

DIFFERENTIAL GEOMETRY

MATH 136

Final Exam Paper

ASSIGNMENT: DUE DECEMBER 15

Write an expository paper about a topic in differential geometry of your choice.

RULES

- The paper must be written by yourself. No human or AI assistance is allowed in writing.
- Acknowledge any references used: books, papers, web, AI chat, discussions, the use of computer algebra.
- Aim for 4 or more pages with references. There can be illustrations. Also for illustrations, either do them yourself or give credit.

GRADING CRITERIA

- Mathematical correctness
- Clarity, readability and elegance
- References and sources
- Adaptation to the course, notation
- Originality or depth or surprise

TOPIC SUGGESTIONS

Pick or modify one of the following topics. If you prefer an other topic you need to check with Oliver.

- (1) **"My favorite manifold"**. i.e. projective plane, the Klein bottle, the projective 3-space $SO(3)$ (a 3 manifold) or $SU(3)$ (a 8 dimensional manifold).
- (2) **"The world of discrete manifolds"**. Explore small examples of discrete manifolds, I.e. Klein bottles, projective planes, or higher genus Klein bottles.
- (3) **"The Hilbert Action"** Outline the proof that critical points of the Hilbert action are Einstein manifolds.
- (4) **"What are Minimal Surfaces?"**. See chapter 3D in Kuehnel.
- (5) **"The rigidity of the sphere"** a compact connected regular surface of constant curvature must be a sphere. There is a section in Do Carmo.
- (6) **"What are Ruled surfaces?"** Section 3C in Kuehnel.
- (7) **"The amazing Moebius strip"**. Explore differential geometry of Moebius band embedded in $\mathbb{R}^3$.

- (8) **"Clairaut's relation for geodesics"** A conserved quantity for geodesics in surfaces of revolution".
- (9) **"The isoperimetric inequality"**. There is an exposition in Do Carmo.
- (10) **"Geodesic rigidity of Gauss"**: if p, q are two points on a manifold and if $\gamma_s(t)$ is a smooth family of geodesics connecting p, q , then all these geodesics must have the same length.
- (11) **"The Willmore energy"** The boy surface is known as "Oberwohlfach surface". A theorem of Bryant-Kusner tells that the Boy surface minimizes the Willmore energy.
- (12) **"Geometry in hyperbolic space"**. Describe the geodesics in a 2 dimensional manifold of constant curvature -1. Gauss-Bonnet for polygons.
- (13) **"Positive Curvature Manifolds"**. The zoo of even dimensional manifolds for which all sectional curvatures are positive.
- (14) **"Fenchel's theorem"** or the Fary-Milnor theorem about total curvature. There is a section in Kuehnel.
- (15) **"The fundamental theorem of Riemannian geometry"** a connection satisfying three axioms must be the Christoffel connection.
- (16) **"Classifying 2-manifolds"**. Describe the classification of 2 manifold as a connected sum of tori or projective planes.
- (17) **"Spherical space forms"**. See for example Theorem 7.30 in Kuehnel.
- (18) **"What are lense spaces"**? What are spherical 3-manifolds?
- (19) **"Synge's theorem"**: an even dimensional orientable positive curvature manifold is simply connected.
- (20) **"The Poincare conjecture"**. How was the Poincare conjecture proven. Especially define the Ricci flow using notation we have used.
- (21) **"Surgery of 3-manifolds"**. Explain how one can use knots to build 3-manifolds. Especially Dehn surgery and the Lickorish-Wallace theorem.
- (22) **"Exotic spheres?"** What are spheres with non-standard differential structure.
- (23) **"The Kerr Metric"**. A solution of the Einstein equations. The Kerr metric" models a rotating non-charged black hole. The Kerr-Newman metric a rotating charged black hole.
- (24) **"Bertrand-Puiseux formulas"** seen in Unit 1. Prove $|S_r(p)| = 2\pi r - \pi K \epsilon^3/3 + \dots$
- (25) **"Chaotic Billiards"**. The geodesic flow in manifolds with boundary. is called a billiard. This involves curvature.
- (26) **"Riemannian metrics on Lie groups"**. Like on $SO(3)$ or $SU(2)$. How does one get a Riemannian metric on such spaces?
- (27) **"The Willmore conjecture"**. What is the Willmore energy? Explain the conjecture and how it was solved.
- (28) **"The Differentiable Jordan Curve theorem"** is a bit easier than the actual Jordan Curve theorem in topology. Prove it. See Do carmo page 400.
- (29) **"The Hopf-Rynov Theorem"**: Two points on a complete surface can be joined by a minimal geodesic. See Do Carmo page 338.
- (30) **"The formulas of Codazzi-Meinardi"**. See section 4.C in Kuehnel.