is a linear map which assigns to a contra-variant vector (column vector) like $\dot{x}$ a number $X\dot{x} = P\dot{u} + Q\dot{v}$.

[1] Both in physics as well in differential geometry, it is important to distinguish between **contra-variant objects** like vector fields $\nabla f = X^i$ and **co-variant objects** like 1-forms $df = X_i = \partial_{x^i} f$.

A 1-form especially can be integrated along a curve $\int_0^L X(x(t))\dot{x}dt$, the **line integral**. The curl of X is denoted by dX. It is a **2-form** which can be integrated over U. The **general Stokes theorem** tells $\boxed{\int_M dX = \int_{\delta M} X}$ if M is a k-manifold with boundary δM and X is a $(k-1)$-form then dX is a k-form. In the case $k = 2$, where X is a 1-form and $U \subset R$ is a region, we have

Theorem 1 (Green)**.** $\int_{\delta U} X = \int_0^{2\pi} X(x(t))\dot{x}(t)\, dt = \iint_U \mathrm{curl}(X)(u,v)\, dudv = \int_U dX.$

14.4. In calculus, you see this using vector fields $F = X^T$, meaning that every point is attached a contra-variant vector. In order to pair this with the velocity vector $\dot{x}$, we had to invoke the **dot product** $v \cdot w = v^T w$ and write a **matrix product** $X(x(t))\dot{x}(t)$. The just formulated version of Green's theorem is completely equivalent.

14.5. The key of Gauss Bonnet is to see that the curvature 2-form $K|r_u \times r_v|$ can be written as the curl dX of a 1-form X. Gauss-Bonnet theorem in the convex case is stated as $\iint_R K|r_u \times r_v|duv = 2\chi(M)$. A second computation will then show that if $M = r(U)$ is a manifold with boundary $r(x) = \delta(M)$, integrating the geodesic curvature along the boundary curve x is a **line integral** of X along x plus 2π. Gauss-Bonnet for surface patches $r(U)$ with boundary $t \to r(x(t))$ will then follow from Green's theorem.

14.6. Assume that $r : R \to \mathbb{R}^3$ is a regular parametrization of the surface M. A simple closed curve $x(t), t \in [0, L]$ encloses a region $U \subset R$ matching orientation. It defines a curve $r(x(t))$ bounding the manifold $r(U) \subset M$. We can assume that $x(t) = (u(t), v(t))$ is parametrized by arc length. At every point $p = r(u, v) \in M$, the vectors $\{r_u, r_v\}$ form a basis of the tangent space T_pM. Let $\{z, w\}$ be the Gram-Schmidt orthonormalized basis obtained from $\{r_u, r_v\}$ and the unit normal vector $n = r_u \times r_v/\sqrt{r_u \times r_v} = z \times w$.

14.7. The following lemma shows that we can attach two vectors z, w to every point p on the surface. It will allow us to define the 1-form $\boxed{X = zdw = [z \cdot w_u, z \cdot w_v]}$.

Lemma 1. *$z = ar_u, w = br_u + cr_v, n = z \times w$ form an orthonormal frame with functions a, b, c that only depend on the first fundamental form.*

Proof. Gram-Schmidt proceeds as follows $z = r_u/\sqrt{r_u \cdot r_u} = r_u/\sqrt{E} = ar_u$ and gets w as the normalization $br_u + cr_v$ of $r_v - (r_v \cdot z)z = r_v - (r_v \cdot r_u)r_u/E = r_v - \frac{F}{E}r_u$. □

14.8. We will see next time that X can be computed from I alone and that

Lemma 2 (Curvature is a curl)**.** *The curl satisfies* $\boxed{dX = Q_u - P_v = K\sqrt{\det(g)}}$.

14.9. For now, this is just an announcement. The computation comes next class. But then we will be close to Gauss-Bonnet: the line integral of X along the boundary will then be related with an integral of geodesic curvature so that we will reach the local Gauss-Bonnet theorem telling that the sum of all curvatures from the interior, the boundary and corners of a triangular region adds up to 2π. And then, by gluing up triangles, we will get the **global Gauss-Bonnet theorem** $\int_M X = 2\pi\chi(M)$. This is the mountain peak we wanted to reach. We are in the middle of the climb right now.

DIFFERENTIAL GEOMETRY

MATH 136

Lecture 15: Theorema Egregium

14.1. In 1827, Karl Friedrich Gauss proved the **"theorema egregium"**. Is curvature determined by distance measurements within the geometry alone, without reference to the ambient space $\mathbb{R}^3$ in which M is embedded? The answer is yes:

Theorem 1 (Theorema Egegium). *Gaussian curvature K is determined by I.*

FIGURE 1. 1794 Map by mathematician Sam Dunn, when Gauss was 17.

14.2. This implies that if two spaces have different curvature, they can not be isometric. That curvature in a surface M can be expressed using the first fundamental form I is not surprising, given that the first fundamental form is used for all distance measurements in M using geodesics curves like light beams. But curvature has been defined as $\det(A)$, using the **shape operator** $A = I^{-1}II$. It invoked both the first and second fundamental form II that made use of the normal vector n in $\mathbb{R}^3$.

14.3. Last time, we saw that we can assign an **orthonormal frame field** $\{z, w\}$ on M which only depends on the first fundamental form. This produces an orthonormal frame $\{z, w, n\}$ at every point $p \in M \subset \mathbb{R}^3$. This "frame field" on M is similar to the Frenet frame field $\{T, N, B\} = \{e_1, e_2, e_3\}$ on a curve, where the Frenet equations told how the frame field moves with time. We are interested in how the field changes when we change u and v. The mathematics is very similar to what we did for curves just that we have now two variables u, v rather than only one variable t. We will need the following formulas to relate the curl of $X = [P, Q] = [z \cdot w_u, z \cdot w_v]$ with curvature.

Lemma 1 (Moving frame lemma)**.**

$$\begin{aligned} z_u &= (z_u \cdot w)w + (z_u \cdot n)n \\ z_v &= (z_v \cdot w)w + (z_v \cdot n)n \\ w_u &= (w_u \cdot z)z + (w_u \cdot n)n \\ w_v &= (w_v \cdot z)z + (w_v \cdot n)n \end{aligned}$$

Proof. We expand each of the vectors z_u, z_v, w_u, w_v in the $\{z, w, n\}$ basis:

$$\begin{aligned} z_u &= (z_u \cdot z)z + (z_u \cdot w)w + (z_u \cdot n)n \\ z_v &= (z_u \cdot z)z + (z_v \cdot w)w + (z_v \cdot n)n \\ w_u &= (w_u \cdot z)z + (w_u \cdot w)w + (w_u \cdot n)n \\ w_v &= (w_v \cdot z)z + (w_v \cdot w)w + (w_v \cdot n)n \end{aligned}$$

and note that $z \cdot z = 1$ implies $z_u \cdot z = z_v \cdot z = 0$. □

Lemma 2 (X is intrinsic)**.** *$X = [P, Q] = [z \cdot w_u, z \cdot w_v]$ is expressible by I alone.*

Proof. **(i)** Let us look at $z \cdot w_u = -(z_u \cdot w)$: We have seen that $z = ar_u, w = br_u + cr_v$, where a, b, c depended only on I. Now $(z_u \cdot w) = (ar_u)_u \cdot w = (a_u r_u + ar_{uu}, br_u + cr_v)$. Multiply out and use $(r_u \cdot r_u) = E$, and $(r_{uu}, r_u) = E_u/2$ and $(r_{uu} \cdot r_v) = F_u - E_v/2$. All these terms involve E, F, G or derivative of those from the first fundamental form I.
(ii) Now do the computation for the second coordinate $z \cdot w_v$. Follow the same steps. You do that in the homework. □

Lemma 3 (Curvature is a Curl)**.** $dX = Q_u - P_v = (z \cdot w_v)_u - (z \cdot w_u)_v = K\sqrt{\det(g)}$.

Proof. The wall is climbed in three pitches:
(Pitch i) $\boxed{(z \cdot w_v)_u - (z \cdot w_u)_v = z_u \cdot w_v - z_v \cdot w_u.}$
Proof. Use the product rule and Clairaut's result $z_{uv} = z_{vu}$.
(Pitch ii) $\boxed{z_u \cdot w_v - z_v \cdot w_u = (n_u \times n_v) \cdot n}$
Proof: Use the **moving frame lemma**, and an identity from the Frenet lecture, as well as $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$ to get

$$\begin{aligned} z_u \cdot w_v - z_v \cdot w_u &= (n_u \cdot z)(n \cdot w_v) - (w_u \cdot n)(z_v \cdot n) \\ &= (n \cdot z_u)(n_v \cdot w) - (w \cdot n_u)(z \cdot n_v) \\ &= (n_u \times n_v, z \times w) = (n_u \times n_v, n) \ . \end{aligned}$$

(Pitch iii) $\boxed{(n_u \times n_v) \cdot n = K\sqrt{\det(I)}}$
Proof. The left hand side is (remember $drA = -dn$ defined the shape operator A), $((A_{11}r_u + A_{21}r_v) \times (A_{12}r_u + A_{22}r_v)) \cdot n = \det(A)|r_u \times r_v| = \det(A)\sqrt{\det(I)} = K\sqrt{\det(I)}$. □

14.4. The "Theorema Egregium" is proven: the 1-form X is intrinsic. So, the curl dX and also $K(x)$ are intrinsic. One can "shoot down" the Theorema Egregium also by expressing K in terms of the intrinsic Γ_{ijk}. The computations here will help however in the proof of Gauss-Bonnet.

Lecture 16: Local Gauss-Bonnet

16.1. We now prove the Gauss-Bonnet theorem in the situation when $U \subset R$ is a polygon. The parametrization $r : R \to M$ plants the polygon $r(U) \subset r(R)$ into the surface M.

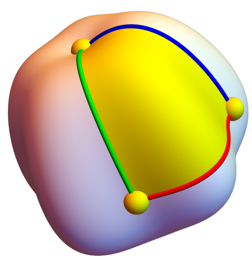

FIGURE 1. The local Gauss-Bonnet theorem tells that face, edge and vertex curvatures of a polygon $r(U)$ in a manifold M add up to 2π.

16.2. A **simple polygon** in M is the image $r(U)$ of a simple polygon $U \subset \mathbb{R}^2$ such that r is smooth and injective on U. Its **Euler characteristic** is $\chi(U) = |V| - |E| + |F| = 3 - 3 + 1 = 1$. As in the discrete Hopf Umlaufsatz, the vertex curvatures are defined as $\kappa_i = \pi - \alpha_i$, where α_i are the polygon angles. The **angles α_i of the polygon** are defined by $\cos(\alpha_i) = \dot{x}_i(1) \cdot \dot{x}_{i+1}(0)$, the dot product of the velocity vectors of the arcs at the end of the incoming and the beginning of the outgoing arc.

16.3. Let U be a simple polygon on M. There are three contributions to curvature: the **face curvature** is the integral of K over the interior, the **geodesic curvature** integrates sectional curvature κ_g over the edges C_j and then there are the **vertex curvatures** $\kappa_j = \pi - \alpha_j$ attached to the vertices.

Theorem 1 (Local Gauss-Bonnet). $\iint_U K dV + \sum_j \int_{C_j} \kappa_g(x_j(t))\, dt + \sum_j \kappa_j = 2\pi$.

16.4. If $x(t) = r(u(t), v(t))$ parametrizes the boundary of the surface $M = r(U)$, we can assume that it is parametrized by arc-length. The velocity vector $\dot{x}$ is a 3-vector tangent to the surface. We look at the orthonormal frame field (z, w) from last time. The **geodesic curvature** of a curve x is defined at points where x is smooth and given as $\boxed{\kappa_g = (n \times \dot{x}) \cdot \ddot{x}}$. Unlike $\kappa = |\dot{x} \times \ddot{x}|$, it is signed. So is the **normal curvature** $\boxed{\kappa_n = n \cdot \ddot{x}}$. Since $\dot{x} \cdot \ddot{x} = 0$, Pythagoras gives $\kappa_g^2 + \kappa_n^2 = \kappa^2$. The velocity vector of the curve can be expressed as an angle so that $\boxed{\dot{x} = \cos(\theta)z + \sin(\theta)w}$. We write $\dot{w}$ for $\frac{d}{dt}w(x(t))$.

Lemma 1 (Geodesic lemma). $\kappa_g = \dot{\theta} - (z \cdot \dot{w})$.

Proof. Fill in the parts of the definition $\kappa_g = (n \times \dot{x}) \cdot \ddot{x}$:
(i) $n \times \dot{x} = \cos(\theta)w - \sin(\theta)z$.
(ii) $\ddot{x} = \dot{\theta}(-\sin(\theta)z + \cos(\theta)w) + \cos(\theta)\dot{z} + \sin(\theta)\dot{w}$.
(iii) So, $\kappa_g = (n \times \dot{x}) \cdot \ddot{x} = \dot{\theta} - z \cdot \dot{w}$ □

16.5. We can now prove the local Gauss-Bonnet theorem:

Proof. (i) Integrating the geodesic lemma gives

$$\int_0^L \kappa_g \, dt = \int_0^L \dot{\theta} \, dt - \int X \, dr$$

(ii) Green's theorem assures that $\int X \, dr = \iint_U K \, dV$ as $K dV = dX$.
(iii) The Hopf Umlaufsatz for curved polygons gives $\int_0^L \dot{\theta}(t) \, dt + \sum_j (\pi - \alpha_j) = 2\pi$.
(iv) Putting (i),(ii),(iii) together gives the proof. □

16.6. Example 1) If K is constant 0 and U is a triangle, Gauss Bonnet is $\kappa_1 + \kappa_2 + \kappa_3 = 2\pi$. This is equivalent to $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ you know from elementary school geometry. For an **n-gon**, the identity $\sum_{i=1}^n \kappa_i = 2\pi$ is equivalent to $\sum_{i=1}^n \alpha_i = (n-2)\pi$.

16.7. Example 2) If $M = \mathbb{S}^2$ is a sphere of radius 1, then curvature is $K = 1$. The integral $\iint_U K \, dV$ is the **area** $|U|$ **of the triangle**. The formula becomes $|U| + \sum(\pi - \alpha_i) = 2\pi$ and so $\boxed{\alpha_1 + \alpha_2 + \alpha_3 = |r(U)| + \pi}$. This is **Girard's theorem** or **Harriot's theorem** in spherical geometry, named after Albert Girard or Thomas Harriot.

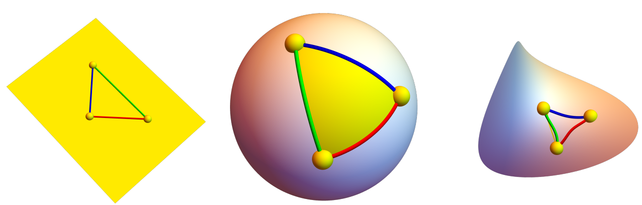

Figure 2. A triangle in the plane has $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. For a spherical triangle of area A, Harriet's theorem gives $\alpha_1 + \alpha_2 + \alpha_3 = \pi + A$. On a hyperbolic space, Lambert's theorem is $\alpha_1 + \alpha_2 + \alpha_3 = \pi - A$.

16.8. Example 3) If M is a surface of constant curvature -1, a triangle is called **hyperbolic** Now, $\iint_U K dV = -|U|$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi - |U|$, a formula found by Johann Heinrich Lambert. The right hand side $\pi - |U|$ is called **spherical defect**.

16.9. Example 4) Take a sphere with a simple closed geodesic on it, integral of K on each half is 2π. The total integral is 4π.

16.10. Example 5) If $K = 0$ and $r(U)$ is a region in the plane bound by a simple smooth curve, we have the **Hopf Umlaufsatz** $\int \kappa_g(t) \, dt = 2\pi$.

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Lecture 17: Global Gauss-Bonnet

17.1. Now we are ready to prove the **global Gauss-Bonnet theorem** for a 2-manifold M without boundary. The surface M is triangulated by a **discrete manifold** $G = (V, E, F)$, where the faces F are the triangles defined by the graph (V, E). The discrete manifold is geometrically realized in M as a collection of points, a collection of curves connecting vertices. The geometrically realized network divides M up into triangular faces $M_i = r(U_i)$. The Euler characteristic of M is $\chi(M) = V - E + F$. As we have seen, $\chi(M)$ does not depend on the triangulation: topological changes like removing a disc and gluing in a new disc (we called this as a **connected sum** $M \to M\#S^2$) or doing a **Barycentric refinement** does not change $\chi(M)$.

Theorem 1 (Gauss-Bonnet theorem)**.** *For a compact* 2*-manifold,* $\iint_M K dV = 2\pi\chi(M)$*.*

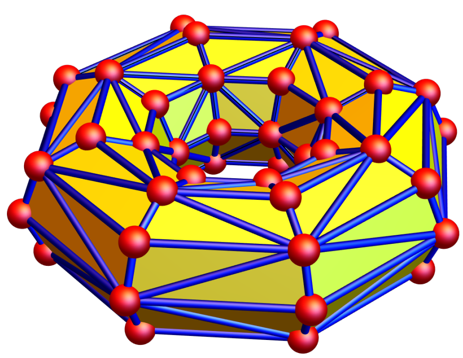

Figure 1. A triangulated manifold M. We apply the local Gauss-Bonnet theorem on each of the triangles. The edge contributions produce with Green's theorem the face curvatures $\iint_{M_i} K \; dV$ as well as $2\pi F$. The vertex contributions produce, using the Euler Handshake lemma, $2\pi(V - E)$. Overall, we have $2\pi(V - E + F) = 2\pi\chi(M)$. The picture shows a $M = \mathbb{T}^2$ with $V = 64$ vertices.

17.2. We will use the local Gauss-Bonnet theorem for each triangle U_i with angles $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$. We first of all want to understand what happens if we glue together triangles such that the frame field $X = [zw_u, zw_v]$ can be define on the union. On the curve obtained by intersecting two adjacent triangles, the line integral of X cancels.

Lemma 1 (Cancellation). *If two triangles M_1, M_2 meet in a curve C and C_1, C_2 are the parametrizations matching the M_1, M_2, then $\int_{C_1} X\, dr + \int_{C_2} X\, dr = 0$.*

Proof. If X is a 1-form and C is a curve and $-C$ is the curve passed backwards, then $\int_C X\ dr + \int_{-C} X\ dr$. This is what happens here. You can see the identity also as a consequence of Green also, noting that $C \cup -C$ encloses an "empty region". □

17.3. We see that all the 1-form contributions from the edges are zero. The contributions from the faces M_i add up:

Lemma 2 (Additivity). $\int_M K\ dV = \sum_i \int_{M_i} K\ dV$.

Proof. The patches M_i are all disjoint. Their union is $\bigcup_i M_i = M$. Areas of disjoint regions add up. □

17.4. The contributions from the vertex degrees $d_i = |S(v_i)|$ add up too. [1]

Lemma 3 (Euler handshake). *If (V, E) has vertex degrees d_i, then $2E = \sum_i d_i$.*

Proof. You prove this in a homework. □

17.5. We still have to look at the contributions from the vertices. At each point P_i we have angles α_{ij} for $j = 1, \dots, d_j$, where d_j is the vertex degree.

Lemma 4 (Adding vertex curvatures). $\sum_{i=1}^{V} \sum_{j=1}^{d_i} \kappa_{ij} = 2\pi E - 2\pi V$.

Proof. (i) $\sum_{i=1}^{V} \sum_{j=1}^{d_i} \kappa_{ij} = \sum_{i=1}^{F} \sum_{j=1}^{3} (\pi - \alpha_{ij})$. (ii) $\sum_{k=1}^{V} \sum_{j=1}^{d_k} \pi = 2\pi E$. (iii) $\sum_{k=1}^{V} \sum_{j=1}^{d_k} \alpha_{kj} = 2\pi V$. □

[2]

17.6. Proof of the global Gauss-Bonnet theorem:
The local Gauss-Bonnet theorem told us $\sum_i [\iint_{U_i} K dV + \sum_j \kappa_{ij} - 2\pi] = 0$. This means that $\iint_U K\ dV + \sum_{i,j} \kappa_{ij} - 2\pi F = 0$. Therefore, using the previous lemma:

$$\iint_U K\ dV = 2\pi V - 2\pi E + 2\pi F = 2\pi \chi(M)\ .$$

17.7. 1) If M is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, then $\iint_M K\ dV = 4\pi$ (HW).
2) A genus k surface has $\iint_M K\ dV = 2\pi(2 - 2k)$. For a torus $\iint_M K\ dV = 0$.
3) A Klein bottle is obtained by gluing two Möbius strips together. $\iint_M K\ dV = 0$. Because each Möbius strip has Euler characteristic 0 (you computed that in an example), and the Möbius strip can be realized so that the boundary curvature χ_g is zero.

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[1]If V, E, F are the vertices, edges and faces. It is custom to write its cardinalities as V, E, F.

[2]See youtube video. This year I fell live (but not "free solo"). Some toe or thum acrobatics: the angles α_{ij} for the triangles F_i are relabeled β_{ij} for vertices. Also use the karate kick $3F = 2E$.

Lecture 18: Riemannian Manifolds

18.1. A **topological manifold** M **of dimension** m is a subset of some $\mathbb{R}^n$ such that every $x \in M$ has a neighborhood U, that is homeomorphic to an open subset $R = \phi(U)$ of $\mathbb{R}^m$. The pair (U, ϕ) is called a **chart**. It produces a **coordinate system** on U: there is a parametrization $r(\phi(x)) = x$, which is a regular map from $R \subset \mathbb{R}^m$ to $\mathbb{R}^n$, meaning that dr has rank m everywhere. A C^k **atlas** on M is a collection $\mathcal{F} = \{U_i, \phi_i\}_{i \in I}$ of charts such that $\bigcup_{i \in I} U_i = M$, and that all coordinate change maps $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are in $C^k(\phi_i(U_j \cap U_i), \mathbb{R}^m)$. An atlas is called **maximal**, if (U, ϕ) is a chart such that $\phi \circ \phi_i^{-1}$ and $\phi_i \circ \phi^{-1}$ are all C^k for all $i \in I$, then $(U, \phi) \in \mathcal{F}$. Two atlases $\mathcal{F}, \mathcal{G}$ are called **equivalent** if their union $\mathcal{F} \cup \mathcal{G}$ is an atlas. Given an atlas $\mathcal{F}$, the union of all atlases equivalent to $\mathcal{F}$ is called a **differentiable structure of** $\mathcal{F}$. [1]

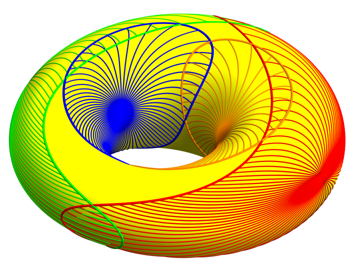

FIGURE 1. A m-manifold $M \subset \mathbb{R}^n$ is shown with part of an atlas $\mathcal{F}$. Each patch U_i is a regularly parametrized by $r : R_i \to U_i$ with $R_i = \phi_i(U_i) \subset \mathbb{R}^m$. The map r has maximal rank m everywhere on R_i.

18.2. A m-dimensional C^k**-differentiable manifold** is a pair $(M, \mathcal{F})$, where M is a m-dimensional topological manifold and $\mathcal{F}$ is a differentiable C^k structure on M. This means is that every point $p \in M$ is contained in a region $U = \phi(R)$. The inverse r of ϕ defines a parametrization from $R \subset \mathbb{R}^m$ to $M \subset \mathbb{R}^n$. We are now in the frame

[1] The concept can be difficult: $\mathcal{F}$ is not unique in general. On S^7, there are 28 different smooth structures. The smooth Poincare conjecture claims that S^4 has a unique differentiable structure.

work used before. We can for example define r_u, r_v and the fundamental forms. We say **smooth** if we just want to be able to differentiate as often as necessary. [2]

18.3. If $E = \mathbb{R}^m$ is the space of **column vectors** of dimension m, its **dual** E^* is defined as the space of all linear maps $f : E \to \mathbb{R}$, It is the space of **row vectors**. If $\{e_1, \dots, e_m\}$ is a basis of E, then $\{e^1, \dots, e^m\}$ denotes a basis of E^*. Every element in E can be written as $v = \sum_i v^i e_i$, every element in E^* can be written as $v = \sum_i v_i e^i$. For $p, q \geq 0$, the linear space T^p_q of all multi-linear maps $(E^*)^p \times E^q \to \mathbb{R}$ is called the space of **tensors of type** (p, q). Column vectors are $(1, 0)$-tensors in $T^1_0 = E$, while row vectors are $(0, 1)$-tensors in $T^0_1 = E^*$, bilinear maps are $(0, 2)$ tensors in T^0_2. A **tensor field of type** (p, q) on a m-manifold M is a smooth assignment of a (p, q) tensor to every point. Such a map is also called a **section** of the tensor bundle, generalizing that a **vector field**. A vector field is a section of the **tangent bundle** TM. For a $(0, 2)$ tensor field g for example, the attribute "smooth" means that for any vector fields X, Y, the function $x \to g(x)(X(x), Y(x))$ is smooth from M to $\mathbb{R}$. If $f : M \to \mathbb{R}^k$ is a smooth map, then df is a $(0, 1)$ tensor field. This is also called a **1-form**. **Vector field** is an abbreviation for a $(1, 0)$ tensor field. The first fundamental form g is by definition a $(2, 0)$ tensor field; think of a symmetric bilinear form attached to every point. A **Riemannian manifold** (M, g) is a smooth manifold M with a positive definite symmetric $(2, 0)$ tensor field g.

18.4. Let M be a m-manifold and $f : M \to \mathbb{R}^k$ be smooth. A point $x \in M$ is called a **critical point** and $f(x)$ a **critical value**, if the rank of $df(x)$ is not m. Non-critical points are called **regular points**.

Theorem 1. *If M is a m-manifold and $f : M \to \mathbb{R}^k$ is smooth and y is a regular value of f, then $M_f = f^{-1}(y)$ is a manifold of dimension $m - k$.*

Proof. At $x \in f^{-1}(y)$, the Jacobean map $df(x)$ has rank k and the kernel $H = \ker(df)$ of $df(x)$ is $(m - k)$-dimensional and $H^\perp$ is k dimensional. Take a chart (U, ϕ) in M which contains x. It defines a parametrization $r = \phi^{-1} : R \to U$. Define $g = f \circ r : R \subset \mathbb{R}^m \to \mathbb{R}^k$. Define eh projection $L : \mathbb{R}^m = H \oplus H^\perp \to \mathbb{R}^{m-k}, (h, h') \mapsto h'$ onto the orthogonal complement of the kernel. The map $F : R \subset \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ defined by $F(z) = (g(z), Lz)$ has the non-singular derivative $dF(u) = (dg(u), Lu)$. By the **inverse function theorem**, a neighborhood of $u = \phi(x)$ is mapped by the diffeomorphism F onto a neighborhood $F(y)$ of $F(u)$. We get so a chart $U_x = r(R_x)$, where $R_x = F^{-1}(H)$. It is a chart of M_f. Doing the same construction at any point $x \in M$ produces an atlas for $f^{-1}(y)$ and verifies that $f^{-1}(y)$ is a manifold. □

18.5. Examples: a) The $\mathbf{d-sphere}$ is the set $M = S^d = \{x \in \mathbb{R}^{d+1} \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\} \subset \mathbb{R}^{d+1}$. The **standard differentiable structure** $\mathcal{F}$ on S^d is generated by $\mathcal{F} = \{(S^d \setminus \{A\}, \phi_A), (S^d \setminus \{B\}, \phi_B)\}$, where $\phi_A \phi_B$ are **stereographic projections** from antipodal points A, B.
b) The set $SL(n, \mathbb{R})$ of $n \times n$ matrices of determinant 1 is a manifold.

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[2]A theorem of Whitney assures that a smooth compact m-manifold M given abstractly as para-compact Hausdorff space can be realized within $\mathbb{R}^n$ if $n = 2m + 1$.

Lecture 19: Discrete Manifolds

19.1. A **discrete m-manifold** is a finite graph $G = (V, E)$ for which every unit sphere $S(v)$ is a discrete $(m-1)$-sphere. A **discrete m-sphere** is a discrete m-manifold which has the property that removing a point renders it contractible. Inductively, a graph is called **contractible**, if both $S(v)$ and $S \setminus v$ are contractible for some $v \in V$. The 1-point space 1 is contractible. The empty graph is the (-1)-sphere. Let F_k denote the set of K_{k+1} subgraphs (k-simplices) and $f_k = |F_k|$. We have $F_0 = V, F_1 = E$. The **Euler characteristic** of M is defined as $\chi(M) = \sum_{k=0}^{m} (-1)^k f_k = f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^m f_m$. This definition of Ludwig Schläfli generalizes $\chi(M) = f_0 - f_1 + f_2 = V - E + F$ for 2-manifolds.

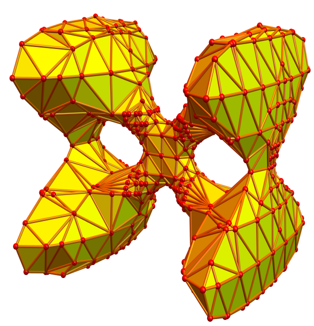

FIGURE 1. This 2-manifold M of genus $g = 2$ has $\chi(M) = 2 - 2g = -2$.

19.2. A graph without edges is a 0-manifold. A 0-manifold with 2 points is a 0-sphere. The reason is that emoving a vertex produces the contractible K_1. Every connected 1-manifold is a 1-sphere, a circular graph C_n with $n \geq 4$. Every finite 2-manifold is either a 2-sphere S^2 or a connected sum of tori or projective planes: $M = S^2, M = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ or $M = \mathbb{P}^2 \# \cdots \# \mathbb{P}^2$. A 2-sphere can be characterized as 2-manifold of Euler characteristic 2. The 16 cell and the 600 cells are examples of 3-spheres. The join of two 1-spheres is a 3-sphere. The join of a k-sphere with a m-sphere is a $(k+m+1)$-sphere. The join of G with the 0-sphere is called **suspension**.

19.3. Euler's formula $\chi(M) = V - E + F = 2$ for 2-spheres generalizes to higher dimension. The 0-sphere has $\chi(M) = V = 2$, every 1-sphere has $\chi(M) = V - E = 0$. Every 2-sphere has $\chi(M) = V - E + F = 2$. This pattern continues:

Theorem 1 (Euler's Gem). *If M is a m-sphere, then $\chi(M) = 1 + (-1)^m$.*

Proof. Use induction with respect to dimension m. For $m = 0$, we have $\chi(M) = 2$. The induction assumption is that all $(m-1)$-spheres S satisfy $\chi(S) = 1 + (-1)^{m-1}$. Pick a vertex v. As the unit sphere $S(v)$ is a $(m-1)$-sphere and $S(v) = B(v) \cap G \setminus v$, where both the unit ball $B(v)$ and $G \setminus v$ are contractible with Euler characteristic 1, we have, using the induction assumption, $\chi(M) = \chi(G \setminus v) + \chi(B(v)) - \chi(G \setminus v \cap B(v)) = 2 - (1 - (-1)^{m-1}) = 1 + (-1)^m$. □

19.4. In the continuum, manifolds can be constructed as level surfaces of functions like $x^2 + y^2 + z^2 = 1$. We can do that also in the discrete. Take an arbitrary function $f : V \to Z_k = \{0, \dots, k\}$. It defines a new graph M_f, where the vertices are the set of complete subgraphs on which f attains all k values. Connect two of these points by an edge, if one is contained in the other. The new graph M_f is a sub-graph of the **Barycentric refinement** of M. There is the analog of what we have seen classically for functions on manifolds.

Theorem 2 (Level Sets). *If M is a m-manifold and $f : M \to Z_k$ is an arbitrary function, then either M_f is empty or then M_f is a $(m-k)$-manifold.*

Proof. Let x be a n-simplex on which f takes all values. This means $f(x) = Z_k$. The graph $S^-(x) = \{y \subset x, y \neq x\}$ is a $(n-1)$-sphere in the Barycentric refinement of M. The simplices in $S^-(x)$ on which f still reaches Z_k is by induction a $(n-1-k)$-manifold and since we are in a simplex, it has to be a $(n-1-k)$-sphere. Every unit sphere $S(x)$ in the Barycentric refinement is a $(m-1)$-sphere as it is the join of $S^-(x)$ with $S^+(x) = \{y, x \subset y, x \neq y\}$. (The join of two spheres is always a sphere.) The sphere $S_f^+(x)$ in M_f is the same than $S^+(x)$ in M because every simplex z in M containing x automatically has the property that $f(z) = Z_k$. So, the unit sphere $S(x)$ in M_f is the join of a $(n-k-1)$-sphere and the $(m-n-1)$-sphere and so a $(m-k-1)$-sphere. Having shown that every unit sphere in M_f is a $(m-k-1)$-sphere, we see that M_f is a $(m-k)$-manifold. □

19.5. Differential geometry works too: define **curvature** as

$$K(v) = \sum_{k=0}^{m} \frac{(-1)^k f_{k-1}(S(v))}{k+1} = 1 - \frac{f_0(S(v))}{2} - \frac{f_1(S(v)))}{3} + \dots .$$

In the case of a 2-manifold, this boils down to $1 - f_0(S(v))/2 + f_1(S(v))/3 = 1 - d(v)/6$, where $d(v)$ is the vertex degree. For odd-dimensional manifolds, the curvature is constant zero.

Theorem 3 (General Gauss-Bonnet). $\sum_{v \in V} K(v) = \chi(M)$

Proof. The proof is the same as in the 2-dimensional case. Again look at the energies $\omega(x) = (-1)^{\dim(x)}$ attached to each simplex x in the graph (complete subgraph with $\dim(x) + 1$ vertices). Then $\chi(M) = \sum_x \omega(x)$. Now distribute all these energies of a k-simplex x equally to the $k+1$ vertices contained in x. As there are f_{k-1} simplices in $S(v)$ which correspond do simplices containing v, this adds $\frac{(-1)^k f_{k-1}(S(v))}{k+1}$ to each vertex v. Now just collect all up at a vertex v to get the curvature $K(v)$. The transactions of energies preserved the total energy = Euler characteristic. □

Lecture 20: The Curvature Tensor

19.1. While $\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}]$ is not a tensor, the **Riemann curvature**

$$R^s_{ikj} = \frac{\partial}{\partial u^k}\Gamma^s_{ij} - \frac{\partial}{\partial u^j}\Gamma^s_{ik} + \sum_r \Gamma^r_{ij}\Gamma^s_{rk} - \sum_r \Gamma^r_{ik}\Gamma^s_{rj}$$

is a $(1,3)$ tensor. The skew-symmetric R^s_i generates an orthogonal change when rotating in the k,j plane. We also define R in the form of the $(0,4)$-tensor $R_{mikj} = \sum_s g_{ms} R^s_{ijk}$.

19.2. Without coordinates: let $X = \sum_i X^i e_i$ and $Y = \sum_i Y^i e_i$ denote **vector fields** $= (1,0)$ tensor fields. [1] The notation $e_i = \partial_{u^i}$ reflects that a vector field $X = \sum_i X^i e_i$ also defines a linear map on functions $Xf = \sum_i X^i f_{u^i} = df X$. It is also known as **Lie derivative**. [2] Since every vector field X also is a linear map, one can look at the commutator $[X,Y] = XY - YX$, which is by Leibniz again a vector field. Proof: in coordinates $X = \sum_j X^j e_j, Y^j = \sum_j Y^j e_j$, this **Lie bracket** is $[X,Y]^i = \sum_j X^j \partial_j Y^i - Y^j \partial_j X^i$.

19.3. The **covariant derivative** $\nabla_X Y$ is a new vector field. Axiomatically it is determined by **Leibniz** $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, **metric compatibility** $g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y,Z)$, and being **torsion free** $\nabla_X Y - \nabla_Y X = [X,Y]$. The **fundamental theorem of Riemannian geometry** assures that there exactly one such derivative: it is $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$ and it determines the Riemann curvature tensor R.

19.4. The curvature tensor now also can be written as $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ so that $R = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. We have $g(e_m, R(e_k,e_j)e_i) = R_{mikj}$. Intuitively, $A = R(e_k, e_j)$ is the Frenet matrix when parallel transporting along a small rectangular loop spanned by e_k, e_j. The $A^s_i \in so(q,\mathbb{R})$ defines then a rotation in $SO(q,\mathbb{R})$ when doing the loop.

19.5. For linearly independent vectors u, v, the **sectional curvature**

$$K(u,v) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2}$$

is an intuitive approach to curvature. It only depends on the plane spanned by u, v and not the coordinate system. It only depends on the plane defined by the tangent vectors u, v. If M is two dimensional, it agrees with the Gauss curvature. But this is not obvious as it includes the Theorema egregium!

[1] The literature uses capital letters for vector fields, $\sum_i X^i e_i$ rather than $\sum_i v^i e_i$.

[2] or directional derivative $\nabla f \cdot v$ in multi-variable calculus, if $|v| = 1$.

Theorem 1. *The sectional curvatures determine the Riemann curvature tensor.*

19.6. The **Ricci curvature** R is a contraction of the curvature tensor $\boxed{R_{ik} = \sum_j R^j_{ijk}}$.
The **scalar curvature** is then the contraction of the Ricci curvature $\boxed{S = \sum_{j,k} g^{jk} R_{jk}}$.
In two dimensions, it is twice the Gauss curvature. The **Einstein tensor** G is defined as $\boxed{R - Sg/2}$. A metric is called an **Einstein metric** if $R = \lambda g$ for some λ. Define the **Hilbert functional** $S(g) = \int_M S_g dV_g$ and the inner product on $(0, 2)$ tensors as $\langle a, b\rangle_g = \int_M \sum_{i,j} a(e_i, e_j)b(e_i, e_j)\ dV$. Under which conditions is Hilbert functional extremal? [3]

Theorem 2. $\frac{d}{dt} S(g + th) = \langle Sg/2 - R), h\rangle_g$.

Theorem 3. *Every 2-manifold is an Einstein manifold: $Sg/2 - R = 0$.*

Proof. The reason is that $K = S/2$ and that the Hilbert functional $S(g) = 2\int_M KdV = 4\pi\chi(M)$ does not depend on the metric by Gauss-Bonnet, so that every g is a critical point. □

We see that for $\dim(M) = 2$, the Ricci tensor R is K times the Riemannian metric tensor g. This is not obvious as it leads to the Theorema egregium.

19.7. To prepare for relativity, generalize Riemannian manifolds. A **metric tensor** on a linear space E is a symmetric $(0, 2)$ tensor which is **non-degenerate** that is $g(u, v) = 0, \forall v \in E \Rightarrow u = 0$. A **metric tensor field** g is a tensor field $g \in T^0_2(M)$ such that $g(x)$ is a metric tensor in $T^0_2(T_xM)$. This means that for any vector fields X, Y the function $x \to g(x)(X(x), Y(x))$ is smooth. A **pseudo Riemannian manifold** (M, g) is a smooth manifold with a metric tensor field g on M. It is a **Riemannian manifold**, if g is positive definite, meaning $g(x)(v, v) \geq 0$ for all v. The **length** of a vector $v \in T_pM$ is defined as $||v|| = \sqrt{|g(p)(v, v)|}$, where $g(p)(u, v) = \sum_{ij} g_{ij}(p)u^i v^j$. [4] A vector in T_pM of length zero is called **null**. Vectors u for which $g(p)(u, u) = \sum_{ij} g_{ij}u^i u^j < 0$ are **time like**, vectors u with $g(p)(u, u) = \sum_{ij} g_{ij}u^i u^j > 0$ **space like**. The **length** of the curve is defined by $\int_a^b ||\dot{x}(t)||\ dt$.

19.8. Which signatures can be realized?

Theorem 4. *On any smooth manifold there exists a Riemannian metric g.*

Proof. We need to show that there exists $g \in T^0_2(M)$ which is symmetric, non-degenerate and positive definite: let $\{U_i, \phi_i\}$ be an atlas for M and let p_i be a **partition of unity**, subordinate to the cover U_i. Let q be a Riemannian metric on $\mathbb{R}^n$. For example $[q] = \text{Diag}(1, 1, 1 \dots, 1)$. Let $q_i = \phi_i^* q$ be the pull back metrics on U_i. Define $g(p) = \sum_i g_i(p)q_i(p)$. This is smooth and positive definite because for $p \in M$ and u in the tangent space T_pM, we have $g(p)(u, u) = \sum_i g_i q_i(u, u) > 0$. □

19.9. It is not always possible to build on a given manifold a metric of a given signature. For example, on the sphere $M = S^2$, there exists **no Lorentzian metric**, that is a metric of signature $(-1, 1)$. The reason is that one can not comb a 2-sphere.

[3] A proof can be found on pages 312-320 in Kuehnel.

[4] Mind the absolute value here!

Unit 21: Relativity

21.1. The **principle of general covariance** states that the pseudo Riemannian manifold (M, g) alone defines the gravitational laws. No preferred basis nor background "aether" concepts are allowed. Physical laws are invariant under smooth coordinate changes. Objects of interests are postulated to be tensorial. Einsteins theory of gravity links **space-time** with **matter**. Matter determines space time (M, g) in that g by minimizing the Hilbert action. Space time determines the paths of particles by minimizing kinetic action. [1]

The Einstein Equations	The Geodesic Equations
$\boxed{R - \frac{1}{2}Sg + \Lambda g = \kappa T}$	$\boxed{\ddot{x} = -\sum_{i,j} \Gamma^k_{ij}\dot{x}^i\dot{x}^j}$
Matter tells Space-time how to curve	Space-time tells Matter how to move

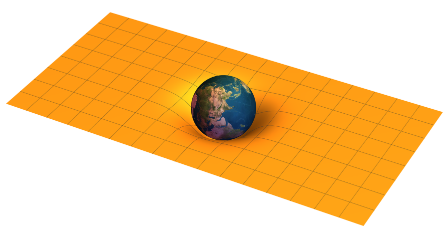

FIGURE 1. Mass deforms space time

21.2. The theory has been experimentally confirmed in various instances like 1) The **perihelion advance of planets like Mercury**, 2) the **gravitational lensing of light** around stars or galaxies, 3) the **time delay in radar probing of planets**, 4) the **spectral shift of light emanating from massive objects**, 5) the **precession of a gyroscope**,freely orbiting the earth, 6) the **detection of gravitational waves**

[1]Even when restricting to gravity and not taking into account quantum mechanics, this is unsatisfactory. In a 2-body problem of two massive particles like a black-hole binary, the masses should contribute to the stress-energy tensor T. In black-hole situations, the paths need to be removed from the manifold. About the math: Yvonne Choquet-Bruhat (1923-2025) dealt with the Cauchy problem in 1969. Demetrios Christodoulou and Sergiu Klainermann proved in 1994 nonlinear gravitational stability. Numerical schemes for black hole binaries in vacuum using **post-Newtonian expansions** exist since 2005. It is a total mess, both from an applied math (engineering) as pure math perspective.

from Black-hole mergers, 7) the **pictures of black holes** like Sagittarius A* and Messier 87* by the **event horizon telescope**.

21.3. In **special relativity**, the metric g is no more required to be positive definite. The **Galilei group** generated by rotations and translations is replaced by the **Poincaré group** generated by **Lorentz transformations** and translations. Any rotation matrix in a (e_0, e_i)-plane with $i = 1, 2, 3$ is replaced by a **hyperbolic rotation**, where cos is replaced by cosh and sin by sinh leading to **Lorentz boosts**. For $M = \mathbb{R}^4$

$$g = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

with speed of light c is called the **flat Lorentz metric**. [2] The null vectors are the vectors in the **light cone** $\{||u|| = 0\}$. Vectors v satisfying $\langle v, v\rangle = \sum_{i,j} g_{ij} v^i v^j < 0$ are **time like**. Vectors $\langle v, v\rangle > 0$ are **space like**. Particles with space like velocity vectors have velocity smaller than the speed of light. Particles with time like velocity vectors are called **tachions**. They have velocity larger than the speed of light.

21.4. The most important example in general relativity is the **Schwarzschild metric**. All known confirmations of general relativity are solely based on this model. On the manifold $M = \mathbb{R}^4 \setminus \{r \leq 2m\}$ we can use the spherical coordinates $t = x^0, r = x^1, \theta = x^2, \phi = x^3$. For $r > 2m$, the **Schwarzschild metric** is given by

$$g = \begin{bmatrix} -c^2(1 - \frac{2m}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2m}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix} .$$

It is a model for the space-time in presence of a single massive object of mass M. [3] The constant m is GM/c^2, where G is the gravitational constant and M the mass. The number $2m$ is called the **Schwarzschild radius** (about 3km for the sun, less than 1cm for the earth). You check in a Pset that (M, g) satisfies the Einstein equations.

21.5. The following establishes why the theory "makes sense". GR is a refinement of the Newton theory of gravitation.

Theorem 1. *Relativity for slow particles in a weak field becomes Newtonian mechanics.*

Proof. If $g = \overline{g} + h$, where $\overline{g}$ is the flat metric and h is small then $\dot{x}^i$ for $i = 1, 2, 3$ can be neglected with respect to $\dot{x}^0$. The geodesic equations $\ddot{x}^k = -\sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j$ is approximated by $-\Gamma^k_{00} = \partial_{u^k} \overline{g}_{00} - \partial_{u^0} h_{0k}$. If the gravitational field does not change in time, the later term goes away and $\ddot{x} = -\frac{1}{2} \nabla h_{00}$ which is the Newtonian equation for $h_{00} = 2V$. We therefore have $g_{00}/c^2 = 1 + 2V/c^2$. Now V/c^2 is 10^{-9} for the earth, 10^{-6} on the sun, 10^{-1} for a neutron star. □

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[2] Also Misner,Thorne, and Wheeler use g to be positive definite on **space-like** hyper-surfaces.

[3] By a **theorem of Birkhoff**, the unique spherically symmetric solution of Einstein's equations.

DIFFERENTIAL GEOMETRY

MATH 136

To Lecture 8: Frenet inverse problem

8.1. Frenet's theorem describes regular curves $r(t)$ in $\mathbb{R}^n$ in terms of the rotation matrix $Q(t)$ that orients the Frenet frame as a function of t. Every Frenet curve is described as such. The reverse is to start with a path of rotation matrices $Q(t)$ and to ask about the properties of the curve. A curve of curvature matrices $K(t) \in so(n)$ leads via the differential equation $Q' = K(t)Q$ to a path in $Q(t) \in SO(n)$. The first row of $\int_0^t Q(s)\, ds$ is the curve $r(t)$. We have now $r'' = K$. One question can be. For which periodic paths $K(t)$ does the curve stay bounded? For which paths $K(t) + tK_0$ does the curve stay bounded?

8.2. Stability of driven systems is a classical problem, for example in celestial mechanics. Here is an other situation. If the curvature grows $K(t) + tK_0$ we get a continuous time version of the **Curlicue problem**. That problem is obtained by taking a sequence $K(t) + tK_0$ in $so(n)$ then look at $Q(t+1) = e^{iK(t)}Q(t)$ with $Q(0) = 1$. This produces **curlicue paths** $R(n) = \sum_{k=0}^{n-1} Q_k$ in the algebra of matrices. Paths can be seen by applying $R(n)$ to a vector. Both the Frenet or Curlicue problem naturally go over to manifolds. In the curlicue problem drive in each step along a geodesic path and in the continuum we get a space dependent Frenet problem $Q' = K(r(t), t)Q$, where now the curvature K is point dependent.

8.3. Let us now focus on a special case in the plane $n = 2$: you drive in a desert and $\kappa(t)$ tells you how to turn your steering wheel. Do you remain in a bounded region? For $n = 2$, we can use complex analysis. The function $\kappa(t)$ gives an explicit curve $r(t) = \int_0^t e^{i\phi(s)}\, ds$, where $\phi(s) = \int_0^s \kappa(u)\, du$ is the phase. We can investigate the case $\kappa(t) = k + a\sin(mt)$ for example We can think of the Frenet equations $\psi' = i\kappa(t)\psi$ as a Schrödinger equation. We prove here that stability happens if and only if k is not a multiple of the oscillator frequency.

8.4. In two dimensions, where $so(2) = \mathbb{R}$ we can ask for which periodic functions $\kappa(t)$ the curve $r(t) = \int_0^t e^{i\phi(s)}\, ds \in \mathbb{C}$ remains bounded, where $\phi(s) = \int_0^s \kappa(u)\, du$ is the phase. An example of a stable curve $\kappa(t) = 1 + \sin(2t)$. An example of an unstable one is $2 + \sin(2t)$. Also for $\kappa(t) = 1 + \sin(2t) + \sin(3t)$ we run away. The case $3 + \cos(2t) + \sin(4t)$ is stable but $2 + \cos(2t) + \sin(4t)$ is unstable. $\kappa[t_] := \sqrt{2} + \sin(2t)$ gives a bounded, non-closed curve.

Theorem 1. *A planar curve with curvature $\kappa(t) = k + a\sin(mt)$ is bounded if and only if k is not a multiple of m.*

8.5.

Proof. This integral is of the form $F(t) = \int_0^t e^{i\phi(s)}\, ds$ where ϕ is the phase function in the theory of stationary phase. For $\kappa(t) = k + am\sin(mt)$ we have is $\phi(s) = ks - a\cos(ms)$ requiring us to integrate $f(s) = \exp(iks)\exp(-ia\cos(ms))$ over $[0, 2\pi]$. Fourier expansion gives

$$f(s) = \sum_{n=-\infty}^{\infty} J_n(a) e^{i(k-mn)s}$$

so that

$$F(t) = \sum_{n=-\infty}^{\infty} J_n(a)(e^{i(k-mn)t} - 1)/(i(k-mn))\ .$$

If $k - mn$ is never zero, then this is a bounded function. If $k + mn = 0$, Hopital shows that that $(e^{i(k-mn)t} - 1)/(i(k-mn)) = t$. The curve therefore grows like $2\pi(-i)^{k/m} J_{k/m}(a)$. □

8.6. With more frequencies like $k+\sin(\alpha t)+\sin(\beta t)$ we have instability if and only if k is in the module generated by α, β. This means that for any 2π periodic trigonometric polynomial $p(t)$ we have stability for $\kappa(t) = k + p(t)$ for almost all k. But there is a dense set of k's for which we have instability.

8.7. There are many related questions. Here are more: A) What is the closure of the path $Q(t) \in SO(3)$. What are the α and ω limit sets?
B) When is the curve C defined by κ and τ bounded?
C) Is it possible that for almost periodic functions κ, τ the curve C in $\mathbb{R}^3$ has an ω limit set that is a fractal?

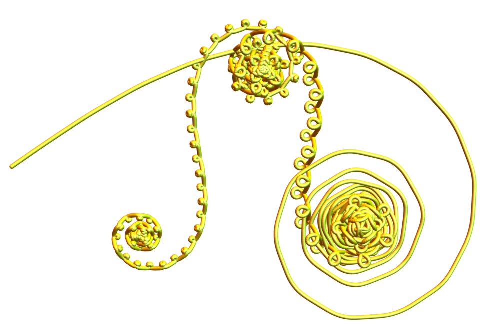

FIGURE 1. An example with $\kappa(t) = t\sin^2(t)/100, \tau(t) = 2\cos(3t)$.

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To Lecture 11: Manifolds as Particles

11.1. All manifolds are assumed to be 2-dimensional, connected and compact without boundary. They form a **monoid** in which addition is the **connected sum** $A\#B$. It means that $\#$ is an associative operation with a **zero element** 0, the **sphere**. Adding a sphere produces a topologically equivalent manifold. Two discrete 2-manifolds are called **homeomorphic** if there exists a sequence of additions or subtractions of spheres that deforms one into the other.

11.2. There is an obvious **asymmetry** between orientable and non-orientable manifolds, as one can see when looking at the **Euler characteristic** $\chi(M)$ which has the property that the genus (spin) $g(M) = 1-\chi(M)/2$ is additive $g(A\#B) = g(A)+g(B)$.

Theorem 1 (Classification of orientable 2-manifolds)**.** *Any nonzero orientable 2-manifold M is a sum of $k \geq 1$ tori $\mathbb{T}^2\#\cdots\#\mathbb{T}^2$ and $\chi(M) = 2-2k$.*

Theorem 2 (Classification of non-orientable 2-manifolds)**.** *Any nonzero non-orientable 2-manifold M is a sum of $k \geq 1$ projective planes $\mathbb{P}^2\#\cdots\#\mathbb{P}^2$ and $\chi(M) = 2-k$.*

11.3. While for non-orientable manifolds, $g(M)$ covers the half integers $\mathbb{N}/2$, orientable manifolds $g(M)$ only cover the natural numbers $\mathbb{N}$. The missing hole in the doughnut forces the introduction of "half-tori". It also solves the group completion, sees manifolds as particles and genus as spin.

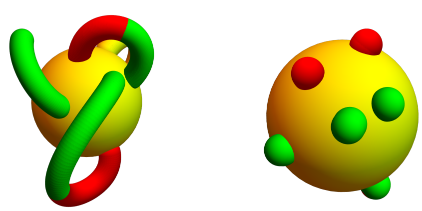

FIGURE 1. A particle with $2g$ Fermions A, B and one with $2g$ Fermions P, Q. Euler characteristic $\chi(G) = 2-2g$ relates with spin g.

11.4. One might try to generate more manifolds like $\mathbb{T}^2\#\mathbb{P}^2$ but this is equivalent to $\mathbb{K}^2\#\mathbb{P}^2 = \mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. One can not embed all manifold in **one single group** because adding a Klein bottle $\mathbb{K}^2$ to a non-orientable M is the same than adding a torus $\mathbb{T}^2$ to it. The **van Dyck's identity** $\mathbb{P}^2\#\mathbb{K}^2 = \mathbb{P}^2\#\mathbb{T}^2$ would imply the false identity $\mathbb{K}^2 = \mathbb{T}^2$. We are forced to consider two groups.

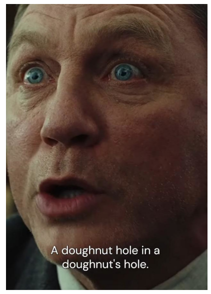

FIGURE 2. *"A doughnut hole in a doughnut's hole! We have to look closer. And when we do, we see the doughnut hole has a hole in its center. "* Inspector Benoit Blanc in "Knives Out".

11.5. If we **symmetrize** we need to introduce **half tori**. In order to Grothendieck complete the monoids to **groups**, we need **anti-projective planes** and **anti-half tori**. Write A for opening a hole and B for opening a second hole and connect it with the already opened hole. Then $A\#B = T^2$. We have naturally extended and completed the monoid to a **finitely presented group** $\langle A, B | A^2 = B^2 = 0 \rangle$. In the non-orientable case, call P the **projective particle** and Q the **anti-projective particle**. Again, the **Pauli principle** postulates, declare P and Q to be involutions. We again get the infinite **dihedral group** $\langle P, Q | P^2 = Q^2 = 0 \rangle$.

11.6. The non-Abelian PQ **group of manifolds** is generated by the projective particle and anti-particle. As Fermions have half spin g, think of P and Q as Fermions.

g=0	0	sphere			
g=1/2	P	projective plane	Q	anti projective plane	g=-1/2
g=1	PQ	Klein bottle	QP	anti Klein bottle	g=-1
g=3/2	PQP	torus with a cap	QPQ	anti torus with a cap	g=-3/2
g=2	$PQPQ$	genus 2 Klein bottle	$QPQP$	anti genus 2 Klein bottle	g=-2

11.7. The non-Abelian AB **group of manifolds** is generated by the torus particle A and torus anti-particle B. Again, one can think of the half spin generators A, B as Fermions.

g=0	0	sphere			
g=1/2	A	half torus	B	anti half torus	g=-1/2
g=1	AB	torus	BA	anti torus	g=-1
g=3/2	ABA	three half torus	BAB	anti three half torus	g=-3/2
g=2	$ABAB$	genus 2 torus	$BABA$	anti genus 2 torus	g=-2

11.8. In both cases, we get the natural **infinite dihedral group** $G = \langle X, Y | X^2 = Y^2 = 1 \rangle$ that is non-Abelian. It is **natural** in the following sense: it can be equipped with a metric such that the group structure on it is determined uniquely by the metric space alone. The integers $\mathbb{Z}$ are not natural because whatever translation invariant metric is picked on $\mathbb{Z}$, it is possible to define both an Abelian and a non-Abelian group structure on it that is group invariant. [1]

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[1]See O. Knill, "On Graphs, Groups and Geometry", 2022

To Lecture 18: What is a manifold?

18.1. We measure distances using light or sound and so observe to live in a **metric space** M, a topological space in which open sets are defined by small open balls $B_r(p)$ defined by the metric. Empirically we also observe our physical space M is a **3-manifold**. A sphere $S_r(p)$ in M given as the set of points in positive distance r from a point is a 2-sphere. Any of these objects $N = S_r(p)$ themselves are a 2-manifold: it has the property that $S_r(p)$ in N is a circle for small r. A circle is a 1-manifold: a small sphere $S(r)$ in a circle is a 0-sphere, a discrete set of 2 points. Every discrete set of points is a 0-manifold because $S_r = \emptyset$ and $\emptyset$ is declared to be a (-1)**-sphere**. The notion of "manifold" is so inductively linked to the notion of "sphere". The induction is with respect to dimension.

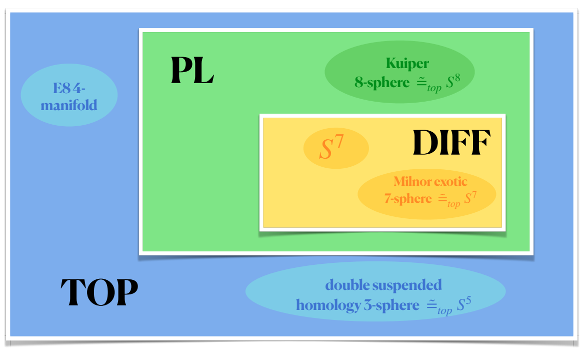

Figure 1. Three categories of manifolds.

18.2. The question **"what is a sphere?"** was first studied by Herman Weyl. Topology tells us that spheres are connected for dimension $q > 0$ and simply connected for $q > 1$. The **standard** q**-sphere** $S_r(x) = \{x \in \mathbb{R}^{q+1}, \sum_{i=1}^{q+1} x_i^2 = 1\}$ defined in Euclidean space is a q-manifold of Euler characteristic $1 + (-1)^q$. It has an atlas with 2 patches given by stereographic projections $\phi_\pm(x) = (x_1, \dots, x_q)/(1 \pm x_{q+1})$. A **sphere** is a manifold which can be covered with 2 contractible subsets. It also admits a Morse function with exactly two critical points. These properties are equivalent and each forces M to be a sphere. Can we define a sphere without Euclidean baggage? We definitely want the Euler characteristic to be $1 + (-1)^q$. For a "Dehn-Sommerville q-manifolds", we only need to ask that $S_r(p)$ is a "Dehn-Sommerville (q-1)manifold" of dimension $(q-1)$ and Euler characteristic $1-(-1)^q$ if r is small enough. The Dupin Cyclide from Homework 1 is such a surface.

18.3. The standard definition of a q-manifold is to define it as a paracompact Hausdorff space that is locally homeomorphic to an open ball in Euclidean space. This leads to the category of **topological manifolds** TOP. Not all topological manifolds can be triangulated. One has therefore also introduced the class PL of **piecewise linear manifolds** that admit a triangulation. Not all triangulated manifolds admit a differentiable structure; others admit many different differentiable structures. The subclass DIFF of **differentiable manifolds** now produces a set of inclusions $\boxed{DIFF \subset PL \subset TOP}$ where each is strict. Kervaire constructed in 1960 a manifold in $PL \setminus DIFF$, the join of a Poincare homology sphere with a circle is in $TOP \setminus PL$ (Edwards and Cannon 1970-79). Milnor constructed in 1959 inequivalent spheres in DIFF. Taubes 1987 got uncountably many differentiable structures in $\mathbb{R}^4$.

18.4. When we talk about discrete manifold we mean PL manifolds as any discrete manifold can be geometrically realized as a PL manifold. It is no surprise therefore that in differential geometry, where we are mostly interested in DIFF, we can do essentially everything using discrete manifolds.

18.5. The three categories can also be extended to the Dehn-Sommerville case. Some Dehn-Sommerville spaces are algebraic varieties given as the intersection of polynomial equations. But in general they fall into the category of **schemes**. We do not need to get fancy however: here is a reasonable definition:

> A **Dehn-Sommerville q-manifold** M is a compact metric space such that for every point $p \in M$ there exists $\rho > 0$ such that every sphere $S_r(p)$ is a Dehn-Sommerville (q-1)-manifold of Euler characteristic $1-(-1)^q$ for $0 < r < \rho$. To start induction, declare $\emptyset$ to be a Dehn-Sommerville (-1)-manifold.

18.6. Note that the just given definition does not use any continuous map and so is extremely elegant. And here is a definition to define a topological manifold without the use of charts. It immediately goes over to combinatorial settings, where no reference to Euclidean spaces is needed any more.

> A metric space (M, d) is called **contractible** if there exists a point $p \in M$ and a continuous map $F(s,x) : [0,1] \times M \to M$ such that $F(0,x) = x$ and $F(1,x) = p$ for all x. A **q-manifold** M is a compact metric space such that for every $p \in M$ there exists $\rho > 0$ such that every $S_r(p)$ is a $(q-1)$-sphere for $0 < r < \rho$. A **q-sphere** M is a q-manifold for which there exists $p \in M$ such that $M \setminus \{p\}$ is contractible. The empty set $\emptyset$ is declared to be the unique (-1)-sphere.

18.7. The just given definition produces the class TOP of compact topological manifolds in finite dimensions. The assumption to have a metric space rather than a paracompact topological space is no a loss of generality in finite dimensions because of the Whitney embedding theorem. The definition produces for every point p a neighborhood chart. One can prove by induction that the ball $B_r(p) = \{x \in M, d(x,p) < r\}$ is homeomorphic to the **standard unit ball** in $\mathbb{R}^q$. Also the reverse holds because a compact $M \in TOP$ is known to be a q-sphere, if $M \setminus \{p\}$ is contractible for some $p \in M$ (actually for all $p \in M$).

DIFFERENTIAL GEOMETRY

MATH 136

To Lecture 21a: Discrete Gravity

21.1. Classical general relativity is based on two fundamental variational principles: **A)** particles move on geodesics, critical points of a variational problem $\mathcal{L}(c)$ on the set of paths c connecting two points. **B)** The metric tensor g is a critical point of the **Hilbert action** $\mathcal{H}(g) = \int_M S_g \; dV_g$. These are **Einstein manifolds** (M, g). They satisfy the Einstein equations $R - \frac{S}{2}g = 0$, where R is the Ricci tensor and S is scalar curvature. In dimension 2, where $S = 2K$, every compact manifold is Einstein by Gauss-Bonnet. In larger dimensions it implies S is constant. Can we model this in the discrete? One of the many approaches was by Regge in 1960. He used an embedding of the finite structure in Euclidean space and so designed a numerical scheme. Can it be done in finite geometry? Finite mathematics is motivated by the possibility that there is no Euclidean space after all. This would happen for example if ZFC was inconsistent. It is well possible that already the real line, the 1-dimensional Euclidean space, is just a pipe dream.

21.2. Replace a Riemannian manifold (M, g) with a discrete manifold $G = (V, E)$ of dimension m. There is no ambient Euclidean space. An edge E is just a pair of vertices. No additional structure like length or distances is assumed. It is just a graph. A naive attempt for part A) is to look at paths (sequences of points in V with adjacent pairs in E) in G and pick the one with the least amount of edges. A path is **simple** if its induced graph is a path graph. This is problematic already in the smallest possible cases. Take for example the icosahedron, a 2-sphere that is one of the smallest 2-dimensional manifolds. There are vertices A,B of distance 2 for which one has already two shortest connection. Even after many Barycentric refinements, we still have always points in distance 2 apart which feature two different geodesics connecting them. We can also not continue a path naturally, something which a geodesic flow should be able to do. Take an edge reaching a vertex of degree 5 in the icosahedron. We would have artificially to stablish a rule on each vertex telling on how an incoming path gets continued or toss a coin. Ugly!

21.3. A better way for part A) is to take maximal ordered simplex $x = (x_0, \dots, x_m)$ and map it into a new simplex $y = (x_1, \dots, x_m, x_0')$, where x_0' is the second point in the 0-sphere $\bigcap_{k=1}^{m} S(x_k) = \{x_0, x_0'\}$. Call this $y = T(x)$. We can continue this process and define $T^2(x) = T(y)$. This is a deterministic. We stack simplices onto each other. It produces a nice dynamical system and an **exponential map**. Having geodesics with exponential map is crucial to define sectional curvature and so to **Ricci curvature** R and so **scalar curvature** S. Again, all this needs to be done without

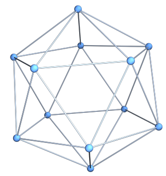

FIGURE 1. Defining a geodesic flow on an icosahedron is problematic if geodesics are paths on the vertex set V. Better: move on triangles.

geometric realization of the structure. The ZFC axiom system might be consistent, forcing mathematics to retrench to the finite.

Theorem 1. *For any m-manifold G, there exists a geodesic flow as a permutation on the finite set P of ordered maximal complete subgraphs of G.*

21.4. We have seen in class that there is for any graph a curvature $K(v)$ that satisfies the Gauss-Bonnet formula $\sum_{v \in V} K(v) = \chi(G)$. In two dimensions, where $K(v) = 1 - \deg(v)/6$, this already encodes the curvature tensor. In higher dimensions, we need to make sense of the Riemann tensor, the Ricci tensor and the scalar curvature. Classically, these curvatures are completely determined from **sectional curvatures**. We therefore need to give a **reasonable notion of sectional curvature** in the finite. The key is that sectional curvature is defined with a reasonable geodesic flow.

21.5. Here is how to define sectional curvature in a m-manifold with $m \geq 2$. Take a triangle $t = (x_1, x_2, x_3) \subset x$ in a maximal ordered simplex x. Every of its edges defines a bone. They are $b_3 = x - (x_1, x_2), b_1 = x - (x_2, x_3)$ and $b_2 = x - (x_1, x_3)$. The dual simplices $(\bigcap_{j=0}^{m-2} S(x_j))$ are cyclic graphs in the dual graph (with maximal simplices are vertices and pairing two if they intersect in a $(m-1)$-simplex). (For $m = 2$, vertices are bones and the unit spheres are the dual bones, for $m = 3$ the edgers are bones and the tetrahedra hinging on it are the dual bone). The bones encode how x is connected to three more maximal simplices y_1, y_2, y_3. Two adjacent bones define new triangle in each of the y_i, allowing to continue the surface. This allows to get sectional curvature for any pair (t, x) where t is a triangle and x is an ordered maximal simplex.

21.6. We can now define Ricci curvature of an edge e of the manifold as the average over all sectional curvatures of triangles t containing e. The scalar curvature S at a vertex v finally is the average over all Ricci curvatures of edges containing v. What property does correspond to the Einstein equations in the continuum? A natural guess is to find a geometry for which a suitably scaled $H = \sum_{v \in V} S(v)$ is maximal or minimal (or to take a short-cut, where $S(v)$ is constant). Looking at such variational problems is pretty unexplored. They could be called the **"Sarumpaet rules"**. [1] One can try different things like minimizing a suitably scaled Hilbert action H. Any local modification of the q-manifold would keep the Hilbert action or make it larger. Encouraging is that any 2-manifold is an Einstein manifold in this sense, because Gauss-Bonnet assures that the Hilbert action is constant for all 2-manifolds of the same type. In general, look for structures with constant S.

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[1] Greg Egan novel "Schild's ladder". Schild's ladder is a numerical scheme to compute geodesics.

DIFFERENTIAL GEOMETRY

MATH 136

1. Homework

This first homework deals with two ways to describe manifolds: implicit equations or parametrization. The core mathematical topic is the implicit function theorem. This homework is a guide from multi-variable calculus to differential topology. This problem set is due Friday September 12.

Dupin Cyclide

Problem 1.1: The level surface $S \subset \mathbb{R}^3$ given by $(x^2+y^2+z^2+7)^2 - 4(3x-1)^2 - 32y^2 = c$ is an example of a **Dupin cyclide**.
a) Draw $\{f = c\}$ for $c = -10, c = 0, c = 10, c = 100$.
b) There are a three parameters c for which S is singular meaning that there are points on S for which the gradient $df^T = \nabla f = \vec{0}$. Find these parameters.
c) Re-derive the multi-variable calculus fact that if the gradient of f is not zero at a point, then the surface can near this point be written as a graph. E.g. $z = g(x, y)$ with $g_x = -f_x/f_z, g_y = -f_y/f_z$.
d) Conclude from c) that if c is not one of the three critical values, the surface has a tangent plane at each point and so is a manifold.
e) A surface is called compact if it is closed and bounded. Prove that S is bounded for any c.
f) Look up and state the Sard theorem about the **critical values**, the set of c is for which $f = c$ is a manifold.

Problem 1.2: Intersecting the surface with $z = 0$ gives the algebraic curve $(x^2+y^2+7)^2 - 4(3x-1)^2 - 32y^2 = c$. It is the solution set of a polynomial equation in two variables.
a) Draw the curves for the parameters $c = -10, c = 0, c = 10, c = 100$.
b) Again verify that there are three parameter values for which the curve is singular.
c) Again verify that in the regular case the curve looks locally like a graph. E.g. $y = g(x)$ with slope $g'(x) = -f_x/f_y$.
d) Conclude that if c is not one of the critical parameters, the we deal with a manifold, a curve that near every point can be written as a graph.
e) Look up the proof of the classification of one dimensional manifolds (without boundary) and give the argument for a proof.

Problem 1.3: a) Verify that for $c = 0$ the surface $f = c$ defined in a) can be parametrized as $\vec{r}(u, v) = [x, y, z]^T$ with

$$\begin{aligned} x &= R(1 + 8\cos(u) - 3\cos(u)\cos(v)) \\ y &= R\sqrt{8}(3 - \cos(v))\sin(u) \\ z &= R\sqrt{8}(\cos(u) - 1)\sin(v) , \end{aligned}$$

where $R = 1/(3 - \cos(u)\cos(v))$.
b) Compute the first fundamental form $g = dr^T dr$.
c) Check that $|r_u \times r_v|^2/R^4 = 256(3 - \cos(v))^2 \sin^4(u/2)$. For which (u, v) values is this zero?

Problem 1.4: Define $f_1(x, y, z) = f(x, y, z)$ from problem 1.1 and define $f_2(x, y, z) = z$ Lets look at the map from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$f(x, y, z) = \begin{bmatrix} f_1(x, y, z) \\ f_2(x, y, z) \end{bmatrix} .$$

The Cassini oval can be written as a system of non-linear equations $f(x, y, z) = C$ with $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The Casini ovals are intersection of two polynomial equations.
a) Compute the **Jacobean matrix**

$$df(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 & \frac{\partial}{\partial z} f_1 \\ \frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 & \frac{\partial}{\partial z} f_2 \end{bmatrix} .$$

b) Look up the implicit function theorem in this case. When does df have not maximal rank?
c) We have seen in problem 1.1 that there are three critical c values. See whether you get the same points.

Problem 1.5: This problem should now more routine after battling problems 1.1-1.4.
a) By using a good picture or coordinates, explain why $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$ is a torus.
b) Find a parametrization for the surface similarly as 1.3 did for the cyclid.
c) Use the implicit description $f = 0$ to show that the manifold is compact.
d) Using the parametrization $S = r(R)$, check the maximal rank condition, again verifying that we have a manifold in the sense that every neighborhood of a point on the surface can be parametrized $\phi(U)$ where U is an open region in $\mathbb{R}^2$.

DIFFERENTIAL GEOMETRY

MATH 136

2. Homework

This is the second homework. It is due Friday September 19th.

Surfaces

Problem 2.1: We have mentioned in class twice the fundamental theorem of linear algebra tells that for any $n \times m$ matrix,

$$\ker(A^T) = \mathrm{im}(A)^\perp .$$

a) Prove this formally and also illustrate your proof with an example.
b) Why does it imply the **rank-nullity theorem**: $\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$ holds for any $n \times m$ matrix. Use the same example to illustrate your proof.
c) If A is a $n \times m$ matrix and $b \in \mathbb{R}^n$. For which k, n, m is it possible that $\{x, Ax - b = 0\}$ is a k-dimensional manifold?

Problem 2.2: a) Verify that the parametrization

$$r(\theta, \psi, \phi) = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \cos(\psi)\sin(\phi) \\ \sin(\psi)\sin(\phi) \end{bmatrix}$$

parametrizes the 3-sphere $S : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ in $\mathbb{R}^4$. It has been used first by Heinz Hopf. Here $\theta \in [0, 2\pi], \psi \in [0, \pi]$ and $\phi \in [-\pi/2, \pi/2]$. (To verify that this is the right choice of parametrization just check that $\theta \to \theta, \psi \to \psi + \pi, \phi \to -\phi$ produces the same point.)
b) Verify that the set $SU(2)$ of complex 2×2 matrices of the form

$$A = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$$

which have determinant 1 also represent the 3-sphere. Verify that there is a $1-1$ correspondence between $SU(2)$ and S.
c) Conclude that there is a multiplicative structure $*$ on the 3-sphere. We can define $x * y$ are points in S and get a new point. This multiplication is associative and each element has a unique inverse. Which point on the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ represents the 1-element?

Problem 3.3: a) Verify that the 2 dimensional surface parametrized as

$$r(u,v) = \begin{bmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \cos(u)\sin(v) \\ \sin(u)\sin(v) \end{bmatrix}$$

is a subset of the 3-sphere. Check that it is a regular surface and so a 2-manifold by checking that dr has rank 2 everywhere.
b) What kind of surface is it? Compute its surface area with the formula given in the notes. Note that u, v both go from 0 to 2π.

Curves

Problem 2.4: a) Verify that for every non-zero integers a, b the curve

$$r(t) = \begin{bmatrix} \cos(at)\cos(bt) \\ \sin(at)\cos(bt) \\ \cos(at)\sin(bt) \\ \sin(at)\sin(bt) \end{bmatrix}$$

t goes from 0 to 2π is contained in the 3-sphere.
b) Find the arc length using the formula you know. The answer depends on a, b.
c) Give an explicit parametrization of a closed curve in the 3-sphere for which the arc length is larger than 1000.

Problem 2.5: a) Given two points A, B in $\mathbb{R}^n$. Prove the **theorem of Archimedes** telling that the straight line gives the shortest connection between A and B. That is, among all smooth curves connecting A with B. We will later call the shortest connection a **geodesic**.
b) Prove that any smooth regular curve $r : [a,b] \to \mathbb{R}^n$ can be parametrized by arc length s: there is a new parametrization $f(s)$ such that the velocity is $|f'(s)| = 1$ at all points. We will need this result next week, when we prove the **fundamental theorem of curves**.

Homework 3

This is due Friday, September 26.

Problem 1: We have seen in class proofs of the formulas

$$\kappa = |r' \times r''|/|r'|^3, \quad \tau = (r' \times r'') \cdot r'''/|r' \times r''|^2 .$$

Prove the easier reverse: if $r(t)$ is parametrized by arc length, then these formulas agree with the formulas for curvature and torsion you have seen in class when we have arc length parametrization and defined in general $\kappa_j = e_j' \cdot e_{j+1}$. So, show that the above formulas simplify to $\kappa = T' \cdot N = e_1' \cdot e_2$ and τ simplifies to $\tau = N' \cdot B = e_2' \cdot e_3$.

Problem 2: a) Look up and write down a proof that if $F(t,x)$ is a differentiable function from $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$, then there exists an open interval $(-a,a)$ and a unique path $x(t) : (-a,a) \to \mathbb{R}^N$ such that $x' = F(t,x)$ and $x(0) = x_0$. We want you to write down the proof in the differentiable case which is a bit more special than the usual assumption assuming a Lipschitz property for F.
b) Justify that if $x(t)$ stays bounded meaning that there is constant such that $|x(t)| \leq M$ for all t, then the solution exists for all t. We call this a global solution. c) Now conclude that if $Q' = K(t)Q$ is a differential equation for a matrix $Q(t)$ with skew symmetric $K(t)$, then there is a global solution $Q(t)$ in $SO(n)$

Problem 3: a) Determine from each of the spaces $SO(n), so(n), SU(n), su(n)$ whether they are linear spaces or not.
b) Check that if $x(t)$ is a differentiable curve in $SO(n)$, then $x(t)$ satisfies the differential equation $x'(t) = A(t)x(t)$, where $A(t) \in so(n)$, the space of skew-symmetric matrices.
c) Show that $A(t) = A$ is a constant skew symmetric matrix, then the **matrix exponential** $Q(t) = e^{At}$ is an orthogonal matrix. What is this matrix $Q(t)$ in the case $n = 2$?

Problem 4: a) First verify that the helix $r(t) = [\cos(at), \sin(at), bt]$ has constant curvature and torsion. What are the values?
b) Now prove that if a curve has constant curvature and torsion, it must be a helix.

Problem 5: a) There is an interval of c values for which the curve

$$r(t) = [\cos(t), \sin(t) + c\sin(3t)]$$

a simple closed curve. What is the interval?
b) Verify the Hopf Umlaufsatz in the case $c = 1/10$ for which the curve is simple. A computer algebra system gives you

$$f(t) = \kappa(t)|r'(t)| = \frac{6c\cos(2t) - 3c\cos(4t) + 1}{(3c\cos(3t) + \cos(t))^2 + \sin^2(t)} .$$

Verify this using a computer algebra system. Research then which integration method can be used to solve $\int_0^{2\pi} f(t)\, dt = 2\pi$. No need to actually do the integral by hand, just describe what you would have to do.
c) Look at the Hopf Umlaufsatz in the example of an asteroid

$$r(t) = [\cos^3(t), \sin^3(t)] .$$

First verify that $|r'(t)| = |3\cos(t)\sin(t)|$ and $\kappa(t) = -(2/3)/|\sin(2t)|$ then compute the rotation index $\int_0^{2\pi} \kappa(t)|r'(t)|\, dt/(2\pi)$. While your result complies with the Hopf Umlaufsatz, there is something strange going on given how you rotate counterclockwise around the region. Figure it out!

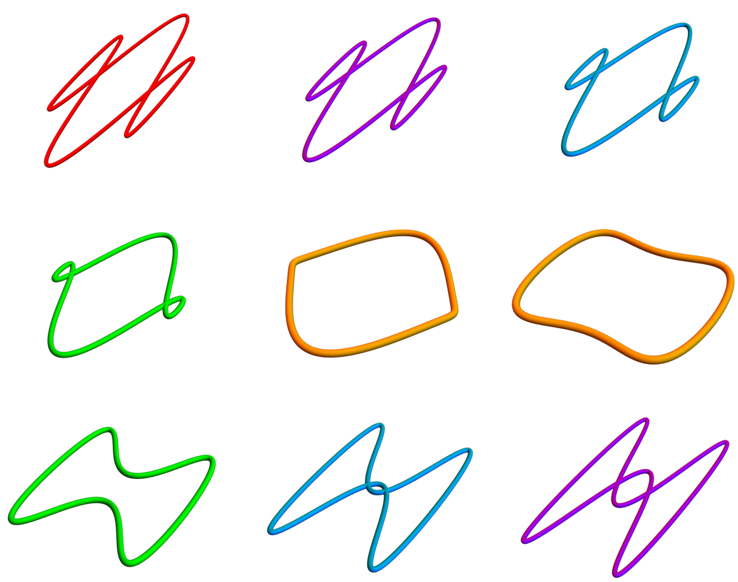

FIGURE 1. A few cycloid curves described in problem 5. There are cases for which the curve is simple and curves for which it is not.

DIFFERENTIAL GEOMETRY

MATH 136

Homework 4

This is the fourth homework. It is due Friday, October 3rd.

Problem 1: There is a **discrete Hopf Umlaufsatz** for polygons.
a) Assume first we have a simple convex polygon with n vertices. Define the curvature at the vertex v_k to be κ_k which is the outer angle $\pi - \alpha_k$ where α_k is the angle you have defined in third grade for polygons. The discrete Hopf Umlaufsatz tells $\sum_{k=1}^{n} \kappa_k = 2\pi$. Prove this.
b) Now formulate the general (not necessarily convex) case. Define suitable curvatures (which are no more positive now in general) such that the result works.

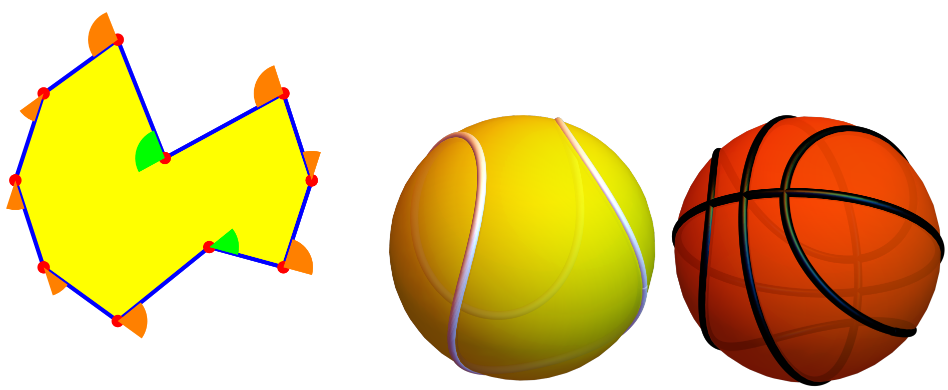

FIGURE 1. In the Umlaufsatz for polygons, curvatures can get positive or negative. The tennis ball and basket ball curve.

Problem 2: Check the four vertex theorem in the example $r(t) = 4[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)]$ The expressions for $\kappa(t)$ and $\kappa'(t)$ are not that bad. Plot the function $\kappa(t)$ and find the critical points, the roots of κ'.

Problem 3: For parameters a, define $c = 2\sqrt{a}$ and the curve $r(t) = [a\cos(t)+\cos(3t), a\sin(t)-\sin(3t), c\sin(2t)]$. For $a = 2$ one has the **tennis ball curve**, for $a = 1/2$ the **base ball curve** and for $a = 1.8$, the **basket ball curve** (Basketballs have two additional grand circles).
a) Verify that these curves are located on a sphere.
b) Look up the tennis-ball theorem, state its content, then write down the main idea on how the theorem is proven.

Problem 4: Let us reformulate the Frenet equations in the plane using complex coordinates. Assume $\kappa(t)$ is an arbitrary continuous function. We have seen that if $K(t) = \begin{bmatrix} 0 & \kappa(t) \\ -\kappa(t) & 0 \end{bmatrix} \in so(2)$, then $Q'(t) = K(t)Q(t)$ defines a path $\begin{bmatrix} \cos(\alpha(t)) & \sin(\alpha(t)) \\ -\sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$ of rotation matrices in $SO(2)$ such that $T(t) = [\cos(t), \sin(t)]$ is the velocity $r'(t)$ and $N(t) = [-\sin(t), \cos(t)]$ is the normal vector. The matrix $Q(t)$ gave us the Frenet frame and so the curve $r(t) = \int_0^t r'(s)\, ds$.
a) Explain why we can identity $K(t)$ with $i\kappa(t)$ and $Q(t)$ with a complex number of length 1. In other words, you explain why $so(2)$ is isomorphic to $u(1)$ and $SO(2)$ is isomorphic to $U(1)$.
b) Now show that we have an explicit formula

$$r(t) = \int_0^t e^{i\int_0^s \kappa(u)\, du}\, ds$$

for the curve.

Problem 5: This is a continuation of Problem 4.4.
a) Lets look at the example $\kappa(t) = 1 + 3\sin(3t)$. We get to the explicit formula assuming $r(0) = 0$ and $r'(0) = 1$

$$r(t) = \int_0^t e^{i\int_0^s (1+3\sin(3u))du}\, ds\ .$$

Draw this curve by integration. Verify that this is a closed bounded curve by checking numerically that $r(2\pi) = 0$.
(If you want to have fun, try something like $\kappa(t) = 1 + 30\sin(5t)$).
b) Now look at $\kappa(t) = 1 + \sin(t)$. Draw this curve by integration. Verify that this produces an unbounded curve because $r(2\pi) \neq 0$.
c) Apropos Hopf theorem. What is the rotation index of the closed curve in a)?
d) Apropos Scheitelsatz: How many vertices does the curve in a) have. How about the curve in b) on the interval from 0 to 2π?

P.S. If you want to have some fun, check out Oliver's recent stability result that $\kappa(t) = k + a\sin(mt)$ for integers k, m and $a \neq 0$ produces a bounded curve if and only if k is not a multiple of m. It appears to be a new result.

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2025

DIFFERENTIAL GEOMETRY

MATH 136

Homework 5

This is the fifth homework. It is due Friday, October 10th.

Problem 1: a) Compute the first and second fundamental form for the surface

$$r(u,v) = \begin{bmatrix} u \\ u^2 - v^2 \\ v \end{bmatrix} .$$

In this problem, we do not want you to use a computer algebra system. You need to write down especially all the matrices dr, dn and A.
b) Use a) to compute the curvature and especially the curvature at $(0,0,0)$.

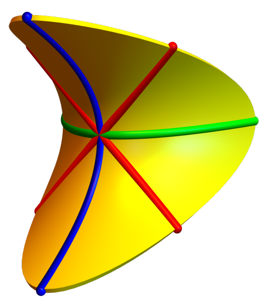

FIGURE 1. A surface with negative curvature.

Problem 2: Compute the first and second and third fundamental form for the torus $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$

$$r(u,v) = \left[\ (5+\cos(v))\cos(u), (5+\cos(v))\sin(u), \sin(v) \ \right] ,$$

where $u, v \in [0, 2\pi)$. We want you to write down explicit expressions for the matrices dr, dn. This problem again can be solved by hand, but you are allowed here to use a computer algebra system to assist you.

Problem 3: a) Compute the matrix A for the shape operator for the same torus.
b) Compute the Gauss curvature K and the mean curvature H.

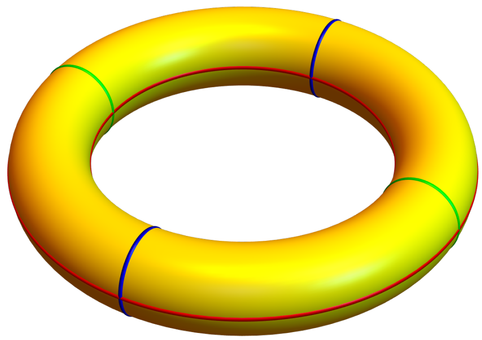

FIGURE 2. The torus for problem 3-4.

Problem 4: a) Again continuing with the torus, integrate $\int_0^{2\pi} \int_0^{2\pi} K dV$. Even if you should use a computer algebra system to integrate, you need to find out what integration method can be used to solve the integral at hand.
b) Your result will be an integer but it will not be compatible with the Gauss Bonnet result that you have seen in unit 9. What was going wrong?

Problem 5: a) Verify that the shape operator matrix A is symmetric in the inner product $\langle v, w \rangle = v^T I w$ so that we can parametrize the surface with a basis such that A is diagonal with respect to the inner product given by I and so that $n_u = -\lambda r_u$ and $n_v = -\mu r_v$. (You assume that I is already diagonal).
b) Use $a)$ to prove the matrix identity $III - 2HII + KI = 0$, where $H = (\lambda + \mu)/2$ and $K = \lambda\mu$ and λ, μ are the eigenvalues of A.

P.S. Here is Mathematica sample code to compute. If you chose to use a computer algebra system, we ask you in problem 1) and 2) to comment what each of the commands you enter does.

```
r={Sin[v] Cos[u],Sin[v] Sin[u],Cos[v]};
ru=D[r,u]; rv=D[r,v];
n=Cross[ru,rv]; n=n/Sqrt[n.n];
nu=D[n,u]; nv=D[n,v];
drt={ru,rv};dr=Transpose[drt];
dnt={nu,nv};dn=Transpose[dnt];
g=drt.dr; h=-dnt.dr; e=dnt.dn;
A=Inverse[g].h; H=Tr[A]/2; K=Det[A];
```

DIFFERENTIAL GEOMETRY

MATH 136

Homework 6

This is the sixth homework. It is due Friday, October 24, 2025

Problem 1: a) Compute the curvatures of the Pentakis Icosahedron (Golden Fullerene) and verify Gauss-Bonnet in this case. b) Divide each face of a cube into 2 triangles, then compute the curvatures and verify Gauss Bonnet.

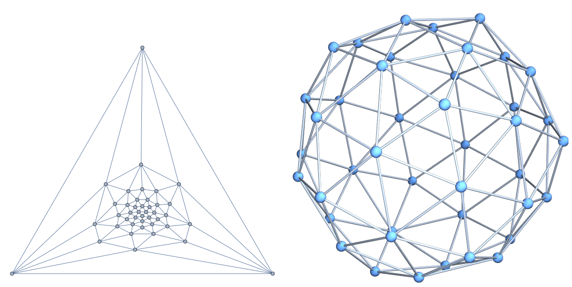

FIGURE 1. The Pentakisicosahedron is a 2-sphere with 42 vertices, 120 edges and 80 triangles. In chemistry it appears as $Au_{30}Si_{12}$.

Problem 2: A graph is called 1-dimensional if it does not contain any triangle. The curvature of a 1-dimensional graph is defined as $1-|S(v)|/2$, where $S(v)$ is the unit sphere. As for 2-manifolds, $|S(v)|$ agrees with the vertex degree $|S_0(v)|$.
a) Along the same line as the Gauss-Bonnet theorem for 2-manifold, prove that $\chi(G) = \sum_{v \in V} K(v)$ for a any 1-dimensional graph.
b) A graph which does not contain any circular subgraph is called a **forest**. A connected component of a forest is called a **tree**. Prove that the Euler characteristic of a forest is equal to the number of trees.
c) Explain why points of positive curvature are called "leaves" and points of negative curvature are "branch points".
d) A "flower" is a circular graph (C_n with $n \geq 4$), where each vertex can be attached a tree. Compute the Euler characteristic of a flower.

Problem 3: A 3-manifold is a finite simple graph for which every unit sphere is a 2-sphere. The classification of 3-manifolds is much more difficult than the classification of 2-manifolds.
a) Look up the 600 cell and the 16 cell and show that they are 3-manifolds.
b) Verify that if H_1, H_2 are 1-spheres, then the join $H_1 \oplus H_2$ obtained by taking the disjoint union of the two graphs and connecting every vertex in H_1 with every vertex in H_2 is a 3-manifold.

Problem 4: A **2-manifold with boundary** is a graph such that every unit sphere is either a circular graph C_n with $n \geq 4$ vertices or a path graph P_n with $n \geq 2$ vertices. The former points are called interior points. The curvature at a general point is $K(v) = 1 - |S_0(v)|/2 + |S_1(v)|/3$, where $S_0(v)$ is the set of vertices of $S(v)$ and $S_1(v)$ is the set of edges in $S(v)$. a) Verify from this definition that for a manifold with boundary, the curvature of a boundary point is $1/2 - |S_1(v)|/6$ and the curvature is $K(v) = 1 - |S_1(v)|/6$ for interior points. b) Check that the graph complement G of C_7 is a 2-manifold without interior. It implements the smallest Möbius strip. Verify that all curvatures of G are zero.

Problem 5: a) The **Barycentric refinement** of a 2-manifold $G = (V, E)$ takes $V' = V \cup E \cup F$ as vertices and takes as E' the set of pairs (x, y) such that $x \subset y$ or $y \subset x$. If $f = [|V|, |E|, |F|]$ is the f-vector of G, then $f(G') = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} f(G)$. Conclude that the Euler characteristic is invariant. (Hint: Show that $[1, -1, 1]$ is an eigenvector of A^T.)
b) The **soft Barycentric refinement** of 2-manifold takes $V' = V \cup F$ as vertices and E' as the set of pairs (x, y) such that $x \subset y$ or $y \subset x$ or $x \cap y$ is in E. Now the f-vectors transform as $f' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} f$. Verify again that the Euler characteristic satisfies $\chi(G) = \chi(G')$.

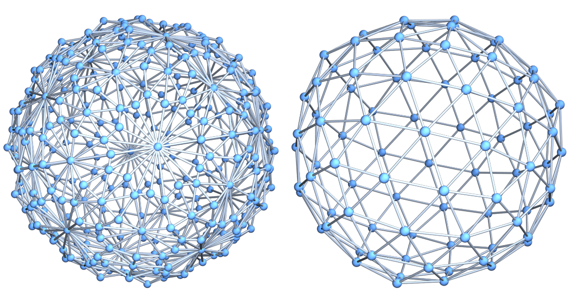

FIGURE 2. The second Barycentric refinement and the second Soft Barycentric refinement of an icosahedron.

DIFFERENTIAL GEOMETRY

MATH 136

Homework 7

This is the seventh homework. It is due Friday, October 31st:

Problem 1: Assume you make a trip and your effort is $F(x, \dot{x}) = x^2\dot{x}^2$ rather than the kinetic energy $\dot{x}^2$ because things are easy initially but get harder later on. What is the best strategy to reach from $x(0) = 0$ to $x(1) = 1$? Go slow first or go fast first? In order to find the best way, solve the Euler-Lagrange problem to minimize the action

$$E(x) = \int_0^1 F(x, \dot{x}) \, dt$$

for $F(x, \dot{x}) = x^2\dot{x}^2$ with $x(0) = 0, x(1) = 1$.

Problem 2: Look at the paraboloid $r(u, v) = [u, v, u^2 + v^2]$.
a) Compute all the Christoffel symbols Γ_{ijk}. These are 8 terms. (Do this by hand). b) Now compute all the Christoffel symbols Γ^k_{ij}. (Also here, do this by hand. There is Mathematica code which can be used for the last problem).

Problem 3: A geodesic $x(t)$ is called **closed**, if there exists L such that $x(L) = x(0)$ and $\dot{x}(L) = \dot{x}(0)$. It is a closed curve on M for which also initial and end velocities agree. Note that the curve x does not have to be simple. Two closed geodesics $x_1(t), x_2(t)$ are called **homotopic**, if one can deform them to each other on the manifold. Formally this means to parametrize both on $[a, b] = [0, 1]$ and then give a continuous $F(t, s)$ of two variables, such that $F(t, 0) = x_1(t)$ and $F(t, 1) = x_2(t)$ and $t \to x_s(t) = F(t, s)$ is on the manifold. We verify here that on any torus, there are infinitely many geodesics that are pairwise not homotopic to each other.
a) Invent and then define a "winding vector" $(n, m) \in \mathbb{Z}^2$.
b) Verify that this number is the same for two closed curves that are homotopic.
c) Draw a torus and a geodesic for winding number $(4, 5)$.
d) Verify that there each homotopy class is not empty by telling how to write one.
e) Argue why there is at least one geodesic in each homotopy class.

Problem 4: We draw some wave fronts $W_r(p)$ on the flat Clifford torus $\mathbb{R}^2/\mathbb{Z}^2$ (Pac-Man square). This manifold can be realized as all point (x, y) in $\mathbb{R}^2$, where points $(x, y), (x + n, x + m)$ identified if $n, m \in \mathbb{Z}$. Work with the point $p = (1/2, 1/2)$.
a) Draw the wave front $W_{1/2}(p)$.
b) Draw the wave front $W_1(p)$.
c) Draw the wave front $W_5(p)$.
You are welcome to become physical part c and use scissor, ruler and compass to do that).

1

Problem 5: The torus

$$r(u, v) = ((a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), b\sin(v)) \ ,$$

has the metric

$$\begin{aligned} g_{11} &= (a + b\cos(v))^2 \\ g_{22} &= b^2 \\ g_{12} &= g_{21} = 0 \end{aligned}$$

Use the following example code (done for the sphere) to compute all the Christoffel symbols Γ^k_{ij}. Make sure to simplify.

```
r={Sin[v] Cos[u],Sin[v] Sin[u],Cos[v]};
ru=D[r,u];          rv=D[r,v];
```

[1]See a project done with Emily Kang during Summer 2024

```
n=Cross[ru,rv];   n=n/Sqrt[n.n];
nu=D[n,u];        nv=D[n,v];
drT={ru,rv};      dr=Transpose[drT];
g=drT.dr;         gi=Inverse[g];
dnT={nu,nv};      dn=Transpose[dnT];
h=-dnT.dr;        e=dnT.dn;
K=Det[h]/Det[g];
X={u,v};  d=2;
c[i_,j_,k_]:=(D[g[[j,k]],X[[i]]]
             +D[g[[k,i]],X[[j]]]
             -D[g[[i,j]],X[[k]]])/2;
Christoffel[i_,j_,k_]:=Sum[gi[[k,l]]*c[i,j,l],{l,d}];
S=Table[Simplify[Christoffel[i,j,k]],{i,d},{j,d},{k,d}];
TableForm[S]
```

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DIFFERENTIAL GEOMETRY

MATH 136

Homework 8

This is the eighth homework. It is due Friday, November 7st:

Problem 13.1: To warm up to Greens theorem. Solve the following problems: a) Use Green to compute the area of the region $|x|^{2/3}/a^2 + |y|^{2/3}/b^2 \leq 1$.
b) Assume F is a smooth 1-form on a 2-torus M, what can you say about $\iint_M dF$?
c) What can you say about $\iint_M K\ dA$ if M is a 2-manifold with a regular parametrization r.

Problem 13.2: a) Green's theorem tells that if $R \subset \mathbb{R}^2$ is a region and $X = [P, Q]$ is a vector field in the plane, then $\iint_R \text{curl} X\ dudv = \int_{\delta R} X(r(t)) \cdot r'(t)\ dt$ where δR is the boundary. Look up and write down a proof of this.
b) Look up the discrete Green theorem and give a proof

1

Problem 13.3: Verify here that Stokes theorem on $S = r(R)$ can be reduced to Green on R:

$$\iint_R \text{curl}(F) r_u \times r_v\ dudv = \iint_R \text{curl}(X) dudv$$

Assume $F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is a vector field in space. Prove the important formula

$$\text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u\ .$$

As we have seen in class, this implies that the 2D field $X = [F.r_u, F.r_v]$ satisfies $\text{curl}(X) = F_u \cdot r_v - F_v \cdot r_u$.

Problem 13.4: We have seen half of the proof that the form X is intrinsic. Verify that also the second component of $X = [z \cdot w_u, z \cdot w_v]$ can be expressed from I alone.

[1] https://people.math.harvard.edu/ knill/teaching/math22b2022/handouts/lecture33.pdf

Problem 13.5: Below you see Gauss's original statement of the theorema Egregium translated into English. Explain what he means with "developing a surface upon any other surface" and why it is not possible for example to find a map of the earth in $\mathbb{R}^3$ which preserves distances.

Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates x, y, z, will correspond a definite point of the latter surface, whose coordinates are x', y', z'. Evidently x', y', z' can also be regarded as functions of the indeterminates p, q, and therefore for the element $\sqrt{(dx'^2 + dy'^2 + dz'^2)}$ we shall have an expression of the form

$$\sqrt{(E' dp^2 + 2F' dp\,.\,dq + G' dq^2)}$$

where E', F', G' also denote functions of p, q. But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically

$$E = E', \quad F = F', \quad G = G'.$$

Thus the formula of the preceding article leads of itself to the remarkable

THEOREM. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

FIGURE 1. Gauss original statement translated into English. Source: Wikipedia

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 136, FALL, 2025

DIFFERENTIAL GEOMETRY

MATH 136

9th Homework

This is the ninth' homework. It is due Friday, November 14th:

Problem 1: a) Prove the **Euler Handshake lemma** $\sum_{i=1}^{V} d_i = 2E$ in graph theory.
b) Assume M is a discrete 2-torus with F faces. How many vertices does it have?
c) In the handout of lecture 17, you see a picture of a discrete torus with $V = 64$ vertices. Determine E and F in that case.

Problem 2: We parametrize a paraboloid M as $r(u,v) = (u, v, u^2 + v^2)$. for $R = \{u^2 + v^2 \leq 1\}$. This is a **2-manifold with boundary**.
a) Compute the curvature K.
b) Compute $|r_u \times r_v| = \sqrt{\det(g)}$.
c) Compute $\iint_R K \, dV$.

Problem 3: We continue with the same paraboloid as before.
a) Compute the curvature of the boundary curve $x(t)$ (parametrized by arc length).
b) Compute the normal curvature $\kappa_n(t) = n(t) \cdot \ddot{x}(t)$ as well as the geodesic curvature $\kappa_g(t) = (n \times \dot{x}) \cdot \ddot{x}$.
c) Verify the local Gauss-Bonnet result. That is show that

$$\iint_R K \, dV + \int_0^L \kappa_g(t) \, dt = 2\pi \; .$$

Problem 4: Use a computer algebra system to verify that

$$\iint_M K \, dV = 4\pi$$

if $M = \{x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ for $a = 2, b = 3, c = 5$. What is the maximal and what is the minimal curvature of this ellipsoid?

Problem 5: The "angular defect" $K(p)$ at a vertex of a convex polyhedron M is the angle needed to add to complete the angle to 2π. For a cube for example, it $2\pi - 3\pi/2 = \pi/2$ at every corner.
a) Descartes theorem states that the total defect of a convex polyhedron is 4π so that the angular defect is a curvature. This is a polyhedral Gauss-Bonnet theorem. Verify this for an icosahedron to see what is going on.
b) Modify the local to global proof of Gauss-Bonnet to see that for a general polyhedron, the value $K(p) = 2\pi - \sum_i \alpha_i$ gives a curvature that adds up to 2π times the Euler characteristic of the surface. Here, α_i are the angle interior angles and the result you want to show is $\sum_p K(p) = 2\pi\chi(M)$. It is a version of Gauss-Bonnet.
c) Illustrate your theorem with the Escher stair polyhedron built in **mine craft** or **Lego**. Compute all the angular defects and add them up. The total curvature should be the Euler characteristic of the stair.

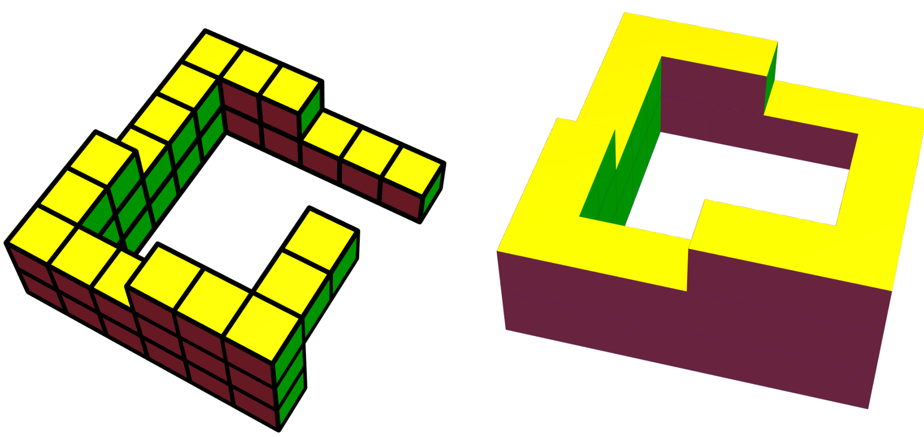


Figure 1. The Escher Stairs built in mine craft. If you look at it from the right angle and do glue the bricks nicely, you see an impossible stair, which always goes down or up depending on whether you are a "wineglass half empty" or "wine glass half full" type of person.

10th Homework

This is the 10th homework. It is due Friday, November 21th:

Problem 1: Tensor field or not?

a) The calculus gradient field $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$.
b) The Jacobian of a function $df = [f_x, f_y, f_z]$
c) The second fundamental form II
d) The shape operator A like $(Av)^k = \sum_j A^k_j v^j$.
e) The inverse I^{-1} of the first fundamental form g^{ij}.
f) The second derivative $r_{u^i u^j}$.
g) The Christoffel symbols $\Gamma_{ijk} = r_{u^i u^j} r_{u^k}$.

Problem 2: Prove that $SU(2)$ is a manifold by explicitly giving the charts.

Problem 3: The following code allows you to experiment with level sets in discrete manifolds. The host manifold is a discrete 4-manifold, the join of a 2-sphere and a 1-sphere. Running the code builds a random function from the vertex list to $\{1, 2, 3\}$. It defines a co-dimension 2 manifold.
a) Run the code as it is, report the numbers V,E,F as well as the Euler characteristic of M_f.
b) Change the code and see what happens if the function takes 4 values rather than 3.
c) Build a 5 manifold as the join of two 2-manifolds and build a 3-manifold by taking a function taking 3 random values.
d) Report the curvature values of your 3-manifold.
e) Report the curvature vales of a 4 manifold by taking a function taking 2 values on the 5 manifold in c).

```
Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]]; w[x_]:=-(-1)^k;
R[G_,k_]:=Module[{},R[x_]:=x->RandomChoice[Range[k]]; Map[R,Union[Flatten[G]]]];
F[G_]:=Delete[BinCounts[Map[Length,G]],1]; Euler[G_]:=F[G].Table[w[k],{k,Length[F[G]]}];
Surface[G_,g_]:=Select[G,SubsetQ[#/.g,Union[Flatten[G] /. g]] &];
S[s_,v_]:=VertexDelete[NeighborhoodGraph[s,v],v];     Sf[s_,v_]:=F[Whitney[S[s,v]]];
```

```
Curvature[s_,v_]:=Module[{f=Sf[s,v]},1+f.Table[(-1)^k/(k+1),{k,Length[f]}]];
Curvatures[s_]:=Module[{V=VertexList[s]},Table[Curvature[s,V[[k]]],{k,Length[V]}]];
J[G_,H_]:=Union[G,H+Max[G]+1,Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1}],0]]]];
ToGraph[G_]:=UndirectedGraph[n=Length[G];Graph[Range[n],
  Select[Flatten[Table[k->l,{k,n},{l,k+1,n}],1],(SubsetQ[G[[#[[2]]]],G[[#[[1]]]]])&]]];
Barycentric[s_]:=ToGraph[Whitney[s]];

G=J[Whitney[Barycentric[CompleteGraph[{2,2,2}]]],Whitney[CycleGraph[7]]];   (* J=Join *)
g=R[G,3]; H=Surface[G,g];     (* A codimension 2 manifold in the 4-sphere G=Oct * C_7 *)
Print["EulerChi=",Euler[H]]; Print["Fvector:",F[H]]; s=ToGraph[H]; GraphPlot3D[s]
Print["Gauss-Bonnet Check:"]; Print[Total[Curvatures[s]]==Euler[H]];
Print["Curvature Values:"];  Print[Union[Curvatures[s]]];
```

Problem 4: The Reissner-Nordstroem metric

$$g = \begin{bmatrix} -\frac{e^2}{r^2}+\frac{2M}{r}-1 & 0 & 0 & 0 \\ 0 & \frac{1}{\frac{e^2}{r^2}-\frac{2M}{r}+1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\phi) \end{bmatrix}$$

is a static solution of the Einstein-Maxwell equations

$$R - \frac{1}{2}Sg = T \, .$$

But it is not a vacuum. It describes the field of a charged, non-rotating body of mass M and charge e. An example is a charged black hole. What is the entry T_{44} for $e = 1, r = 1, \phi = \pi/3$?

Problem 5: a) Verify that the metric on $SU(2) = S^3$ given by the parametrization

$$r = [\cos(u)\cos(w), \sin(u)\cos(w), \cos(v)\sin(w), \sin(v)\sin(w)]$$

satisfies equation $R-2g = 0$. What is S? b) Finally check that the pseudo sphere given by the parametrization

$$r = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v) + \log\left(\tan(\frac{v}{2}))\right]$$

satisfies the vacuum Einstein equations. What is the curvature K?

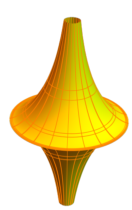


Figure 1. The Pseudo sphere

DIFFERENTIAL GEOMETRY

MATH 136

Midterm Part I (in class October 16, 2025)

Sign your name. You have 30 minutes. Each question is one point. No half points.

1) If $r(u,v)$ parametrizes a surface, then $|r_u \times r_v|^2$ is [   ] in terms of g.
2) If $r(t)$ parametrizes a curve, then $|r'(t)| = \sqrt{\det(g)}$ is called the [   ]
3) The integral $\int_a^b \sqrt{\det(g)}\ dt$ for a curve $r(t)$ was called [   ] integral.
4) The curvature of a circle of radius $1/7$ is [   ]
5) The result that the shortest connection between two points in the plane is a line is the [   ] theorem.
6) The curve with name [   ] has constant curvature and non-zero constant torsion.
7) The curve $[0, \cos(3t), \sin(3t), 0]$ has curvature [   ] and torsion [   ]
8) The Frenet frame in 3 dimensions is also called the [   ] frame.
9) For a curve in $\mathbb{R}^n$, the curvature κ_{n-1} is called [   ]
10) The curve with name [   ] has infinitely many vertices.
11) The evolute of a curve is $r(t)+N(t)/\kappa(t)$, where $\kappa(t)$ is the [   ]

12) Tennis ball: a curve dividing the erea of S^2 equally has [ ] vertices at least.

13) The Euler characteristic of a projective plane is [ ]

14) The Euler characteristic of a Klein bottle is [ ]

15) The Euler characteristic of a discrete 2-manifold is defined as [ ]

16) The constant curvature discrete 2-sphere with 12 vertices is also called a [ ]

17) We needed the [ ] process to produce general Frenet frames.

18) The matrix $I^{-1}II$ is always a symmetric matrix. True or False? [ ]

19) The mean curvature of the Clifford Klein bottle is [ ]

20) The Gauss curvature of the round unit sphere is [ ]

21) Which blind mathematician first looked at discrete curvature? [ ]

22) If $e_k(t) \cdot e_k(t) = 1$ for all t then $e_k(t)'$ is [ ] to $e_k(t)$.

23) $so(5)$ consists of orthogonal matrices. True or False? [ ]

24) $SU(3)$ is associated to the strong force. It is a special [ ] group.

25) It is possible that a matrix entry of $SO(5)$ is 5. True or False? [ ]

26) It is possible that a matrix entry of $so(5)$ is 5. True or False? [ ]

27) The fundamental theorem of linear algebra tells $\ker(A)^{\perp} =$ [ ]

28) If r parametrizes an ellipsoid then $\iint_R |n_u \times n_v| \, dudv =$ [ ]

29) True or false? If you know I and II, then you know III. [ ]

30) The curvature of a discrete 2-manifold is given as [ ]

DIFFERENTIAL GEOMETRY

MATH 136

Midterm part II. Due Friday 10/17/2025, midnight

Handwritten. No internet, no correspondence, no computer algebra system, closed book, no internet no AI. One single page of handwritten notes is allowed. Put your name on each page of your paper and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."** This part has 70 points. Part I done in class could give 30 points. In total, the full midterm counts 100 points.

Problem A (20 points): The graph $G = (V, E)$ in the picture is an example of a **positive curvature 2-manifold**.
a) (4 points) Verify that this is a 2-manifold. Describe how to do this.
b) (4 points) What does positive curvature mean for 2-manifolds?
c) (4 points) Compute all curvatures of G.
d) (4 points) Compute the sum of all curvatures of G.
e) (4 points) Count V, E and F and the Euler characteristic $\chi(G)$ of G.

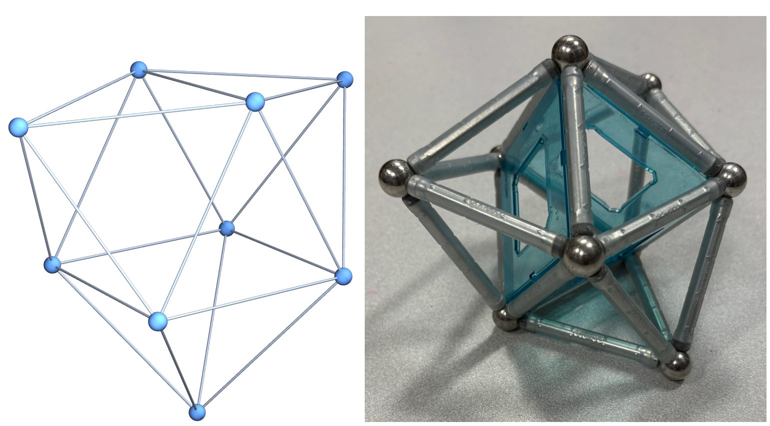


FIGURE 1. There are exactly 6 positive curvature manifolds of dimension 2, the octahedron and icosahedron are two examples. The species given here can be obtained by gluing three pyramids to a triangular cylinder. The picture to the right is a build-up with Oliver's geomag sticks.

Problem B (20 points): The **Cissoid of Diocles** C is the contour curve

$$f(x,y) = x^3 + xy^2 - y^2 = 0$$

It is an example of a **cubic algebraic curve**.

a) (2 points) Verify that C has the parametrization

$$r(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{t^2}{1+t^2} \\ \frac{t^3}{1+t^2} \end{bmatrix} .$$

b) (2 points) Is C a regular curve with the parametrization in a)?
c) (2 points) Is C a Frenet curve with the parametrization in a)?
d) (2 points) It C a closed curve ?
e) (2 points) Is C a simple curve?
f) (2 points) Is the curvature $\kappa(t)$ of C defined everywhere?
g) (2 points) Compute $df(x,y)$, then $df(r(t))$.
h) (2 points) What is $\frac{d}{dt}f(r(t))$?
i) (2 points) Use the chain rule to get $df(r(t))r'(t)$.
j) (2 points) Write down the arc length integral for $t \in [0,1]$. No evaluation is needed!

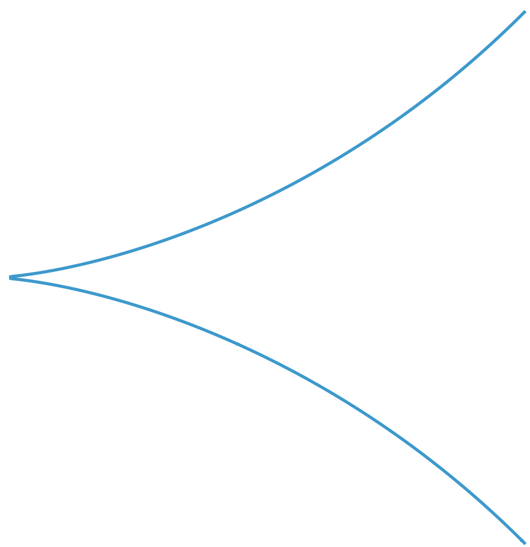


Figure 2. The Cissoid of Diocles.

Problem C (10 points): The planar curve

$$r(t) = \begin{bmatrix} \sin(3t) \\ \sin(2t) \end{bmatrix}$$

with $t \in [0, 2\pi)$ is an example of a **Lissajoux figure**.
a) (2 points) Compute the signed curvature function $\kappa(t)$.
b) (2 points) How is the rotation index ρ defined in general? Do not compute the integral but find the index from the picture.
c) (2 points) State the Hopf Umlaufsatz. Does it apply for this curve?
d) (2 points) You are told that the function κ has 6 local maxima. How many vertices does this curve have?
e) (2 points) If a curve has points, where $|r'(t)| = 0$, can you conclude that the curve is **not** arc-length parametrizable?

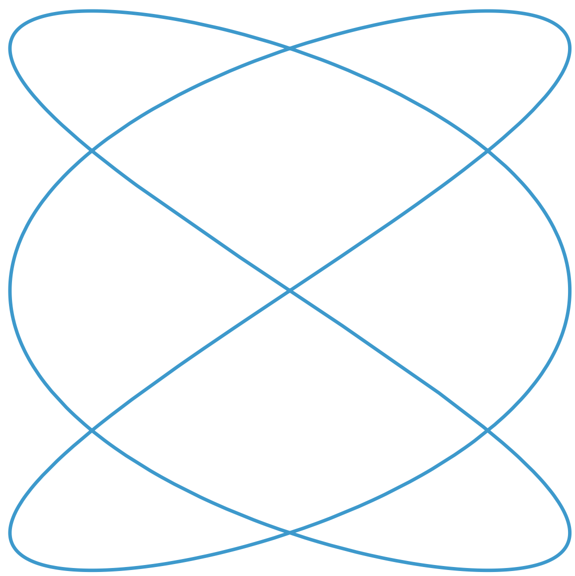


FIGURE 3. Lissajoux figure

Problem D (20 points): Define the surface by the parametrization

$$r(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix} = \begin{bmatrix} u \\ v \\ uv \end{bmatrix} .$$

a) (4 points) Write down the first fundamental form $g = I$.
b) (4 points) Write down the second fundamental form $h = II$.
c) (4 points) Write down the third fundamental form $e = III$.
d) (2 points) How is the shape operator A defined from I, II (in general)?
e) (2 points) How are K and H obtained, if you know I, II, III (in general)?
f) (4 points) We know $e - 2Hh + Kg = 0$ (as you have proven it in a homework). Check this matrix identity in the present case.
Note: In d),e) just state the general definitions without actually computing it for $z = xy$. As for f), an oracle tells you that with $B = 1/\sqrt{1+u^2+v^2}$ curvatures are $\boxed{K = -B^4}$ and $\boxed{H = uvB^3}$. There is no need to verify these two boxed formulas for K, H. The quantity B can be useful in parts a),b),c) using for example $B_u = -uB^3, B_v = -vB^3$.

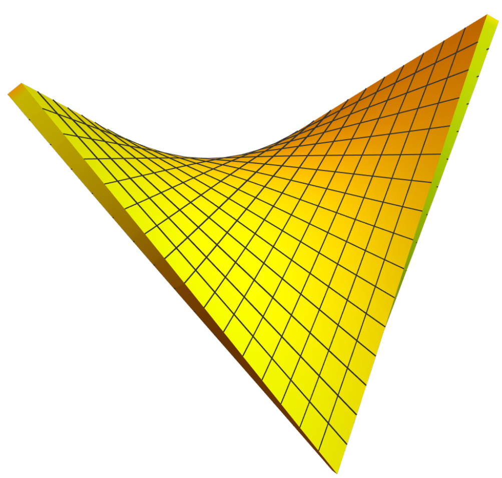


Figure 4. The hyperbolic paraboloid.

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2025

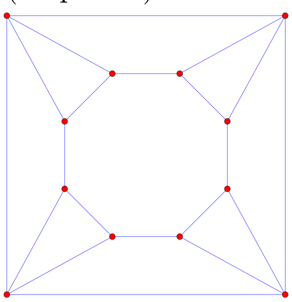
DIFFERENTIAL GEOMETRY

MATH 136

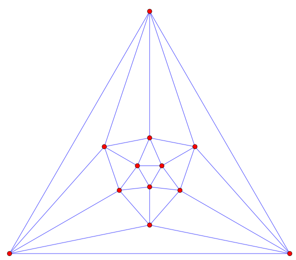
Final Part I (December 4, 2025)

Sign your name. You have 60 minutes. Closed book. No justifications are needed.

1) (10 points) Fill out the following table about graphs:

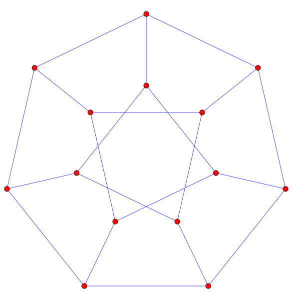


1

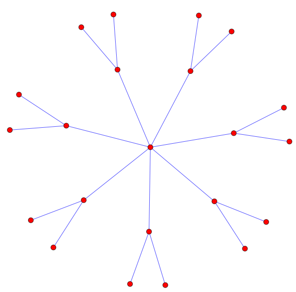


2

Graph	Contractible?	Manifold?
1		
2		



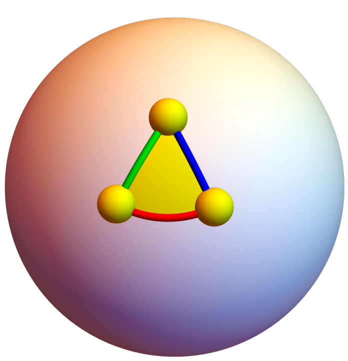
3



4

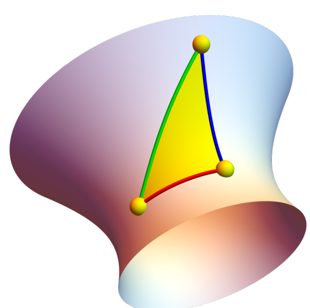
Graph	Contractible?	Manifold?
3		
4		

2) (10 points) You see triangles in a sphere of curvature 1 or part of a hyperbolic plane with curvature -1.The edges are geodesics and you are given the angles α, β, γ in radians. Find the area:



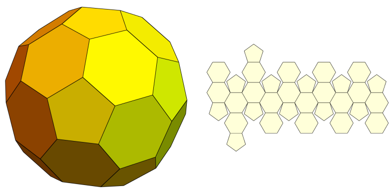
A

B

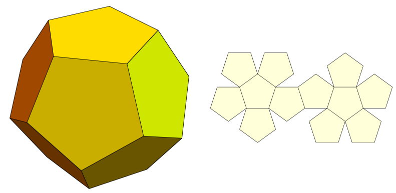


Triangle	α	β	γ	Area
A (in sphere)	$\pi/3$	$7\pi/18$	$7\pi/18$	
B (in pseudo sphere)	$\pi/6$	$\pi/2$	$5\pi/18$	

3) (10 points) You see polyhedra and their nets (which could be folded to build the polyhedron). In each case, check Descartes theorem, which is a Gauss-Bonnet theorem, where curvatures are located on vertices:



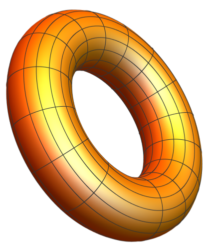
A



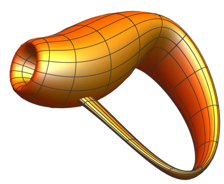
B

Polyhedron	excess $\kappa(v)$	$\|V\|$	$\|E\|$	$\|F\|$	$\chi(M)$	$\sum_{v \in V} \kappa(v)$
A						
B						

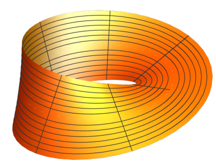
4) (10 points) Fill out the following table



1



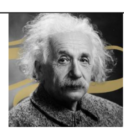
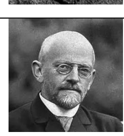
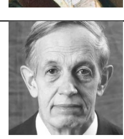
2

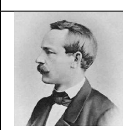
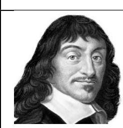
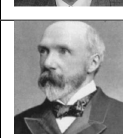


3

Surface	Orientable?	Manifold with boundary?	Manifold without boundary?	Euler characteristic
1				
2				
3				

5) (10 points) Name the mathematicians or fill in their achievements

Name	Picture	Known for
		Relativity
Hilbert		
		Discrete relativity
		Theorema Egregium
Nash		

Name	Picture	Known for
		Symbols Γ_{ijk}
		Polyhedral Gauss-Bonnet
		$\frac{d}{dt} F_{\dot{x}} = F_x$
Hopf		
		Fund. theorem of curves

6) (10 points) Name the objects

	$\Gamma_{ijk} = r_{u^iu^j} \cdot r_{u^k}$
	$A = I^{-1}II$
	$K = \det(A)$
Mean curvature	$H =$ ....
	$\kappa = T' \cdot N$
	$\tau = N' \cdot B$
	$\kappa_g(t) = (n \times \dot{x}) \cdot \ddot{x}$
Normal curvature	$\kappa_n =$ ....
Euler characteristic	$\chi(G) =$ ....
	$R^s_{ikj} = \frac{\partial}{\partial u^k}\Gamma^s_{ij} - \frac{\partial}{\partial u^j}\Gamma^s_{ik} + \sum_r \Gamma^r_{ij}\Gamma^s_{rk} - \sum_r \Gamma^r_{ik}\Gamma^s_{rj}$

7) (10 points) Match the following important identities or write in the formulas. If something needs to be filled to the left, use from the following: Cauchy-Binet identity, Gauss Bonnet for surfaces, Frenet equations, Stokes theorem, Greens theorem, Einstein equations, Identity for Fundamental forms, Hopf Umlaufsatz, Gauss-Bonnet for geodesic triangle, Karate Kick identity, Geodesic equations, Stokes theorem, Curvature identity.

	$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$
Einstein equations	
	$III - 2HII + KI = 0$
	$\int_C \kappa(t)\, dr(t) = 2\pi$
	$\int\int_U K dV + \sum_j \kappa_j = 2\pi$
Gauss-Bonnet for surfaces	
	$\ddot{x}^k + \sum_{i,j=1}^m \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$
Curvature relations $\kappa_g, \kappa_n, \kappa$	
	$\|r_u \times r_v\|^2 = \|r_u\|^2\|r_v\|^2 - \|r_u \cdot r_v\|^2 = \det(g)$
Karate-Kick identity	

8) (30 points)

1) The third fundamental form is a [ ] tensor.

2) The Schwarzschild metric describes the gravity of a [ ] hole.

3) A 90-90-X geodesic triangle on the unit sphere $x^2 + y^2 + z^2 = 1$ has area $\pi/2$. What is the angle X? [ ]

4) The Christoffel symbol Γ^k_{ij} is a tensor. True or false?

5) If F is a vector field which has $\text{curl}(F) = 0$, and G is the unit disk $x^2 + y^2 \leq 1, z = 0$ with boundary C then the line integral is $\int_C F\, dr =$ [ ] .

6) The shape operator matrix A is is a [ ] tensor.

7) A discrete manifold has the property that every unit sphere is a [ ] .

8) The 0-dimensional sphere has [ ] vertices and [ ] edges.

9) A connected 2-manifold with Euler characteristic 1 must be a [ ] .

10) The principle that action and length functional have the same critical points is called the [ ] principle.

11) It is possible that the Klein bottle has constant zero curvature. True or false?

12) The Möbius strip has Euler characteristic [ ] .

13) The projective plane is orientable. True or False? [ ] .

14) True or false: there are only finitely many closed periodic geodesics for a torus [ ] .

15) The Euler characteristic of a 2-manifold in terms of V, E, F is given by the formula [ ] .

16) If G is a discrete m manifold and $f : V(G) \to \{0, \dots, k\}$ is a map, then M_f is a [ ] manifold or [ ] .

17) True or false: the Ricci tensor is symmetric: [ ] .

18) A key in the proof of the local Gauss-Bonnet theorem is [ ] 's theorem.

19) If M is $x^2/9 + y^2/4 + z^2/16 = 1$, then $\iint_M K \; dV =$ [ ] .

20) A 3-dimensional manifold has Euler characteristic [ ] .

21) The Euler's gem formula is [ ] .

22) The geodesic curvature of a geodesic is [ ] .

23) The normal curvature of a grand circle in a sphere of radius 5 is [ ] .

24) The geodesic curvature of a grand circle in a sphere of radius 5 is [ ] .

25) In a triangle on a sphere the sum of the angles is [ ] than π.

26) In a triangle on a flat torus, the sum of the angles is equal to [ ] .

27) Lambert's theorem deals with the sum of the angles of a triangle on a [ ] .

28) The sum of the angular defects $K(p)$ of a convex polyhedron is equal to [ ] .

29) True or False? Geodesics extremize the energy functional [ ] .

30) The Einstein equations can be derived by extremizing the [ ] functional.

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

Score:

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2025

DIFFERENTIAL GEOMETRY

MATH 136

Final Exam Paper

Assignment: due December 14, 2025

Write an expository paper about a topic in differential geometry of your choice.

Rules

- The paper must be written by yourself. No human or AI assistance is allowed in writing.
- Acknowledge any references used: books, papers, web, discussions, the use of computer algebra.
- Aim for 4 or more pages with references. There can be illustrations. Also for illustrations, either do them yourself or give credit.

Grading criteria

(1) Mathematical correctness
(2) Clarity, readability and elegance
(3) References and sources
(4) Adaptation to the course, notation
(5) Originality or depth or surprise

The paper is 60 points, the in class exam is 40 points. In total: 100 points.

Topic suggestions

Pick or modify one of the following topics. If you prefer an other topic, check with Oliver. The topic needs to be related to this course - of course!

(1) **"Positive Curvature Manifolds"**. Examples of even dimensional manifolds for which all sectional curvatures are positive.
(2) **"Clifton-Pohl torus"** A pseudo Riemannian manifold on which the geodesic flow is not complete.
(3) **"Caustics in Riemannian geometry?"**
(4) **"What is a Jacobi field?"**
(5) **"Differential geometry of evolutes."**
(6) **"Principal curvatures and umbillic points"**
(7) **"The symmetries of the Riemann tensor"** Like Bianchi identities.
(8) **"Minimal surfaces"**
(9) **"Chaotic geodesic flows"** (like on surfaces of constant negative curvature).

(10) **"My favorite Riemanninan manifold"**. i.e. projective plane, the Klein bottle, the hyperbolic plane, the projective 3-space $SO(3)$ (a 3 manifold) or $SU(3)$ (a 8 dimensional manifold).
(11) **"The Differentiable Jordan Curve theorem"**. Easier than the topological Jordan Curve theorem. Prove it. See Do carmo page 400.
(12) **"The Hopf-Rynov Theorem"**: Two points on a complete surface can be joined by a minimal geodesic. See Do Carmo page 338.
(13) **"The formulas of Codazzi-Meinardi"**. See section 4.C in Kuehnel.
(14) **"The world of discrete manifolds"**. Explore small examples of discrete manifolds, I.e. Klein bottles, projective planes, or higher genus Klein bottles.
(15) **"The Brachistochrone problem"** An example of a variational problem.
(16) **"The Hilbert Action"** Outline the proof that critical points of the Hilbert action are Einstein manifolds.
(17) **"What are Minimal Surfaces?"**. See chapter 3D in Kuehnel.
(18) **"Complex manifolds"** or **Kähler manifolds**.
(19) **"The complex projective plane $\mathbb{CP}^2$.**
(20) **"What is a Kalabi-Yau manifold?**
(21) **"How can Morse theory help to understand Riemannian geometry?"**
(22) **"What are Ruled surfaces?"** Section 3C in Kuehnel.
(23) **"The rigidity of the sphere"** a compact connected regular surface of constant curvature must be a sphere. There is a section in Do Carmo.
(24) **"Clairaut's relation for geodesics"** A conserved quantity for geodesics in surfaces of revolution".
(25) **"The isoperimetric inequality"**. There is an exposition in Do Carmo.
(26) **"Geodesic rigidity of Gauss"**: if p, q are two points on a manifold and if $\gamma_s(t)$ is a smooth family of geodesics connecting p, q, then all these geodesics must have the same length.
(27) **Gauss Lemma in Riemannian geometry**: why are geodesics are perpendicular to wave fronts.
(28) **"Exotic spheres?"** What are spheres with non-standard differential structure.
(29) **"What is Willmore energy?"** The boy surface is known as "Oberwohlfach surface". A theorem of Bryant-Kusner tells that the Boy surface minimizes the Willmore energy.
(30) **"Geometry of hyperbolic space"**. i.e. geodesics, in a 2 dimensional manifold of constant curvature -1. Gauss-Bonnet for polygons.
(31) **"Fenchel's theorem"** or the Fary-Milnor theorem about total curvature of curves. There is a section in Kuehnel.
(32) **"The fundamental theorem of Riemannian geometry"** a connection satisfying three axioms must be the Christoffel connection.
(33) **"Classifying 2-manifolds"**. Describe the classification of 2-manifold as a connected sum of tori or projective planes.
(34) **"Spherical space forms"**. See for example Theorem 7.30 in Kuehnel.
(35) **"What are lense spaces"**? What are spherical 3-manifolds?
(36) **"Synge's theorem"**: an even dimensional orientable positive curvature manifold is simply connected.
(37) **"The Poinare conjecture"**. How was the Poincare conjecture proven. Especially define the Ricci flow using notation we have used.

(38) **"Surgery of 3-manifolds"**. Explain how one can use knots to build 3-manifolds. Especially Dehn surgery and the Lickorish-Wallace theorem.
(39) **"The Kerr Metric"**. A solution of the Einstein equations. The Kerr metric" models a rotating non-charged black hole. The Kerr-Newman metric a rotating charged black hole.
(40) **"Bertrand-Puiseux formulas"**. Prove $|S_r(p)| = 2\pi r - \pi K \epsilon^3/3 + ...$
(41) **"Chaotic Billiards"**. The geodesic flow in manifolds with boundary. is called a billiard. This involves curvature.
(42) **"Riemannian metrics on Lie groups"**. Like on $SO(3)$ or $SU(2)$. How does one get a Riemannian metric on such spaces?
(43) **"The Willmore conjecture"**. What is the Willmore energy? Explain the conjecture and how it was solved.
(44) **"The tensor algebra"**. Especially the exterior algebra of skew symmetric tensors.
(45) **"Counting Periodic geodesics."**
(46) **"The isometric embedding problem R^3."**
(47) **Open questions in Riemannian geometry**
(48) **Look at the inverse Frenet problem in the case of piecewise linear periodic curvatures**.
(49) **"Curvature of convex polyhedra in four dimensions."**
(50) **"Curvature in a conformal metric.**

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2025

DIFFERENTIAL GEOMETRY

MATH 136

Unit 0: Syllabus

Administrative

```
Instructor: Oliver Knill, Office SC 432
Course assistants: Zak Adams (zakadams),
Jayanti Leslie-Iyer  (jleslieiyer), Eric Sui (ericsui)
Luke Zhu  (lukezhu) and Hugo Nunez (hugonunez).
Class hours:    TuTh 12-1
Office hours Oliver:  MWF, 11-12
Weakly homework, individually hand written. Submitted on Canvas
Midterm inclass quiz, and midterm paper on a topic of the course.
Final inclass quiz and final project paper.
Grades: HW 50 percent, midterm 20, final 30
Reading suggestions: Kuehnel and DoCarmo
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Synopsis

1.1. This is an introduction to Riemannian geometry and in particular to Riemannian geometry of curves and surfaces. We also develop less technical discrete differential geometry. Low dimensional Riemannian geometry is an active area of mathematics with many open problems and applications, especially in computer science and computer graphics. It is not only the language of gravity, it is an inspiration for art and architecture The subject features many unsolved problems.

Policies

1.2. The landscape of teaching changes rapidly. Generative AI's are already strong. Autonomous thinking, problem solving skills and creativity become more important. Those who rely on AI for thinking are the first to be replaced: using it only requires minimal abilities:a monkey can submit a PDF and submit the result.

```
No AI for homework or exams
Computer algebra systems can be used, if acknowledged
Handwritten work for home and inclass work
Final paper should be typed
Class attendance is necessary
Collaboration for Psets is ok
But each submit individual work
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Lectures

1.3. The goal is to reach 4 mountain peaks: Frenet-Serret theorem, Gauss-Bonnet theorem, Theorema egregium, Riemannian manifold setup, Einstein equations.

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W1    1 Tuesday   September 2th:   What is differential geometry? Notations.
      2 Thursday  September 4:     Parametrized and implicit surfaces.

W2    3 Tuesday   September 9:     Jacobean map, Surface area
      4 Thursday  September 11:    Parametrization of Curves, Arc length, Curvature

W3    5 Tuesday   September 16:    The Frenet theorem in two and 3 dimensions
      6 Thursday  September 18:    The Frenet theorem in arbitrary dimensions

W4    7 Tuesday   September 23:    A global result: The Hopf Umlaufsatz
      8 Thursday  September 25:    A global result: The four vertex theorem

W5    9 Tuesday   September 30:    The fundamental forms I,II,III
     10 Thursday  October 2:       The Gauss Map A and Curvature K

W6   11 Tuesday  October 7:       Euler characteristics
     12 Thursday October 9:       Discrete Differential geometry

W7   13 Tuesday    October 14:    Review for midterm
     14 Thursday October 16:      Inclass part Midterm

W8   15 Tuesday    October 21:    Calculus of variations and Geodesics
     16 Thursday October 23:      Exponential map and geodesic coordinates

W9   17 Tuesday    October 28:    Differential forms and Green's Theorem
     18 Wedneday  October 30:     The Theorema Egregium

W10  19 Tuesday    November 4:    Local Gauss Bonnet theorem
     20 Thursday November 6:      Global Gauss Bonnet theorem

W11  21 Tuesday    November 11:   Riemannian manifolds
     22 Thursday November 13:     Discrete manifolds

W12  23 Tuesday    November 18:   The curvature and Ricci tensors
     24 Thursday Novemer  20:     General relativity

W13  25 Tuesday    November 25:
        November 27 until December 1th  Thanksgiving

W14  26 Tuesday   December 2:    Final review
     27 Thursday December 4:     In class part final
```