Unit 20: Relativity

20.1. The **principle of general covariance** states that the pseudo Riemannian manifold (M, g) alone defines gravitational laws. No preferred basis nor background "aether" concepts are allowed. Physical laws are invariant under smooth coordinate changes. Objects of interests are tensorial. Einsteins theory of gravity links **space-time** with **matter**. Matter determines space time (M, g) in that g by minimizing the Hilbert action. Space time determines the paths of particles by minimizing kinetic action. [1]

The Einstein Equations	The Geodesic Equations
$\boxed{R - \frac{1}{2}Sg + \Lambda g = \kappa T}$	$\boxed{\ddot{x} = -\sum_{i,j} \Gamma_{ij}^k \dot{x}^i \dot{x}^j}$
Matter tells Space-time how to curve	Space-time tells Matter how to move

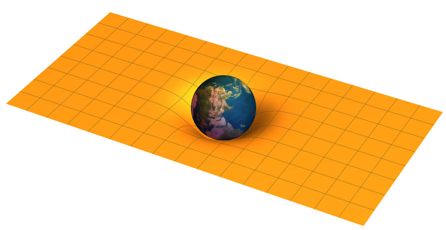

FIGURE 1. Mass deforms space time

20.2. The theory has been experimentally confirmed in various instances like 1) The **perihelion advance of planets like Mercury**, 2) the **gravitational lensing of light** around stars or galaxies, 3) the **time delay in radar probing of planets**, 4) the **spectral shift of light emanating from massive objects**, 5) the **precession of a gyroscope**,freely orbiting the earth, 6) the **detection of gravitational waves** from Black-hole mergers, 7) the **pictures of black holes** like Sagittarius A* and Messier 87* by the **event horizon telescope**.

[1]Even when restricting to gravity and not taking into account quantum mechanics, this is unsatisfactory. In a 2-body problem of two massive particles like a black-hole binary the masses should contribute to the stress-energy tensor T. In black-hole situations, the paths need to be removed from the manifold. About the math: Yvonne Choquet-Bruhat (1923-) dealt with the Cauchy problem in 1969. Demetrios Christodoulou and Sergiu Klainermann proved in 1994 nonlinear gravitational stability. Numerical schemes for black hole binaries in vaccum using **post-Newtonian expansions** exist since 2005. It is a total mess, both from an applied math (engineering) as pure math perspective.

20.3. In **Special relativity** the metric is no more required to be positive definite. The **Galilei group** generated by rotations and translations is replaced by the Poincaré group generated by **Lorentz transformations** and translations. Any rotation matrix in a (e_0, e_i)-plane with $i = 1, 2, 3$ is replaced by a hyperbolic rotation, where cos is replaced by cosh and sin by sinh leading to **Lorentz boosts**. For $M = \mathbb{R}^4$

$$g = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with speed of light c is called the **flat Lorentz metric**. [2] The null vectors are the vectors in the **light cone** $\{||u|| = 0\}$. The vector $(1, 0, 0, 0)$ is time like and $(0, 1, 0, 0)$ is an example of a space like vector. Particles with space like velocity vectors have velocity smaller than the speed of light, particles with time like velocity vectors are called **tachions**. They have velocity larger than the speed of light.

20.4. The most important example in general relativity is the **Schwarzschild metric**. All the known confirmations of general relativity are just based on this model. On the manifold $M = \mathbb{R}^4 \setminus \{r \leq 2m\}$ we can use the spherical coordinates $t = x^0, r = x^1, \theta = x^2, \phi = x^3$. For $r > 2m$, the **Schwarzschild metric** is given by

$$g = \begin{bmatrix} -c^2(1 - \frac{2m}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2m}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix}.$$

It is a model for the space-time in presence of a single massive object of mass M. [3] The constant m is GM/c^2, where G is the gravitational constant and M the mass. The number $2m$ is called the **Schwarzschild radius**. You verify that (M, g) satisfies the Einstein equations in the homework.

20.5. The following establishes why the theory "makes sense". It is a refinement of the Newton theory of gravitation.

Theorem 1. *Relativity for slow particles in a weak field becomes Newtonian mechanics.*

Proof. If $g = \overline{g} + h$, where $\overline{g}$ is the flat metric and h is small then $\dot{x}^i$ for $i = 1, 2, 3$ can be neglected with respect to $\dot{x}^0$. The geodesic equations $\ddot{x}^k = -\sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j$ is approximated by $-\Gamma^k_{00} = \partial_{u^k} \overline{g}_{00} - \partial_{u^0} h_{0k}$. If the gravitational field does not change in time, the later term goes away and $\ddot{x} = -\frac{1}{2} \nabla h_{00}$ which is the Newtonian equation for $h_{00} = 2V$. We therefore have $g_{00} = 1 + 2V/c^2$. Now V/c^2 is 10^{-9} for the earth, 10^{-6} on the sun, 10^{-1} for a neutron star. □

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[2] Also Misner,Thorne, and Wheeler use g to be positive definite on space-like hyper-surfaces.

[3] By a **theorem of Birkhoff**, the unique spherically symmetric solution of Einstein's equations.