

# DIFFERENTIAL GEOMETRY

MATH 136

## Unit 16: Global Gauss-Bonnet

**16.1.** Now we are ready to prove the **global Gauss-Bonnet theorem** for a 2-manifold  $M$  without boundary. The surface  $M$  is triangulated by a **discrete manifold**  $G = (V, E, F)$ , where the faces  $F$  are the triangles defined by the graph  $(V, E)$ . The discrete manifold is geometrically realized in  $M$  as a collection of points, a collection of curves connecting vertices. The geometrically realized network divides  $M$  up into triangular faces  $M_i = r(U_i)$ . The Euler characteristic of  $M$  is  $\chi(M) = V - E + F$ . As we have seen,  $\chi(M)$  does not depend on the triangulation: topological changes like removing a disc and gluing in a new disc (we called this as a **connected sum**  $M \rightarrow M \# S^2$ ) or doing a **Barycentric refinement** does not change  $\chi(M)$ .

**Theorem 1** (Gauss-Bonnet theorem). *For a compact 2-manifold,  $\iint_M K dV = 2\pi\chi(M)$ .*

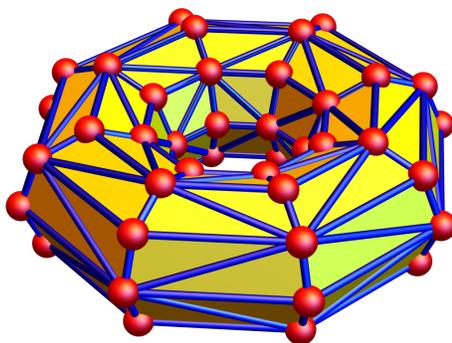


FIGURE 1. A triangulated manifold  $M$ . We apply the local Gauss-Bonnet theorem on each of the triangles. The edge contributions produce with Green's theorem the face curvatures  $\iint_{M_i} K dV$  as well as  $2\pi F$ . The vertex contributions produce, using the Euler Handshake lemma,  $2\pi(V - E)$ . Overall, we have  $2\pi(V - E + F) = 2\pi\chi(M)$ . The picture shows a  $M = \mathbb{T}^2$  with  $V = 64$  vertices.

**16.2.** We will use the local Gauss-Bonnet theorem for each triangle  $U_i$  with angles  $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$ . We first of all want to understand what happens if we glue together triangles such that the frame field  $X = [zw_u, zw_v]$  can be defined on the union. On the curve obtained by intersecting two adjacent triangles, the line integral of  $X$  cancels.

**Lemma 1** (Cancellation). *If two triangles  $M_1, M_2$  meet in a curve  $C$  and  $C_1, C_2$  are the parametrizations matching the  $M_1, M_2$ , then  $\int_{C_1} X dr + \int_{C_2} X dr = 0$ .*

*Proof.* If  $X$  is a 1-form and  $C$  is a curve and  $-C$  is the curve passed backwards, then  $\int_C X dr + \int_{-C} X dr$ . This is what happens here. You can see the identity also as a consequence of Green also, noting that  $C \cup -C$  encloses an “empty region”.<sup>1</sup>  $\square$

**16.3.** We see that all the 1-form contributions from the edges are zero. The contributions from the faces  $M_i$  add up:

**Lemma 2** (Additivity).  $\int_M K dV = \sum_i \int_{M_i} K dV$ .

*Proof.* The patches  $M_i$  are all disjoint. Their union is  $\bigcup_i M_i = M$ . Areas of disjoint regions add up.  $\square$

**16.4.** The contributions from the vertex degrees  $d_i = |S(v_i)|$  add up too.<sup>2</sup>

**Lemma 3** (Euler handshake). *If  $(V, E)$  has vertex degrees  $d_i$ , then  $2E = \sum_i d_i$ .*

*Proof.* You prove this in a homework.  $\square$

**16.5.** We still have to look at the contributions from the vertices. At each point  $P_i$  we have angles  $\alpha_{ij}$  for  $j = 1, \dots, d_j$ , where  $d_j$  is the vertex degree.

**Lemma 4** (Adding vertex curvatures).  $\sum_{i=1}^F \sum_{j=1}^{d_i} \kappa_{ij} = 2\pi E - 2\pi V$ .

*Proof.* Three comfortable pitches (this is called “Genusskletterei”).

(Pitch i)  $\sum_{i=1}^F \sum_{j=1}^{d_i} \kappa_{ij} = \sum_{i=1}^F \sum_{j=1}^3 (\pi - \alpha_{ij})$ .

(Pitch ii)  $\sum_{k=1}^V \sum_{j=1}^{d_k} \pi = 2\pi E$ .

(Pitch iii)  $\sum_{k=1}^V \sum_{j=1}^{d_k} \alpha_{kj} = 2\pi V$ .  $\square$

**16.6. Proof of the global Gauss-Bonnet theorem:**

The local Gauss-Bonnet theorem told us  $\sum_i [\iint_{U_i} K dV + \sum_j \kappa_{ij} - 2\pi] = 0$ . This means that  $\iint_U K dV + \sum_{i,j} \kappa_{ij} - 2\pi F = 0$ . Therefore, using the previous lemma:

$$\iint_U K dV = 2\pi V - 2\pi E + 2\pi F = 2\pi \chi(M).$$

**16.7. 1)** If  $M$  is  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ , then  $\iint_M K dV = 4\pi$  (HW).

**2)** A genus  $k$  surface has  $\iint_M K dV = 2\pi(2 - 2k)$ . For a torus  $\iint_M K dV = 0$ .

**3)** A Klein bottle is obtained by gluing two Möbius strips together.  $\iint_M K dV = 0$ .

Because each Möbius strip has Euler characteristic 0 (you computed that in an example), and the Möbius strip can be realized so that the boundary curvature  $\chi_g$  is zero.

<sup>1</sup>An easier way is to assume that the connections geodesics so that the line integrals are zero anyway. That works also in the non-orientable case

<sup>2</sup>If  $V, E, F$  are the vertices, edges and faces. It is custom to write its cardinalities as  $V, E, F$ .