# Unit 19: Curvature Tensor

**19.1.** While $\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}]$ is not a tensor, the **Riemann curvature**

$$R^s_{ikj} = \frac{\partial}{\partial u^k}\Gamma^s_{ij} - \frac{\partial}{\partial u^j}\Gamma^s_{ik} + \sum_r \Gamma^r_{ij}\Gamma^s_{rk} - \sum_r \Gamma^r_{ik}\Gamma^s_{rj}$$

is a $(1,3)$ tensor. Think of it as a matrix $R^s_i$ describing a linear transformation when rotating around a small square in the $k,j$ plane. We can also look at at the $(0,4)$-tensor $R_{mikj} = \sum_s g_{ms} R^s_{ijk}$.

**19.2.** To gain more intuition, let us rewrite this without coordinates. Let $X = \sum_i X^i e_i$ and $Y = \sum_i Y^i e_i$ be vector fields (1,0) tensor fields. [1] Reflected in the notation $e_i = \partial_{u^i}$ is that a vector field $X = \sum_i X^i e_i$ also defines a linear map on functions $Xf = \sum_i x^i f_{u^i} = dfX$, the **directional derivative**. [2] Since every vector field $X$ also is a linear map, one can look at the commutator $[X,Y] = XY - YX$, which is by Leibniz again a vector field. Proof: in coordinates $X = \sum_j X^j e_j, Y^j = \sum_j Y^j e_j$, this **Lie bracket** is $[X,Y]^i = \sum_j X^j \partial_j Y^i - Y^j \partial_j X^i$.

**19.3.** The **covariant derivative** $\nabla_X Y$ is a new vector field. Axiomatically it is determined by **Leibniz** $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, **metric compatibility** $g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y,Z)$, and being **torsion free** $\nabla_X Y - \nabla_Y X = [X,Y]$. The **fundamental theorem of Riemannian geometry** assures that there exactly one such derivative: and this is $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$ determining the Riemann curvature tensor $R$.

**19.4.** The curvature tensor now also can be written as $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ meaning $g(e_s, R(e_k,e_j)e_i) = \sum_r g_{sr} R^r_{ikj} e_r = R_{sikj}$. Intuitively, $R(e_k,e_j)$ tells what happens if one parallel transports along a small rectangular loop spanned by $e_k, e_j$. A linear transformation $A^s_i$ results from looping in the $k,j$ plane.

**19.5.** For linearly independent vectors $u,v$, the **sectional curvature**

$$K(u,v) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2}$$

is probably the most intuitive approach to curvature as it does not depend on the coordinate system. It only depends on the plane defined by the tangent vectors $u,v$. If $M$ is two dimensional, it agrees with the Gauss curvature. But this is not obvious as it reestablishes the Theorema egregium!

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[1] Most of the literature uses capital letters for vector fields. $\sum_i X^i e_i$ rather than $\sum_i v^i e_i$.

[2] The directional derivative is written as $\nabla f \cdot v$ in multi-variable calculus.

**Theorem 1.** *The sectional curvatures determine the Riemann curvature tensor.*

**19.6.** The **Ricci curvature** $R$ is a contraction of the curvature tensor $\boxed{R_{ik} = \sum_j R^j_{ijk}}$.

The **scalar curvature** is then the contraction of the Ricci curvature $\boxed{S = \sum_{j,k} g^{jk} R_{jk}}$. In two dimensions, it is twice the Gauss curvature. The **Einstein tensor** $G$ is defined as $\boxed{R - Sg/2}$. A metric is called an **Einstein metric** if $R = \lambda g$ for some $\lambda$. Define the **Hilbert functional** $S(g) = \int_M S_g dV_g$ and the inner product on $(0,2)$ tensors as $\langle a, b\rangle_g = \int_M \sum_{i,j} a(e_i, e_j) b(e_i, e_j)\ dV$. Under which conditions is Hilbert functional extremal? [3]

**Theorem 2.** $\frac{d}{dt} S(g + th) = \langle Sg/2 - R), h\rangle_g$.

**Theorem 3.** *Every 2-manifold is an Einstein manifold:* $Sg/2 - R = 0$.

*Proof.* The reason is that $K = S/2$ and that the Hilbert functional $S(g) = 2\int_M K dV = 4\pi\chi(M)$ does not depend on the metric by the global Gauss-Bonnet theorem. □

We see that in the 2 dimensional case the Ricci tensor $R$ is $K$ times the Riemannian metric tensor $g$. Again, this is not obvious as it re-establishes the Theorema egregium.

**19.7.** In order to prepare for relativity, we also need to generalize Riemannian manifolds. A **metric tensor** on a linear space $E$ is a symmetric $(0,2)$ tensor which is **non-degenerate** that is $g(u,v) = 0, \forall v \in E \Rightarrow u = 0$. A **metric tensor field** $g$ is a tensor field $g \in T^0_2(M)$ such that $g(x)$ is a metric tensor in $T^0_2(T_x M)$. This means that for any vector fields $X, Y$ the function $x \to g(x)(X(x), Y(x))$ is smooth. A **pseudo Riemannian manifold** is a smooth manifold with a metric tensor field $g$ on $M$. A pseudo Riemannian manifold $(M, g)$ is a **Riemannian manifold**, if $g$ is positive definite, meaning $g(x)(v,v) \geq 0$ for all $v$. The **length** of a vector $v \in T_p M$ is defined as $||v|| = \sqrt{|g(p)(u,u)|}$, where $g(p)(u,v) = \sum_{ij} g_{ij}(p) u^i v^j$. [4] A vector of length zero is called **null**. Vectors $u$ for which $\sum_{ij} g_{ij} u^i u^j < 0$ are **time like**, vectors $u$ with $\sum_{ij} g_{ij} u^i u^j > 0$ **space like**. The **length** of the curve is defined by $\int_a^b ||\dot{x}(t)||\ dt$.

**19.8.** Does every manifold allow a pseudo Riemannian manifold of a certain signature?

**Theorem 4.** *On any Riemannian manifold there exists a Riemannian metric* $g$.

*Proof.* There is a tensor field $g \in T^0_2(M)$ which is symmetric, non-degenerate and positive definite: let $\{U_i, \phi_i\}$ be an atlas for $M$ and let $p_i$ be a **partition of unity**, subordinate to the cover $U_i$. Let $q$ be a Riemannian metric on $\mathbb{R}^n$. For example $[q] = \text{Diag}(1, 1, 1 \dots, 1)$. Let $q_i = \phi_i^* q$ be the pull back metrics on $U_i$. Define $g(p) = \sum_i g_i(p) q_i(p)$. This is smooth and positive definite because for $p \in M$ and $u$ in the tangent space $T_p M$, we have $g(p)(u,u) = \sum_i g_i q_i(u,u) > 0$. □

**19.9.** It is not always possible to build on a given manifold a metric of a given signature. For example, on the sphere $M = S^2$, there exists **no Lorentzian metric**, that is a metric of signature $(-1, 1)$. The reason is that one can not comb a 2-sphere.

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[3] A proof can be found on pages 312-320 in Kuehnel.

[4] Note the appearance of an absolute value.