ENTRY TOPOLOGY

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Alexander compactification

The [Alexander compactification] Y of a Hausdorff space (X, O) is the topological space $(Y = X \cup x, P)$, where x is an additional point. The topology P consists of the elements in O and the complements of closed subsets as neighborhoods of that point. The new topological space Y is compact.

Alexander's subbase theorem

The [Alexander's subbase theorem]: if every open cover of a topological space X has a finite sub-cover then X is compact.

arc-connected

A topological space is called [arc-connected] if any two points can be connected by a path, a continuous image of an interval. Path connected is stronger than connected but not equivalent: the subset $\{(x, \sin(1/x)),\ x \in R^+\} \cup \{(0, y), -1 \leq y \leq 1\ \}$ of the plane with topology induced from the plane is connected but not path connected. Arc-connected is also called path-connected.

Baire category

[Baire category] is a measure for the size of a set in a topological space. Countable unions of nowhere dense sets are called of the first categorie or meager, any other set of second category. Complements of meager sets are called residual. Baire category is used to quantify certain sets. For example it is known that "most" numbers are Liouville numbers in the sense that they form a residual set among all real numbers.

Baire space

A [Baire space] is a topological space with the property that the intersection of countable family of open dense subsets is dense.

Baire category theorem

The [Baire category theorem]: a complete metric space is a Baire space.

ball

A [ball] in a metric space is a set of the form $\{y \mid \ d(x,y) < r\ \}$. The closure of an open ball is a closed ball. To make clear that a ball is open, one sometimes calls it also open ball.

barrier function

A [barrier function] for a set S in a topological space (X,O) is a nonnegative continuous function f defined on X which is zero in S and positive in the complement of S. A barrier function is sometimes also called a penalty function.

basis

A [basis] of a topological space (X,T) is a subset B of T such that

- the empty set is in B,
- arbitrary unions of sets in B are in B,
- the intersection of two sets in B is a union of sets in B.

A basis B defines the topology (X,T). Every $A \in T$ is a union of elements in B. An example: if (X,d) is a metric space then the set of all balls $\{y \mid d(x,y) < 1/k\ \}$, where x is taken from a dense set in X and k is a positive integer form a basis.

bicontinuous

A function is [bicontinuous] if it is continuous invertible and has a continuous inverse. A bicontinuous function is also called a homeomorphism.

bounded

A subset of a metric space is [bounded] if it is contained in some ball of finite radius.

boundary

The [boundary] of a set A in a topological space (X,T) is the the set $C \setminus B$, where C is the closure of A and B is the interior of A. Examples:

- if A is the open unit disc in the plane, then the boundary is the unit circle.
- in a discrete topological space, the boundary of any set is empty.

Cantor set

A [Cantor set] is a topological space which is homeomorphic to the Cantor middle set.

Cantor middle set

The [Cantor middle set] is the subset of the unit interval which is the complement of $\bigcup_{n=1}^{\infty} Y_n$, where $Y_1 = (1/3, 2/3), Y_2 = (1/9, 2/9) \cup (7/9, 8/9)$ etc. are sucessive middle sets. It is a fractal with Hausdorff dimension $\log(2)/\log(3)$.

Cantor middle set

A topological space homeomorphic to a ball in Euclidean space is called a [cell]. Examples of cells are polyhedra in three dimensional space.

closure

The [closure] of a set A in a topological space (X, T) is the intersection of all closed sets in X, which contain A. One writes $\overline{Y}$ for the closure of Y.

dense

A set A is called [dense] in a topological space (X, T), if every open set $Y \in O$ in X contains at least one point in A.

finer

A topology (X, T) is [finer] than a topology (X, S) if S is a subset of T. In that case, (X, S) is called a coarser topology than (X, T). Examples:

- the discrete topology on X is finer than any other topology on X.
- A set S of subsets of X defines a topology, the coarsest topology O which contains S.

topological space

A [topological space] is a pair (X,T) where T is a set of subsets of X satisfying

- $\emptyset \in T$,
- if $A, B \in T$, then $A \cap B \in T$,
- an arbitrary union of subsets in T is in T.

Elements in T are called open sets. The complement of an open set is called a closed set. Examples:

- the discrete topology on X: T is all subsets of X
- the indiscrete topology on X: T contains only X and the empty set,
- the cofinite topology: T is the set of all subsets A such that their complement in X is a finite set.
- (X, d) metric space: T is the set of sets A such that for x in A, also a small ball $B = \{|y - x| < a\}$ is contained in A.

open set

An [open set] of a topological space (X, T) is an element of T.

closed set

A [closed set] is the complement of an open set in a topological space (X, T). A closed set contains all its limit points.

continuous

A map f between two topological spaces (X, T) and (Y, S) is called [continuous] if the inverse image of any open set is open: for all $A \in S$ one has $f^(-1)(A) \in T$. Note that f does not need to be invertible: one defines $f^{-1}(A) = \{x \in X | f(x) \in A\}$. Examples of results known:

- The composition of two continuous maps is continuous.
- Every map on the discrete topological space is continuous.
- A map between metric spaces is continuous if and only if it is sequential continuous: for any $x_n \to x$, one has $f(x_n) \to f(x)$.
- A map between topological spaces is continuous if for every net $x_t \to x$, the net $f(x_t)$ converges to $f(x)$.

homeomorphism

An invertible map f between two topological spaces (X, T) and (Y, S) is called a [homeomorphism] if f and the inverse of f are both continuous.

homeomorphic

[homeomorphic] If there exists a homeomorphism between two topological spaces, the topological spaces are called homeomorphic.

connected

A topological space (X, T) is called [connected], if there are no two disjoint open sets U, V whose union is X. For a connected topological space, the empty set $\emptyset$ or X are the only sets which are both open and closed. A subset A of a topological space is connected if it is connected with the on A induced topology: there are no disjoint open sets U, V whose union contains A.

locally connected

A topological space is [locally connected] if every point has arbitrarily small neighborhoods which are connected. Examples.

- A union of disjoint open intervals on the real line is locally connected but not connected.
- The union of the graphs of $f(x) = 2\sin(1/x)$ and $g(x) = 1$ and the y-axes all intersected with the set $\{y > 1\}$ is connected but not locally connected because small neighborhoods of the point $(0, 1)$ are not connected.

Hausdorff

A topological space (X, T) is called [Hausdorff] if for every two points $x, y \in X$, there are disjoint open sets $U, V \in T$ such that $x \in U$ and $y \in V$. This is refined through seperation axioms, $T0, ..., T4$. Hausdorff is also called $T2$. Any metric space is Hausdorff: if d is the distance betwen x and y, then balls of radius $d/3$ around x and y seperate the points. The plane X with semimetric $d(x, y) = |x_1 - y_1|$ is not Hausdorff: the points $x = (0, -1)$ and $y = (0, 1)$ can not be seperated by open sets.

seperation axioms

[seperation axioms] define classes of topological spaces with decreasing seperability properties: $T4 \Rightarrow T3 \Rightarrow T2 \Rightarrow T1 \Rightarrow T0$.

T0 space:	for two different points x, y in X one of the points has an open neighborhood U not containing the other point.
T1 space:	for two different points x, y in X there exists an open neighborhood U of x and an open neighborhood V of y. such that x is not in V and y is not in U.
T2 space:	also called Hausdorff" two different points x, y can be seperated with disjoint neighborhoods U, V.
T3 space:	T1 and regular: any point x and any closed set F not containing x can be seperated by two disjoint neighborhood.
T4 space:	T1 and normal: any two disjoint sets F,G can be separated by two disjoint open sets.

It is known that a T4 space with a countable basis is metrizable.

Hausdorff topology

The [Hausdorff topology] is a metric on the set of closed bounded subsets of a complete metric space. The distance between two sets A and B is the infimum over all r for which A is contained in a r-neighborhood of B and B is contained in a r-neighborhood of A.

Lindeloef

A topological space is called [Lindeloef] if every open cover of X contains a countable subcover.

compact

A topological space is called [compact] if every open cover of X contains a finite subcover. Examples of results known about compactnes:

- Heine-Borel theorem: a closed interval in the real line is compact.
- If $f : X \to Y$ is continuous and onto and X is compact, then Y is compact. As a consequence, a continouous function on a compact subspace has both a maximum and a minimum.
- In a Hausdorff space, compact sets are closed.
- In a metric space, compact sets are closed and bounded.
- Closed subsets of compact spaces are compact.
- Tychonof theorem: the product of a collection of compact spaces is compact.

countably compact

A topological space is called [countably compact] if every countable open cover of X contains a finite subcover.

locally compact

A topological space is called [locally compact] if every point has a neighborhood, which has a compact closure. Examples.

- The real line is compact but not locally compact.
- A compact Hausdorff space is locally compact.
- The n-dimensional Euclidean space R^n is lcally compact but not compact.

locally compact

A set U of open sets in a topological space (X, O) is called locally finite if every point $x \in X$ has a neighborhood V, such that V has a nonempty intersection with only finitely many elements in U.

paracompact

A topological space (X,O) is called [paracompact] if every open cover has a countable, locally finite subcover.

relatively compact

A subset A of a topological space (X, T) is called [relatively compact] if the closure of A is compact.

filter

A [filter] on a nonempty set X is a set of subsets F satisfying

- X is in F, but the empty set $\emptyset$ is not in F.
- If A and B are in F, then their intersection is in F.
- If A is in F and B is a subset of A, then B is in F.

Examples:

- Principal filter for a nonempty subset A consists of all subsets of X which contain A.
- Frechet filter for an infinite set consists of all subsets of X such that their complement is finite.
- Neighborhood filter of a point x in a topological space (X,T) is the set of open neighborhoods of x.
- Elementary filter for a sequence x_n in X consists of all sets A in X such that x_n is in A for large enough n.

converges

A sequence x_n in a topological space [converges] to a point x, if for every neighborhood U of x, there exists an integer m, such that for $n > m$ one has $x_n \in U$.

Filter convergence

[Filter convergence] A filter F converges to x in a topological space (X, T) if F contains the neighborhood filter G of x, that is if F contains all neighborhoods of x. For example, an elementary filter to a sequence x_n converges to a point x, if and only if x_n converges to x.

accumulation point

A point y is called an [accumulation point] of a filter F, if there exists a filter G containing F such that G converges to x.

directed

A set M is called [directed] if there exists a partial order $(M, <)$ on M satisfying for every two points $a, b \in M$ there exists c, with $a < c$ and $b < c$.

interior

The [interior] of a set A in a topological space (X, T) is the union of all open sets in X, which are contained in A.

Koch curve

The [Koch curve] is a fractal in the plane. It has Hausdorff dimension $\log(4)/\log(3)$. It is constructed by building an equilateral triangle on the middle third of each side of a given equilateral triangle K_0 leading to a curve K_1 and recursively build K_{n+1} from K_n by replacing each middle third of a line segment in K_n with a triangle. The curve is the limit of K_n, when n goes to infinity.

metrizable

A topological space (X, T) is called [metrizable] if there exists a metric on X such that the topology generated by the metric is T.

metric space

A [metric space] (X, d) is a set X with a function d from $X \times X \to [0, \infty)$ satisfying $d(x, y) = d(y, x), d(x, y) = 0 \Leftrightarrow x = y$ and $d(x, z) \leq d(x, y) + d(y, z))$. The set $T = \{U \subset X \mid \forall x \in X, \exists r > 0, B_r(x) = \{y \mid d(x, y) < r\} \subset U\}$ defines a topological space (X, T).

metric space

A [metric space] (X, d) is a set X with a nonnegative function d from $X \times X$ satisfying $d(x, y) = d(y, x)$,$d(x, y) = 0$ if and only if $x = y$ and $d(x, z) \leq d(x, y) + d(y, z)$. A metric space defines a topological space (X,T): the topology T is the set of subsets A of X such that for all points $x \in A$, there is a small ball $d(y, x) < r$ which is also contained in A.

net

A [net] with values in a topological space X is a function f: D -¿ X, where D is a directed set. For example: if D is the set of natural numbers, then a net is a sequence. A net defines a filter F: it is the set of all sets A such that x_t is eventually in A. A net x_t converges to a point x if and only if the associated filter converges to x.

open cover

[open cover] A subset U of O, where (X, O) is a topological space is called an open cover of X if the union of all elements in U is X. If U and V are open covers and $V \subset U$, then V is called a subcover of U.

product space

The [product space] between topological spaces is defined as $(X \times Y, O \times P)$, where $X \times Y$ is the set of all pairs $(x, y), x \in X, y \in Y$ and $O \times P$ is the coarsest topological space which contains all products $A \times B$, where $A \in O$ and $B \in P$. For example, if $(X, O) = (Y, P)$ are both the real line with the topology generated by $d(x, y) = |x - y|$, then the product space is homeomorphic to the plane with the metric $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

second countable

A topological space is called [second countable], if it has a countable basis. Example. Every seperable metric space is second countable. Especially, every finite-dimensional Euclidean space is second countable.

metrizable

A topological space is called [metrizable] if there exists a metric d on the set X that induces the topology of X. Any regular space with a countable basis is metrizable.

homotopic

[homotopic] If f and g are continuous maps from the topological space X to a topological space Y, we say that f is homotopic to g if there is a continuous map F from $X \times I$ to Y, such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all x. For example, the maps $f(x) = x^2$ and $g(x) = sin(x)$ on the real line are homotopic, because we can define $F(x,t) = (1-t)x^2 + t\sin(x)$. The maps $f(x) = x$ and $g(x) = \sin(2\pi x)$ on the circle are not homotopic. While g is homotopic to the constant function $h(x) = 0$, the map $f(x)$ can not be deformed to a constant without breaking continuity.

induced topology

The [induced topology] on a subset A of X, where (X,T) is a topological spoace is the the topological space $(A, \{Y \cap A\}_{Y \in T})$.

path homotopic

[path homotopic] If f and g are and continuous homotopic maps from an interval to a space X, we say f and g are path homotopic if their images have the same end points. For instance, the maps $f(x) = x^2$ and $g(x) = x^3$ are path homotopic on the closed interval from 0 to 1. The maps $f(x) = 2x^2$ and $g(x) = x^3$ are homotopic on the unit interval but not path homotopic.

loop

A [loop] is a path in a topological space that begins and ends at the same point. A loop is also called a closed curve. Loops play a role in definitions like simply connected: a topological space is simply connected if every loop is homotopic to a constant loop which is a fancy way telling that every closed path can be collapsed inside X to a point.

fundamental group

The [fundamental group] of a topological space at a point is the set of homotopy classes of loops based at that point.

Topologist's Sine Curve

The [Topologist's Sine Curve] is the union S of the graph of the function $\sin(1/x)$ on the positive real axes R^+ with the y-axes. It an example of a topological space which is connected but not path-connected. Proof: if S were path-connected, there would exist a path $r(t) = (x(t), y(t))$ connecting the two points $(0, 1)$ and $(0, \pi)$. The set $\{t|r(t) \in S\}$ is closed. Let T be the largest t in that set for which $r(t)$ is in the y-axes. Then x(T)=0 and r(t)=(x(t),sin(1/x(t)) for $t > T$. Because there are times $t_n > t_{n-1} > T, t_n \to T$ for which $y(t_n) = (-1)^n$, the function $r(t)$ can not be continuous at $t = T$.

Urysohn lemma

The [Urysohn lemma] tells that if X is a normal space and A and B are disjoint closed subsets of X, then there exists a continuous map f from X to the unit interval such that $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

Proof: use the normality of X to construct a family U_p of open sets of X indexed by the rational numbers P in the unit interval so that for $p < q$, the closure of U_p is contained in U_q. These sets are simply ordered in the same way that their subscripts are ordered in the real line. Given some enumeration of the rationals, where 1 and 0 are the first two elements of the enumeration, define $U_1 = X \setminus B$. Because A is a closed set contained in the open set U, there is by normality an open set U_0 such that $A \subset U_0$ and the closure of U_0 is a subset of U_1. In general, let P_n denote the set consisting of the first n rational numbers in the sequence. Suppose that U_p is defined for all rational numbers p in a set P_n, then $p < q$ implies that the closure of U_p is a subset of U_q. If r is the next rational number in the sequence; we define U_r: the set $P_{n+1} = P_n \cup \{r\}$ is a finite subset of the unit interval and has a simple ordering induced by the ordering of the real line. In a finite simply ordered set, every element, other than the largest and smallest, has an immediate predecessor and and immediate successor. 0 is the smallest and 1 is the largest element of the simply ordered set P_{n+1}, and $r \notin \{0, 1\}$. So r has an immediate predecessor $p \in P_{n+1}$ and an immediate successor $q \in P_{n+1}$. The sets U_p and U_q are already defined, and the closure of U_p is contained in U_q by the induction hypothesis. Because X is normal, we can find an open set U_r such that the closure of U_p is contained in U_r and the closure of U_r is contained in U_q. Now the induction condition holds for every pair of elements of P_{n+1}.

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