

DIFFERENTIAL GEOMETRY

MATH 136

Unit 17-18 Homework

This is the 10th homework. It is due Friday, November 22th:

Problem 1: a) Give a definition of what a **Riemannian C^k -manifold with boundary** is. There are very little changes needed to the definition given in the course. You just need to use besides \mathbb{R}^m also $H^m = \{x \in \mathbb{R}^m, x_1 \geq 0\}$.
b) Conclude from the definition that the boundary is again a C^k manifold by giving a concrete atlas for this manifold.

Problem 2: From each of the following objects determine whether it is a tensor field or not

- a) The calculus gradient field $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$.
- b) The Jacobian of a function $df = [f_x, f_y, f_z]$
- c) The second fundamental form II
- d) A linear transformation field of TM like the shape operator A like $(Av)^k = \sum_j A_j^k v^j$.
- e) The inverse I^{-1} of the first fundamental form g^{ij} .
- f) The second derivative $r_{u^i u^j}$.
- g) The Christoffel symbols $\Gamma_{ijk} = r_{u^i u^j} r_{u^k}$.

Problem 3: Prove that $SU(2)$ is a manifold by explicitly giving the charts.

Problem 4: Draw a 2-dimensional discrete manifold M of your choice by hand. You can find manifolds which are planar, meaning that they can be realized in the plane without crossings of the edges. Also chose a random function f on the vertices taking values in $\{0, 1\}$. Now look at the level curve M_f as defined in the text.

Problem 5: The following code allows you to experiment with level sets in discrete manifolds. The host manifold is a discrete 4-manifold, the join of a 2-sphere and a 1-sphere. Running the code builds a random function from the vertex list to $\{1, 2, 3\}$. It defines a co-dimension 2 manifold.

- Run the code as it is, report the numbers V, E, F as well as the Euler characteristic of M_f .
- Change the code and see what happens if the function takes 4 values rather than 3.
- Build a 5 manifold as the join of two 2-manifolds and build a 3-manifold by taking a function taking 3 random values.
- Report the curvature values of your 3-manifold.
- Report the curvature vales of a 4 manifold by taking a function taking 2 values on the 5 manifold in c).

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Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]],1]]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]]; w[x_-]:=-(-1)^k;
R[G_,k_-]:=Module[{},R[x_-]:=x->RandomChoice[Range[k]]; Map[R,Union[Flatten[G]]];
F[G_-]:=Delete[BinCounts[Map[Length,G],1]; Euler[G_-]:=F[G].Table[w[k],{k,Length[F[G]]}];
Surface[G_,g_-]:=Select[G,SubsetQ[#/./g,Union[Flatten[G]/./g]]&];
S[s_,v_-]:=VertexDelete[NeighborhoodGraph[s,v],v]; Sf[s_,v_-]:=F[Whitney[S[s,v]]];
Curvature[s_,v_-]:=Module[{f=Sf[s,v]},1+f.Table[(-1)^k/(k+1),{k,Length[f]}];
Curvatures[s_-]:=Module[{V=VertexList[s]},Table[Curvature[s,V[[k]],{k,Length[V]}];
J[G_,H_-]:=Union[G,H+Max[G]+1,Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1},0]]]];
ToGraph[G_-]:=UndirectedGraph[n=Length[G];Graph[Range[n],
Select[Flatten[Table[k->1,{k,n},{1,k+1,n}],1],(SubsetQ[G[#[[2]]],G[#[[1]]]])&]];
Barycentric[s_-]:=ToGraph[Whitney[s]];

G=J[Whitney[Barycentric[CompleteGraph[{2,2,2}]],Whitney[CycleGraph[7]]]; (* J=Join *)
g=R[G,3]; H=Surface[G,g]; (* A codimension 2 manifold in the 4-sphere G=Oct * C7 *)
Print["EulerChi=-",Euler[H]]; Print["Fvector:-",F[H]]; s=ToGraph[H]; GraphPlot3D[s]
Print["Gauss-Bonnet-Check:-"]; Print[Total[Curvatures[s]]==Euler[H]];
Print["Curvature-Values:-"]; Print[Union[Curvatures[s]]];

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