ENTRY POTENTIAL THEORY

[ENTRY POTENTIAL THEORY] Authors: Oliver Knill: jan 2003 Literature: "T. Ransford","Potential theory in the complex plane".

Analytic

[Analytic] Let $D \subset C$ be an open set. A continuous function $f : D \to C$ is called analytic in D, if for all $z \in D$ the complex partial derivative

$$\frac{\partial f}{\partial z} := \lim_{|h| \to 0} \frac{1}{h}(f(z+h) - f(z))$$

exists and is finite. Analytic functions are also called holomorphic. Properties: the sum and the product of analytic functions are analytic. If f_n is a sequence of analytic maps which converges uniformly on compact subsets of D to a function f, then f is analytic too.

complex partial derivative

Define the [complex partial derivative] of a complex function $f(z) = f(x + iy)$ in the complex plane is defined as $\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f$.

conformal map

A [conformal map] is a differentiable map from the complex plane to the complex plane which preserves angles.

- every conformal map which has continuous partial derivatives is analytic.
- An analytic function f is conformal at every point where its derivative $f'(z)$ is different from 0.

Dirichlet problem

Solution of the [Dirichlet problem]. If D is a regular domain in the complex plane and f is a continous function on the boundary of D, then there exists a unique harmonic function f on D such that $h(z) = f(z)$ for all boundary points of D.

Dirichlet problem

Let K be a compact subset of the complex plane. Let $P(K)$ the set of all Borel probability measure on K. A measure ν maximizing the potential theoretical energy in $P(K)$ is called an [equilibrium measure] of K. Properties:

- every compact K has an equilibrium measure.
- if K is not polar then the equilibrium measure is unique.

fine topology

The [fine topology] on the complex plane is defined as the coarsest topology on the plane which makes all subharmonic functions continuous.

Frostman's theorem

[Frostman's theorem]: If ν is the equilibrium measure on a compact set K, then the potential p_ν of ν is bounded below by $I(\nu)$ everywhere on C. Furthermore, $p_\nu = I(\nu)$ everywhere on K except on a F_sigma polar subset E of the boundary of K.

Frostman's theorem

A function h on the complex plane is called harmonic in a region D if it satisfies the mean value property on every disc contained in D.

harmonic measure

A [harmonic measure] w_D on a domain D is a function from D to the set of Borel probability measures on the boundary of D. The measure for z is defined as the functional $g \mapsto H_D(g)(z)$, where $H_D(g)$ is the Perron function of g on D.

- if the boundary of D is non-polar, there exists a unique harmonic measure for D.
- if $D = Im(z) < 0$, then $w_D(z, a, b) = \arg((z-b)/(z-a))/\pi$

Harnack inequality

The [Harnack inequality] assures that for any positive harmonic function h on the disc $D(w, R)$ and for any $r < R$ and $0 < t < 2Pi$

$$h(w)(R-r)/(R+r) \leq h(w + re^{it} \leq h(w)(R+r)/(R-r)$$

extended Liouville theorem

The [extended Liouville theorem]: if f is subharmonic on the complex plane $C - E$, where E is a closed polar set and f is bounded above then f is constant.

generalized Laplacian

The [generalized Laplacian] $\Delta(f)$ of a subharmonic function f on a domain D is the Radon measure μ on D defined as the linear functional $g \mapsto \int_D u \Delta g \; dA$. The Laplacian of a subharmonic function is also called the Riesz measure. The Laplacian is known to exist and is unique. If p_μ is the potential associated to μ, then $\Delta p_\mu = \mu$.

Hadamard's three circle theorem

[Hadamard's three circle theorem] assures that for any subharmonic function f on the annulus $\{r < |z| < R\}$ the function $M(f,r) = sup_{|z|=r} f(z)$ is an increasing convex function of $\log(r)$.

Jensen formula

[Jensen formula] If f is holomorphic in the disc $D = B(0,R), r < R$ and $a_1, ..., a_n$ are the zeros of f in the closure of D counted with multiplicity, then $\int_0^{2pi} \log|f(rexp(it))|\, dt = \log|f(0)| + N\log(r) - \sum_{j=1}^n log|a_j|$.

Jensen formula

If f is a subharmonic function in a neighborhood of a point z in the complex plane, then the limit $\lim_{rto0} M(f,r)/\log(r)$ exists and is called the [Lelong number] of f at z. Here $M(f,r) = \sup_{|z|=r} f(z)$.

hyperbolic domain

An open set in the extended complex plane is a [hyperbolic domain] if there is a subharmonic function on G that is bounded above and not constant on any component of G. A domain which is not hyperbolic is called a parabolic domain. Known facts:

- every bounded region is a hyperbolic domain (take f(z)=Re(z)).
- an open not connected set is hyperbolic.
- the complex plane is not a hyperbolic domain

Perron function

The [Perron function] for a domain D is defined as the functional assigning to a continous function g on the boundary of D the value $H_D(g)$, which is the supremum of all subharmonic functions u satisfying $\sup_{z \to w} u(z) \leq g(w)$.

potential

A subharmonic function f is called a [potential] if $f = p_\mu$, where $\mu = \Delta f$ is the Laplacian of f and $p_m u(z) = -\int_D \log|z-w| d\mu(w)/(2\pi)$ is the potential defined by μ.

logarithmic capacity

The [logarithmic capacity] of a subset E of the complex plane is defined as $c(E) = \sup_\mu \exp(-I(\mu))$, where $I(\mu)$ is the potential theoretical energy of μ and the supremum is taken over all Borel probability measures mu on C whose support is a compact subset of E. Known facts:

- $c(E) = 0$ if and only if E is polar.
- a disc of radius r has capacity r
- a line segment of length h has capacity $h/4$.
- if K has diameter d, then $c(K) \leq d/2$.
- if K has area A, then $c(A) \geq (A/\pi)^{1/2}$.

mean value property

The [mean value property] tells that if h is a harmonic function in the disc $D(w, R)$ and $0 < r < R$, then $h(w) = \int_0^{2pi} h(w + re^{it})dt/(2\pi)$.

polar set

A subset S of the complex plane is called a [polar set] if the potential theoretical energy $I(\mu)$ is $-\infty$ for every finite Borel measure μ with compact support $\text{supp}(\mu)$ in S. Properties of polar sets:

- every countable union of polar sets is polar.
- every polar set has Lebesgue measure zero.

Poisson integral formula

The [Poisson integral formula]: if h is harmonic on the disk $D(w, R')$, then for all $0 < r < R < R'$ and $0 < t < 2\pi$, $h(w + re^{it}) = \int_0^{2pi} h(w + Re^{is})(R^2 - r^2)/(R^2 - 2Rrcos(s - t) + r^2)ds/(2\pi)$

potential theoretical energy

The [potential theoretical energy] $I(\mu)$ of a finite Borel measure μ of compact support on the complex plane is defined as

$$I(\mu) = \int_C \int_C \int \int log|z - w| d\mu(w) d\mu(z) .$$

potential theoretical energy

A function f on an open subset U of the complex plane is called [subharmonic] if it is upper semicontinous and satisfies the local submean inequality. Examples:

- if g is holomorphic then $f = \log|g|$ is subharmonic
- if μ is a Borel measure of compact support, then $f(z) = \int \log|z-w|\, d\mu(w)$ is subharmonic.
- any harmonic function is subharmonic.
- if g is subharmonic, then $\exp(g)$ is subharmonic.

regular

A boundary point w of a domain D is called [regular] if there exists a barrier at w. A barrier is a subharmonic function f defined in a neighborhood N of w which is negative on $D \cap N$ and such that $\lim_{z \to w} f(z) = 0$. It is known that z is a regular boundary point if and only if the complement of D is non-thin at z.

irregular

A boundary point w of a domain D is called [irregular] if it is not regular. It is known that if z has a neighborhood N such that N intersected with the boundary of D is polar, then z is irregular.

irregular

A domain D for which every point is regular is called a [regular domain]. For example, a simply connected domain D such that the complement of D in the Riemann sphere contains at least two points, is regular.

Riemann mapping Theorem

The [Riemann mapping Theorem]: if D is a simply connected proper subdomain of the complex plane, there exists a conformal map of D onto the unit disc.

Riesz decomposition theorem

The [Riesz decomposition theorem] tells that every subharmonic function f can be written as $f = p_\mu + h$, where $\mu = \Delta f$ is the Laplacian of f, $2\pi p_\mu$ is the potential of μ and where h is harmonic.

submean inequality

The local [submean inequality] for a function in the complex plane tells that there exists $R > 0$ such that for all $0 < r < R$ one has

$$f(w) \leq \int_0^{2pi} f(w + re^{it})\, dt/(2pi) \, .$$

submean inequality

Let f be a subharmonic function on a domain D. The [maximum principle] says that if f attains a global maximum in the interior of D then f is constant.

thin set

A subset S of the complex plane is called a [thin set] if for all w in the closure of $S - w$ and all subharmonic functions f, $\limsup_{z \to w} f(z) = f(w)$. Examples:

- every single point in the interiour of S is thin.
- F_σ polar sets S are thin at every point.
- connected sets of cardinality larger than 1 are non-thin at every point of their closure
- A domain S is thin at a point $z \in S$ if and only if z is regular.

Wiener criterion

The [Wiener criterion] gives a necessary and sufficient condition for a set S to be thin at a point w. Let S be a F_σ subset of C and let w be in S. Let $a < 1$ and define $S_n = zinS, a^n < |z - w| < a^(n-1)$. The criterion says that S is thin at w if and only if $\sum_{n \geq 1} n/log(2/c(S_n)) < infinity$, where

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