CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 22: Power series, 10/27/2021

Power series

22.1. A series

$$S(x) = \sum_{k=0}^{\infty} a_k x^k \ .$$

is a **power series**. More generally one can center them at a point c

$$S(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \ .$$

22.2. An important class of power series are **Taylor series**

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \ .$$

Example: $\sum_{k=0}^{\infty} x^k$ which is the Taylor series of $1/(1-x)$.

Example: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ which is the Taylor series of $\exp(x)$.

Radius of convergence

22.3. For any power series, there is an interval $(c-R, c+R)$ centered at c on which the series converges. This is called the **interval of convergence**. The number R is called the **radius of convergence**.

22.4. If R is the radius of convergence then for $|x-c| < R$, the series converges for $|x-c| > R$ the series is divergent.

Example: For the series $\sum_{k=1}^{\infty} \frac{x^k}{k}$ for example, the series converges for $|x| < 1$ by the ratio test we have seen before. For $|x| > 1$ the series does not converge by the n'th term test.

For $x = 1$, we have the **Harmonic series**, where the series diverges. For $x = -1$ we have an **alternating series** which converges. The limit is $\log(2)$. We say then the"maximal interval of convergence" is $[-1, 1)$ as the left point is included.

22.5. Here is a formula for the radius:

$R = \lim_{k \to \infty} |a_k|/|a_{k+1}|$, if the limit exists.

22.6. Indeed, for $|x-c| < R$, the terms $b_k = a_k(x-c)^k$ satisfy $|b_{k+1}|/|b_k| = |x-c|/R < 1$ so that $\sum_k b_k$ converges by the **ratio test**. For $|x-c| > R$, the terms satisfy $|b_{k+1}|/|b_k| = |x-c|/R > 1$ so that $\sum_k b_k$ diverges by the **n'th term test**.

Why are they useful?

22.7. Power series are mostly used for **Taylor series**. This allows us to work with functions by taking **polynomial approximations** with control about the error.

22.8. A power series not coming from a named function f is the **prime function**

$$S(x) = \sum_{p \text{ prime}} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + \cdots .$$

The coefficients of $S(x)^2 = x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + \dots$ tell in how many ways one can write a given number as a sum of two primes. The **Goldbach conjecture** states that every even number can be written as such. The **Goldbach comet** plotting the coefficients of the power series of $S(x)^2$ illustrates that.

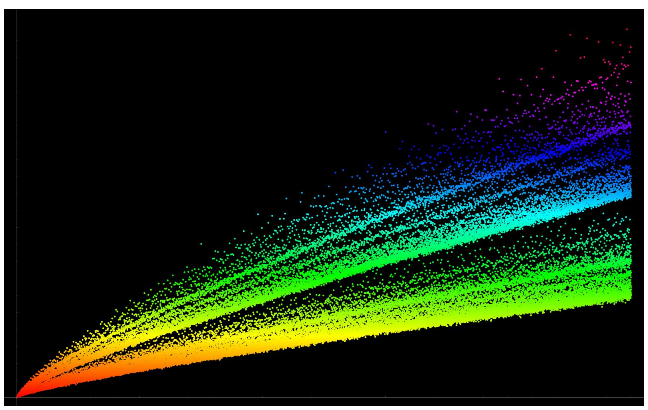

Figure 1. The Goldbach comet gives the coefficients of b_k.

22.9. An other application are **moment generating function** in **statistics**.

$$S(x) = \sum_{n=0}^{\infty} \frac{E[X^n]x^n}{n!}$$

where $E[X]$ denotes the expectation or mean of a random variable X. Moments are important because for example the variance of a random variable is $E[X]^2 - E[X]^2$ and so expressible using moments.

Examples

- HW 21 is due Friday