DIFFERENTIAL GEOMETRY

MATH 136

Unit 17: Riemannian Manifolds

17.1. A **locally Euclidean space** M **of dimension** m is a subset of some $\mathbb{R}^n$ such that every $x \in M$ has a neighborhood U, that is homeomorphic to an open subset $R = \phi(U)$ of $\mathbb{R}^m$. The pair (U, ϕ) is called a **chart** producing a **coordinate system** on U: there is a parametrization $r(\phi(x)) = x$, which is a regular map from $R \subset \mathbb{R}^m$ to $\mathbb{R}^n$, meaning that dr has rank m everywhere. A C^k **atlas** on a locally Euclidean space M is a collection $\mathcal{F} = \{U_i, \phi_i\}_{i \in I}$ of charts such that $\bigcup_{i \in I} U_i = M$, and that all $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are in $C^k(\phi_i(U_j \cap U_i), \mathbb{R}^m)$. An atlas is called **maximal**, if (U, ϕ) is a chart such that $\phi \circ \phi_i^{-1}$ and $\phi_i \circ \phi^{-1}$ are C^k for all $i \in I$, then $(U, \phi) \in \mathcal{F}$. Two atlases $\mathcal{F}, \mathcal{G}$ are called **equivalent** if their union $\mathcal{F} \cup \mathcal{G}$ is an atlas. Given an atlas $\mathcal{F}$, the union of all atlases equivalent to $\mathcal{F}$ is called a **differentiable structure generated by $\mathcal{A}$**. [1]

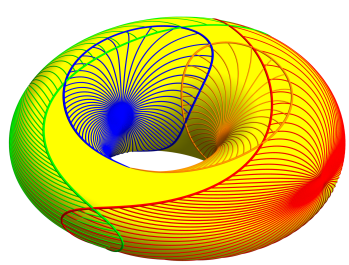

FIGURE 1. A m-manifold $M \subset \mathbb{R}^n$ is shown with part of an atlas $\mathcal{F}$. Each patch U_i is a regularly parametrized by $r : R_i \to U_i$ with $R_i = \phi_i(U_i) \subset \mathbb{R}^m$. The map r has maximal rank m everywhere on R_i.

17.2. A m-dimensional C^k**-differentiable manifold** is a pair $(M, \mathcal{F})$, where M is a m-dimensional locally Euclidean space and where $\mathcal{F}$ is a differentiable C^k structure on M. What this means is that near every point we are in a parametrized region $r(U)$

[1] The concept can be difficult: $\mathcal{F}$ is not unique in general. On S^7, there are 28 different smooth structures. The smooth Poincare conjecture claims that S^4 has a unique differentiable structure.

and so can use what we have done in this course like forming r_u, r_v, define fundamental forms etc. Instead of C^k, we usually just say **smooth**. [2]

17.3. If $E = \mathbb{R}^m$ is the space of **column vectors** of dimension m, its **dual** E^* is defined as the space of all linear maps $f : E \to \mathbb{R}$, It is the space of **row vectors**. If $\{e_1, \dots, e_m\}$ is a basis of E, then $\{e^1, \dots, e^m\}$ denotes a basis of E^*. Every element in E can be written as $v = \sum_i v^i e_i$, every element in E^* can be written as $v = \sum_i v_i e^i$. For $p, q \geq 0$, the linear space T^p_q of all multi-linear maps $(E^*)^p \times E^q \to \mathbb{R}$ is called the space of **tensors of type** (p, q). Column vectors are $(1, 0)$-tensors in $T^1_0 = E$, while row vectors are $(0, 1)$-tensors in $T^0_1 = E^*$, bilinear maps are $(0, 2)$ tensors in T^0_2. A **tensor field of type** (p, q) on a m-manifold M is a smooth assignment of a (p, q) tensor to every point. Such a map is also called a **section** of the tensor bundle, generalizing that a **vector field** is a section of the **tangent bundle** TM. For a $(0, 2)$ tensor field g for example smooth means that for any vector fields X, Y, the function $x \to g(x)(X(x), Y(x))$ is smooth. If $f : M \to \mathbb{R}^k$ is a smooth map, then df is a $(0, 1)$ tensor field. This is also called a **1-form**. A **vector field** means a $(1, 0)$ tensor field. The first fundamental form g is by definition a $(2, 0)$ tensor field, a bilinear form attached to every point. A **Riemannian manifold** (M, g) is a smooth manifold M with a positive definite symmetric $(2, 0)$ tensor field g.

17.4. Let M be a m-manifold and $f : M \to \mathbb{R}^k$ be smooth. A point $x \in M$ is called a **critical point** and $f(x)$ a **critical value**, if the rank of $df(x)$ is not m. Non-critical points are called **regular points**.

Theorem 1. *If M is a m-manifold and $f : M \to \mathbb{R}^k$ is smooth and y is a regular value, then $M_f = f^{-1}(y)$ is a manifold of dimension $m - k$.*

Proof. If $x \in f^{-1}(y)$ is given, the Jacobean map $df(x)$ has rank k and the kernel $H = \ker(df)$ of $df(x)$ is $(m - k)$-dimensional and $H^\perp$ is k-dimensional. Take a chart (U, ϕ) in M which contains x. Define

$$g = f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^m \to \mathbb{R}^k .$$

The projection $L : \mathbb{R}^m = H \oplus H^\perp \to \mathbb{R}^{m-k}, (h, h') \mapsto h'$ onto the orthogonal complement which is non-singular on H. The map $F : \phi(U) \to \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ as $F(z) = (g(z), Lz)$ has derivative $dF(u) = (dg(u), Lu)$ and which is non-singular. By the **inverse function theorem**, a neighborhood $\phi(U)$ of $\phi(x)$ is mapped by the diffeomorphism F onto a neighborhood $F(\phi(U))$ of $F(\phi(x))$. We get so a chart $U_x = \phi^{-1} \circ F^{-1}(F(\phi(U)) \cap \{\psi(y)\} \times \mathbb{R}^{m-k})$ on $f^{-1}(y)$ which is mapped by $\phi_x = F \circ \phi$ into a $(m - k)$-dimensional space. Doing the same construction at any point $x \in M$ produces an atlas for $f^{-1}(y)$ and verifies that $f^{-1}(y)$ is a manifold. □

17.5. Examples: a) The $\mathbf{d-sphere}$ is the set $M = S^d = \{x \in \mathbb{R}^{d+1} \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\} \subset \mathbb{R}^{d+1}$. Take two points $A = (0, \dots, 0, 1)$ and $B = (0, \dots, 0, -1)$. The **standard differentiable structure** $\mathcal{F}$ on S^d is generated by $\mathcal{F} = \{(S^d \setminus \{A\}, \phi_A), (S^d \setminus \{B\}, \phi_B)\}$, where ϕ_A are the **stereographic projections** from A to $\{x_{d+1} = 0\}$. b) The set $SL(n, \mathbb{R})$ of $n \times n$ matrices of determinant 1 is a manifold.

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[2] A theorem of Whitney assures that any smooth compact m-manifold M (defined more abstractly using fancy-schmancy paracompact Hausdorff spaces) is part of $\mathbb{R}^n$ with $n = 2m + 1$.