Lecture 22: Power series, 10/27/2021

Power series

22.1. A series

\[ S(x) = \sum_{k=0}^{\infty} a_k x^k. \]

is a power series. More generally one can center them at a point \( c \)

\[ S(x) = \sum_{k=0}^{\infty} a_k (x - c)^k. \]

22.2. An important class of power series are Taylor series

\[ S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k. \]

Example: \( \sum_{k=0}^{\infty} x^k \) which is the Taylor series of \( 1/(1 - x) \).

Example: \( \sum_{k=0}^{\infty} \frac{x^k}{k!} \) which is the Taylor series of \( \exp(x) \).

Radius of convergence

22.3. For any power series, there is an interval \((c - R, c + R)\) centered at \( c \) on which the series converges. This is called the interval of convergence. The number \( R \) is called the radius of convergence.

22.4. If \( R \) is the radius of convergence then for \( |x - c| < R \), the series converges for \( |x - c| > R \) the series is divergent.

Example: For the series \( \sum_{k=1}^{\infty} \frac{x^k}{k!} \) for example, the series converges for \( |x| < 1 \) by the ratio test we have seen before. For \( |x| > 1 \) the series does not converge by the \( n \)’th term test.

For \( x = 1 \), we have the Harmonic series, where the series diverges. For \( x = -1 \) we have an alternating series which converges. The limit is \( \log(2) \). We say then the “maximal interval of convergence” is \([-1, 1)\) as the left point is included.
22.5. Here is a formula for the radius:

\[ R = \lim_{k \to \infty} |a_k|/|a_{k+1}|, \text{ if the limit exists.} \]

22.6. Indeed, for \(|x - c| < R\), the terms \(b_k = a_k(x - c)^k\) satisfy \(|b_{k+1}|/|b_k| = |x - c|/R < 1\) so that \(\sum_k b_k\) converges by the ratio test. For \(|x - c| > R\), the terms satisfy \(|b_{k+1}|/|b_k| = |x - c|/R > 1\) so that \(\sum_k b_k\) diverges by the n’th term test.

Why are they useful?

22.7. Power series are mostly used for Taylor series. This allows us to work with functions by taking polynomial approximations with control about the error.

22.8. A power series not coming from a named function \(f\) is the prime function

\[ S(x) = \sum_{p \text{ prime}} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + \cdots. \]

The coefficients of \(S(x)^2 = x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + \cdots\) tell in how many ways one can write a given number as a sum of two primes. The Goldbach conjecture states that every even number can be written as such. The Goldbach comet plotting the coefficients of the power series of \(S(x)^2\) illustrates that.

![Figure 1. The Goldbach comet gives the coefficients of \(b_k\).](image)

22.9. An other application are moment generating function in statistics.

\[ S(x) = \sum_{n=0}^{\infty} \frac{E[X^n]x^n}{n!} \]

where \(E[X]\) denotes the expectation or mean of a random variable \(X\). Moments are important because for example the variance of a random variable is \(E[X^2] - E[X]^2\) and so expressible using moments.

Examples

- HW 21 is due Friday

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