ENTRY LINEAR ALGEBRA

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adjacency matrix

The [adjacency matrix] of a graph is a matrix A_{ij}, where $A_{ij} = 1$ whenever there is an edge from node i to node j in the graph. Otherwise, $A_{ij} = 0$. Example: the graph with three nodes with the shape of a V has the adjacency matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, where node 2 is conneced to both node 1 and 3 and node 1 and 3 are not connected to each other.

affine transformation

An [affine transformation] is the composition of a linear transformation with a shift like for example: $T(x, y) = (2x + y, 3x + 4y) + (2, 3)$.

Algebra

[Algebra] was originally the art of solving equations and systems of equations. The word comves from the Arabic "al-jabr" meaning "restauration". The term was used by Mohammed al-Khowarizmi, who worked in Bhagdad.

algebraic multiplicity

The [algebraic multiplicity] of a root y of a polynomial p is the maximal integer k for which $p(x) = (x-y)^k q(x)$. The algebraic multiplicity is bigger or equal than the geometric multiplicity.

angle

The [angle] between two vectors v and w is $\arccos((x \cdot y)/(||x||||y||))$, where $x\dot{y}$ is the dot product between x and y and $||x|| = \sqrt{x \cdot x}$ is the length of x. The inverse of cos gives two angles in $[0, 2\pi]$. One usually choses the smaller angle.

argument

The [argument] of a complex number $z = x + iy$ is ϕ if $z = re^{i\phi}$. The argument is determined only up to addition of 2π. It can be determined as $\phi = \arctan(y/x) + A$, where $A = 0$ if $x > 0$ or $x = 0, y > 0$ and $A = \pi$ if $x < 0$ or $x = 0$ and $y < 0$. For example, $\arg(i) = \pi/2$ and $\arg(-i) = 3pi/2$. The argument is the imaginary part of $\log(z)$ because $\log(re^{iphi}) = \log(r) + i\phi$.

associative law

The [associative law] is $(AB)C = A(BC)$. It is an identity which some mathematical operations satisfy. For example, the matrix multiplication satisfies the associative law. One says also, that the operation is associative. An example of a product which is not associative is the cross product $v \times w$: if i, j, k are the standard basis vectors, then $i \times (i \times j) = i \times k = -j$ and $(i \times i) \times j = 0 \times j = 0$.

augmented matrix

The [augmented matrix] of a linear equation $Ax = b$ is the $n \times (n+1)$ matrix $\begin{pmatrix} A & b \end{pmatrix}$. One considers the augmented matrix when solving a linear system $Ax = b$. The reduced row echelon form rref $\begin{pmatrix} A & b \end{pmatrix}$ contains the solution vector x in the last column, if a solution exists. More generally, a matrix which contains a given matrix as a submatrix is called an augmented matrix.

basis

A [basis] of a linear space is a finite set of vectors $v_1, ..., v_n$, which are linearly independent and which span the linear space. If the basis contains n vectors, the vector space has dimension n.

basis theorem

The [basis theorem] states that d linearly independent vectors in a vector space of dimension d forms a basis.

block matrix

A [block matrix] is a matrix A, where the only non-zero elements are contained in a sequence of smaller square matrices arranged along the main diagonal of A. Such matrices are also called block diagonal matrices. The matrix $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \end{pmatrix}$. is an example of a block-diagonal matrix containing a two $2x2$, and a $1x1$ matrix in its diagonal.

Cauchy-Schwarz inequality

The [Cauchy-Schwarz inequality] tells that $|x \cdot y|$ is smaller or equal to $||x||\ ||y||$. Equality holds if and only if x and y are parallel vectors.

Cayley-Hamilton theorem

The [Cayley-Hamilton theorem] assures that every square matrix A satisfies $p(A) = 0$, where $p(x) = \det(A - x)$ is the characteristic polynomial of A and the right hand side 0 is the zero matrix.

change of basis

A [change of basis] from an old basis
v_j to a new basis
w_j is described by an invertible matrix S which relates the coordinates $(a_1, ..., a_n)$ of a vector $a = \sum_i a_i v_i$ in the old v-basis with the coordinates $(b_1, ..., b_n)$ of the same vector $b = \sum_i b_i w_i$ in the new w-basis. The relation of the coordinates is $b = Sa$ In that case, one has $v_j = \sum_j S^T_{ij} w_j$, where S^T is the transpose of S. For example if $v_1 = (1,0), v_2 = (0,1), w_1 = (3,4) w_2 = (2,3)$, then $a = (a_1, a_2) = (5,7)$ in the v-basis has the coordinates $b = (b_1, b_2) = (1,1)$ in the w-basis. With $S = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $S^T = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ we have $b = Sa$ and $w_1 = 3v_1 + 4v_2, w_2 = 2v_1 + 3v_2$.

characteristic matrix

The [characteristic matrix] of a square matrix A is the matrix $A(x) = (xI - A)$, where I is the identity matrix. The characteristic matrix is a function of the free variable x.

characteristic polynomial

The [characteristic polynomial] of a matrix A is the polynomial $p(x) = \det(xI - A)$, where I is the identity matrix. It has the form $p(x) = x^n - \mathrm{tr}(A) x^(n-1) + ... + (-1)^n \det(A)$ where tr(A) is the trace of A and $\det(A)$ is the determinant of the matrix A. The eigenvalues of A are the roots of the characteristic polynomial of A.

Cholesky factoriztion

The [Cholesky factoriztion] of a symmetric and positive definite matrix A is $A = R^T R$, where R is upper triangular with positive diagonal entries.

circulant matrix

A [circulant matrix] is a square matrix, where the entries in each diagonal are constant. If S is the shift matrix which has 1 in the side diagonal and 0 everywhere else like in the 3x3 case: $S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, then a circular matrix can be written as $A = a_0 + a_1 S + ... + a_{n-1} S^{n-1}$. A general $3x3$ circulant matrix has the form $A = a + bS + cS^2$ which is $S = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$.

classical adjoint

The [classical adjoint] $\mathrm{adj}(A)$ of a $n \times n$ matrix A is the $n \times n$ matrix whose entry a_{ij} is $a_{ij} = (-1)^(i+j) \det(A_{ji})$, where A_{ji} is a minor of A. The classical adjoint plays a role in Cramer's rule $A^(- 1) = \mathrm{adj}(A)/\det(A)$. The name "adjoint" comes from the fact that we have a change indices like in the adjoint. However, the classical adjoint has nothing to do with the adjoint.

codomain

The [codomain] of a linear transformation $T : X \to Y$ is the target space Y. The name has its origin from naming X the domain of A.

cofactor

A [cofactor] C_{ij} of a $n \times n$ matrix A is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A and multiplying the result with $(-1)^{i+j}$.

coefficient

A [coefficient] of a matrix A is an entry A_{ij} in the i'th row and the j'th column. For a real matrix, all entries are real numbers, for a complex matrix, the entries can be complex numbers.

column

A [column] of a matrix is one of the vectors $\begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \dots \\ A_{m1} \end{pmatrix}, \begin{pmatrix} A_{1n} \\ A_{2n} \\ A_{3n} \\ \dots \\ A_{mn} \end{pmatrix}$ of a $m \times n$ matrix A. Column vectors are in the image of the transformation $x \mapsto A(x)$.

column

The matrix A defining a linear equation Ax=b or

$$\begin{aligned} A_{11}x_1 + \dots A_{1n}x_n &= b_1 \\ \dots &= \dots \\ A_{m1}x_1 + \dots A_{mn}x_n &= b_m \end{aligned}$$

is called the [coefficient matrix] of the system. The augmented matrix is the $m \times (n+1)$ matrix $\begin{pmatrix} A & b \end{pmatrix}$, where b forms an additional column.

column picture

The [column picture] of a linear equation $Ax = b$ is that the vector b becomes a linear combination of the columns of A. The linear equation is solvable if the vector b is in the column space of A.

column space

The [column space] of a matrix A is the linear space spanned by the columns of A.

commuting matrices

Two [commuting matrices] A, B satisfy $AB = BA$. In that case, if A is diagonalizable, then also B is diagonalizable and both A and B share the same n eigenvectors.

commutative law

The [commutative law] $A * B = B * A$ for some operation $*$ is an identity which holds for certain operations like the addition of matrices. Other operations like the multiplication of matrices does not satisfy the commutative law. One says: matrix multiplication is not commutative.

complex conjugate

The [complex conjugate] of a complex number $z = x + iy$ is the complex number $x - iy$. It has the same length $|z|$ as z.

Complex numbers

[Complex numbers] form an extension of the real numbers. They are obtained by introducing $i = (-1)^{1/2}$ and extending the rules of addition $(a + ib) + (c + id) = (a + c) + i(b + d)$ and multiplication $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$. The absolute value $r = |x + iy|$ is the length of the vector (a, b). The argument of z, $\phi = \arg(z)$ is defined as the angle in $[0, 2\pi)$ between the x axes and the vector (a, b). Using these polar coordinates one can see the Euler identity $z = r\exp(i\phi) = r\cos(\phi) + ir\sin(\phi)$.

consistent

A system of linear equations $Ax = b$ is called [consistent], if there exists for every vector b a solution vector x to the equation Ax=b. If the system has no solution, the system is called inconsistent.

continuous dynamical system

A [continuous dynamical system] is defined by an ordinary differential equation $d/dtu = f(u)$ where $u = u(t)$ is a vector valued function and $f(u)$ is a vector field. If $f(u)$ is linear, the equation has the form $d/dtu = Au$. The name "continuous" comes from the fact that the time variable t is taken in the continuum. This distinguishes the system from discrete dynamical systems $u(t+1) = f(u(t))$ determined by a map f and where t is an integer. For linear continuous dynamical systems, the origin 0 is invariant. The origin is called asymptotically stable if $x(t) \to 0$ for all initial conditions $x(0)$. For continuous dynamical systems $u_t = Au$, this is equivalent with the requirement that all eigenvalues of A have a negative real part. In two dimensions, where the trace and the determinant determine the eigenvalues, linear stability is characterized by $\det(A) > 0, \mathrm{tr}(A) < 0$ (stability quadrant).

covariance matrix

A [covariance matrix] A of two finite dimensional random variables x, y with expectation $E[x] = E[y] = 0$ is defined as $A_{ij} = E[x_i y_j]$, where $E[x] = (x_1 + ... + x_n)/n$ is the mean or expectation of x. The covariance matrix is always symmetric. If the covariance matrix is diagonal, the random variables x, y are called uncorrelated.

Cramer's rule

[Cramer's rule] tells that a solution x of a linear equation $Ax = b$ can be obtained as $x_i = \det(A_i)/\det(A)$, where A_i is the matrix obtained by replacing the column i of A with the vector b.

de Moivre formula

The [de Moivre formula] is $(\cos(x)+i\sin(x))^n = \cos(nx)+i\sin(nx)$. It is useful to derive trigonometric identities like $cos(x)^3 - 3\sin(x)^2\cos(x) = \cos(3x)$.

determinant

The [determinant] of a $n \times n$ square matrix A is the sum over all products $A[1, \pi(1)]...A[n, \pi(n)](-1)^\pi$, where π runs over all permutations of $\{1, 2, ..., n\}$ and $(-1)^\pi$ is the sign of the permutation π. Example: for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant is $\det(A) = ad - bc$. Example: For a $3x3$ matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the determinant is $\det(A) = aei + bfg + cdh - ceg - bfg - cdh$. Properties of the determinant are $\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det(A)$, $\det(A^{-1}) = 1/\det(A)$.

differential equation

A [differential equation] is an equation for a function f in one or several variables which involves derivatives with respect to these variables. An ordinary differential equation is a differential equation, where derivatives appear only with respect to one variable. By adding new variables if necessary (for example for t, or derivatives $u_t, u_t t$ etc, one can write an ordinary differential equation always in the form $x_t = f(x)$.

dilation

A [dilation] is a linear transformation $x \to bx$. Dilations scale each vector v by the factor b but leave the direction of v invariant.

dimension

The [dimension] of a vector space X is the number of basis vectors in a basis of X.

distributive law

The [distributive law] is $A * (B + C) = A * B + A * C$. The set of matrices with matrix multiplication * and addition + is an example where the distributive law applies.

dot product

The [dot product] $v \cdot w$ of two vectors v and w is the sum of the products $v_i w_i$ of their components v_i, w_i. For complex vectors, the dot product is defined as $\sum_i \overline{v}_i w_i$. Examples:

- $(3, 2, 1) \cdot (1, 2, -1) = 6$.
- if $v \cdot w = 0$, then the vectors are orthogonal.
- the length of the vector $|v|$ is the square root of $v \cdot v$.
- $v \cdot w = |v||w| \cos(\alpha)$, where α is the angle between v and w.
- if A, B are two $n \times n$ matrices, then $(AB)_{ij}$ is the dot product of the i'th row of A with the j'th column of B

echelon matrix

The [echelon matrix] of a matrix A is a matrix $\mathrm{rref}(A)$, where the pivot in each row comes after the pivot in the previous row. The pivot is the first nonzero entry in each row. The echelon matrix is also called a matrix in reduced row echelon form.

eigenbasis

An [eigenbasis] to a matrix A is a basis which consists of eigenvectors of A.

eigenvalue

An [eigenvalue] λ of a matrix A is a number for which there exists a vector v such that $Av = \lambda v$. Example: $A = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}$ has the eigenvector $v = (0, 1)$ with eigenvalue $x = 4$.

eigenvector

An [eigenvector] v of a matrix A is a nonzero vector v for which $Av = \lambda v$ with some number λ (called eigenvalue). Example: $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ has the eigenvector $v = (1, 1)$ with eigenvalue $\lambda = 0$.

Elimination

[Elimination] is a process which reduces a matrix A to its echelon matrix $\text{rref}(A)$. See row reduced echelon form.

ellipsoid

An [ellipsoid] can be written as the set of points x which satisfy $x\dot{A}x = 1$, where A is a positive definit matrix. The axes v_i of the ellipse are the eigenvectors of A have the length $1/x_i$, where x_i are the eigenvalues of A.

entry

An [entry] or coefficient of a matrix is the number or the variable $A(i,j)$ of a matrix.

expansion factor

The [expansion factor] of a linear map is the absolute value of the determinant of A. It is the volume of the parallelepiped obtained as the image of the unit cube under A.

exponential

The [exponential] $\exp(A)$ of a matrix A is defined as the sum $\exp(A) = 1 + A + A^2/2! + A^3/3! + ...$ The linear system of differential equations $x' = Ax$ for $x(t)$ has the solution $x(t) = \exp(At)x(0)$.

factorization

The [factorization] of a polynomial $p(x)$ is the representation $p(x) = a(\lambda_1 - x)...(\lambda_n - x)$, where $lambda_i$ are the n roots of the polynomials whose existence is assured by the fundamental theorem of algebra.

Fourier coefficients

The [Fourier coefficients] of a 2π periodic function $f(x)$ on $[-\pi,\pi]$ is $c_n = (1/2\pi)\int_{-pi}^{p} if(x)exp(-inx)$. One has $f(x) = \sum_n c_n \exp(inx)$. By writing $f(x) = g(x) + h(x)$, where $g(x) = [f(x) + f(-x)]/2$ is even and $h(x) = [f(x) - f(-x)]/2$ is odd one can obtain real versions: the even function can be written as a cos-series $g(x) = sum_{n=0}^{\infty} a_n \cos(nx)$, where $a_n = (2/\pi)\int_0^{\pi} g(x)\cos(nx)dx$ for $n > 0$ and $a_0 = (1/\pi)\int_0^{\pi} g(x)\, dx$. The odd function can be written as the sin-series $h(x) = sum_{n=1}^{\infty} b_n \sin(nx)$, where $b_n = (2/\pi)\int_0^{\pi} h(x)\sin(nx)dx$. The complex Fourier coefficients c_n are coordinates of $f(x)$ with respect to the orthonormal basis $\exp(inx)$. The real Fourier coefficients are the coordinates of $f(x)$ with respect to orthogonal basis $1, \cos(nx), \sin(nx), n > 0$. The [Fourier series] of a function f is $f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(inx)$ or $f(x) = \sum_n a_n \cos(nx)$ for even functions or $f(x) = sum_{n=1}^{\infty} b_n \sin(nx)$ for odd functions.

fundamental theorem of algebra

The [fundamental theorem of algebra] states that a polynomial $p(x) = x^n + ...a_1x + a_0$ of degree n has exactly n roots.

Gauss Jordan elimination

The [Gauss Jordan elimination] is a method for solving linear equations. It was already known by the Chinese 2000 years ago. Gauss called it "eliminatio vulgaris". The method does linear combinations of the rows of a $n \times (n+1)$ matrix until the system is solved. Example:

$$\left| \begin{array}{ccc} \text{2 x + 4y} & = & 2 \\ \text{3 x + y} & = & 12 \end{array} \right|$$

$$\left| \begin{array}{ccc} \text{x + 2y} & = & 1 \\ \text{3 x + y} & = & 13 \end{array} \right|$$

$$\left| \begin{array}{ccc} \text{x + 2y} & = & 1 \\ \text{- 5y} & = & 10 \end{array} \right|$$

$$\left| \begin{array}{ccc} \text{x + 2y} & = & 1 \\ \text{y} & = & \text{-2} \end{array} \right|$$

$$\left| \begin{array}{ccc} \text{x} & = & 5 \\ \text{y} & = & \text{-2} \end{array} \right|$$

First, the top equation was scaled, then three times the first equation was subtracted from the second equation. Then the the second equation was scaled. Finally, twice the the second equation was subtracted from the first.

geometric multiplicity

The [geometric multiplicity] of an eigenvalue λ is the dimension of $\ker(\lambda - A)$. The geometric multiplicity is smaller or equal to the algebraic multiplicity.

Gibbs phenomenon

The [Gibbs phenomenon] describes the error when doing a Fourier approximation of the discontinuous Heavyside function $f(x) = \begin{cases} -1 \text{if} -\pi \leq x \leq 0 \\ 1 \text{if} 0 < x < \pi \end{cases}$ The Fourier approximation is $s_n(x) = (4/\pi)\sum_{k=1}^n \sin((2k-1)t)/(2k-1)$. The derivative d/dxs_n can be computed: (differentiate and sum up the geometric series): $(2/\pi)\sin(2nx)/\sin(x)$ which vanishes at $x = \pm\pi/2n$. These are the extrema for s_n. Now, $s_n(\pi/2n) = (4/\pi)\sum \sin((2k-1)(\pi/2n))/((2k-1)(\pi/2n))$ is a Riemann sum approximation for $(2/\pi)\int_0^\pi \sin(t)/tdt = \pi(1-\pi^2/(3!3)+\pi^4/(5!5)-...) = 1.1793...$ This overshoot is called the Gibbs phenomenon. It was first discoverd by Wilbraham in 1848 then by Gibbs in 1899. The human eye can recognize the Gibbs phenomenon as "ghosts" on a TV screen, unless it is corrected for.

Gramm-Schmidt process

The [Gramm-Schmidt process] is an algorithm which constructs from a basis $v_1, ..., v_n$ an orthonormal basis $w_1, ..., w_n$. The procedure goes by induction: if $w_1, ..., w_{k-1}$ are orthonormal, then the next vector w_k is $w_k = u_k/||u_k||$, where $u_k = v_k - (w_1 \cdot v_k)w_1 - (w_2 \cdot v_k)w_2 \cdots - (w_{k-1} \cdot v + k)w_{k-1}$.

graph

A [graph] is a set of n nodes connected by m edges. It is completely defined by its adjacency matrix. In a directed graph, the nodes are oriented. Examples:

- a complete graph has all nodes connected. There are $n(n-1)/2$ edges. Its adjacency matrix is $E - I$, where E is the $n \times n$ matrix with all entries equal to 1 and I is the identity matrix.
- a tree has $m = n - 1$ edges and no closed loops. An example is the graph with the adjecency matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.
- The directed graph with two edges $1 \to 2$ has the adjacency matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Hankel matrix

A [Hankel matrix] is a square matrix A, where the entries are constant along the antidiagonal. In other words, each entry A_{ij} depends only on $i + j$. A general 3×3 Hankel matrix is of the form $A = \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix}$.

heat equation

The [heat equation] is the linear partial differential equation $u_t = \mu u_{xx}$. The heat equation on a finite interval $[0, \pi]$ with boundary conditions $u(0, t) = 0$, $u(\pi, t) = 0$ and initial conditions $u(x, 0) = f(x)$ can be solved with the Fourier series $u(x, t) = \sum_{n>0} a_n \sin(nx) e^{-\mu n^2 t}$, where $a_n = (2/\pi) \int_0^p i f(x) \sin(nx) \, dx$ are the Fourier coefficients.

Hermitian matrix

A [Hermitian matrix] satisfies $A^* = \overline{A}^T = A$, where A^T is the transpose of A and $\overline{A}$ is the complex conjugate matrix, where all entries are replaced by their complex conjugates.

Hessenberg matrix

A [Hessenberg matrix] A is an upper triangular matrix with only one extra nonzero adjacent diagonal below the diagonal. Example: a general 3×3 Hessenberg matrix is $A = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & g & h \end{pmatrix}$.

Hilbert matrix

A [Hilbert matrix] is a symmetric square matrix, where $A_{ij} = 1/(i+j-1)$. It is an example of a Hankel matrix and positive definit. Hilbert matrices are examples of matrices which are difficult to invert, because their determinant is small. For example, for $n = 3$, the Hilbert matrix $A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$ has determinant 1/2160. A [hyperplane] in n-dimensional space V is a $(n-1)$-dimensional linear subspace of V.

identity matrix

The [identity matrix] is the matrix I which has 1 in the diagonal and zero everywhere else. The identity matrix I satisfies $IA = A$ for any matrix A. The $3x3$ identity matrix is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

incidence matrix

The [incidence matrix] of a directed graph with n nodes and m edges is a $m \times n$ matrix which has a row for each edge connecting nodes i and j with entries -1 and 1 in columns i, j. Example. The directed graph $1 \Rightarrow 2 \Leftarrow 3$, $1 \rightarrow 4$ with 3 edges and 4 nodes has the 3×4 incidence matrix $A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.

inconsistent

A system of linear equations $Ax = b$ is called [inconsistent] if the system has no solutions.

indefinite matrix

An [indefinite matrix] is a matrix, whith eigenvalues of different sign. A positive definite matrix is an example of a matrix which is not indefinite.

Independent vectors

[Independent vectors]. If no linear combination $a_1v_1 + ... + a_nv_n$ is zero unless all a_i are zero, then the vectors $v_1, ..., v_n$ are called independent. If A is the matrix which contains the vectors v_i as columns, then the kernel of A is trivial. A basis consists of independent vectors.

Independent vectors

The linear transformation corresponding to the identity matrix is called the [identity transformation].

image

The [image] of a linear transformation $T : X \to Y, T(x) = Ax$ is the subset of all vectors $y = Ax, xinX$ in Y. The image is denoted by $\mathrm{im}(T)$ or $\mathrm{im}(A)$ and is a subset of the codomain Y of T. The image is also called the range. The dimension of the image of T is equal to the rank of A and the dimension satisfies $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n$, where n is the dimension of the linear space X.

index

The [index] of a linear map T is defined as $\mathrm{ind}(A) = \dim(\ker A) - \dim(\mathrm{coker} A)$, where $\mathrm{coker}(A)$ is the orthogonal complement of the image of A. Examples are:

- The index of a $n \times n$ matrix A is $\dim(\ker A) - \dim(\mathrm{coker} A) = 0$.
- The index of the differential operator $Df = f'$ acting on smooth functions on the real line is $1 - 0 = 1$ because D has a one dimensional kernel (the constant functions) and a zero dimensional cokernel (all functions can be obtained as the image of D). The index of D^n is n.
- The index of the differential operator $Df = f'$ acting on smooth functions on the circle is $1 - 1 = 0$ because D has a one dimensional kernel (the constant functions) and a one-dimensional cokernel (the constant functions, one can not find a periodic function g such that g'=1).
- The Atiyah-Singer index theorem relates topological properties of a surface M with the index of a "Dirac operator" T on it. The previous two examples exemply that. $T = D$ is a Dirac operator and the topology of the circle or the line are different.

inverse

The [inverse] of a square matrix A is a matrix B satisfying $AB = I$ and $BA = I$ where I is the identity matrix. For example, the inverse of the transformation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the transformation $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} /(ad - bc)$.

invertible

A square matrix A is [invertible] if there exists a matrix B such that $AB = I$. A matrix A is invertible if and only if the determinant of A is different from zero.

Jordan normal form

The [Jordan normal form] $J = S^{-1}AS$ of a square matrix A is a block matrix $J = \mathrm{diag}(J_1, ..., J_k)$, where each block is of the form $J_k = x_k I_k + N_k$, where x_k is an eigenvalue of A, I_k is an identity matrix and N_k is a matrix with 1 in the first sidediagonal. If all eigenvalues of A are different, then the Jordan normal form is a diagonal matrix.

kernel

The [kernel] of a linear transformation $T : X \to Y$, $T(x) = Ax$ is the linear space $\{xinXsuchthatAx = 0\}$. The kernel is denoted by $\ker(T)$ or ker(A) and is a subset of the domain X of T. The dimension of the kernel $\dim(\ker(A))$ and the dimension of the image $\dim(\mathrm{im}(A))$ are related by $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n$, where n is the dimension of the linear space X. The kernel of a transformation is computed by building $\mathrm{rref}(A)$, the reduced row echelon form of A. Echelon = "series of steps". Every vector in the kernel of $\mathrm{rref}(A)$ is also in the kernel of A.

Leontief

Wassily [Leontief]. A Russian-born US economist who was working also at Harvard University. Leontief was a winner of the 1973 Economics Nobel prize for the development of the input-output method and for its application to important economic problems. Linear algebra students find the following problem in textbooks: two industries A and B produce output with value x and y (in millions of dollars). Assume that the consumer demand is a for the product of A and b for the product of B. Assume also an industry demand: p x is transfered from A to B, and q y is transfered from B to A. For which x and y are both the industry and conumer demand satisfied? The problem is equivalent to solving the linear system

$$\begin{aligned} x - qy &= a \\ -px + y &= b \end{aligned}$$

Laplace equation

The [Laplace equation] in a region G is the linear partial differential equation $u_{xx} + u_{yy} = 0$. A solution is determined if $u(x,y)$ is prescribed on the boundary of G. On the square $[0,\pi] \times [0,\pi]$ with boundary conditions 0 except at the side $y = \pi$, where one has $u(x,\pi) = f(x)$, one can find a solution via Fourier series: $u(x,y) = \sum_{n>0} a_n \sin(nx)\sinh(ny)/sinh(n\pi)$, where $a_n = (2/\pi)\int_0^\pi f(x)\sin(nx)\,dx$. The case with general boundary conditions can be solved by adding corresponding solutions $u(x,y), u(y,x), u(x,\pi-y), u(\pi-y,x)$ for the other 3 sides of the square.

Laplace expansion

The [Laplace expansion] is a formula for the determinant of A: $\det(A) = (-1)^{i+1}a_{i1}\det(A_{i1}) + ... + (-1)^{i+n}a_{in}\det(A_{in})$.

leading one

A [leading one] is an entry of a matrix in reduced row echelon form which is contained in a row with this element as the first nonzero entry.

leading variable

A [leading variable] is a variable which corresponds to a leading one in rref(A).

least-squares solution

A vector $x \in R^n$ is called a [least-squares solution] of the system $Ax = b$ where A is a $m \times n$ matrix, if $||b - Ay||$ is less or equal then $||b - Ax||$ for all $y \in R^n$. If x is the least-squares solution of $Ax = b$ then Ax is the orthogonal projection of b onto the image im(A). The explicit formula is $x = (A^T A)^(-1) A^T b$ and derived from that $A^T(Ax - b) = 0$ which itself just means that $Ax - b$ is orthogonal to the image of A.

length

The [length] of a vector v is $||v|| = (v \cdot v)^{1/2} = (v_1^2 + ... + v_n^2)^(1/2)$. The length of a complex number $x + iy$ is the length of (x, y). The length of a vector depends on the basis, usually it is understood with respect to the standard basis.

linear combination

A [linear combination] of n vectors $v_1, ..., v_n$ is a vector $a_1 v_1 + ... + a_n v_n$.

Linearly dependent vectors

[Linearly dependent vectors]. If there exist $a_1, ..., a_n$ which are not all zero such that $a_1 v_1 + ... + a_n v_n = 0$, then the vectors $v_1, ..., v_n$ are called linearly dependent.

Linearity

[Linearity] is a property of maps between linear spaces: it means that lines are maped into lines and the image of the sum of two vectors is the same as sum of the images. For example: $T(x, y, z) = (2x + z, y - x)$ is linear. $T(x, y) = (x^2 - y, x)$ is nonlinear.

linear dynamical system

A [linear dynamical system] is defined by a linear map $x \mapsto Ax$. The orbits of the dynamical system are $x, Ax, A^2x, ...$

linear space

A [linear space] is the same as a vector space. It is a set which is closed under addition and multiplication with real numbers.

linear combination

A sum $a_1 v_1 + ... + a_n v_n$ is called a [linear combination] of the vectors $v_1, ..., v_n$.

linear subspace

A [linear subspace] of a vector space V is a subset of V which is also a vector space. In particular, it is closed under addition, scalar multiplication and contains a neutral element.

linear system of equations

A [linear system of equations] is an equation of the form Ax=b, where A is a $m \times n$ matrix, x is a n-vector and b is a m-vector. There are three possibilities:

- consistent with one solution: no row vector $(0...0\|1)$ in rref$(A|b)$. There is exactly one solution if there is a leading one in each column of rref(A).
- consistent with infinitely many solutions: there are columns with no leading one.
- Inconsistent with no solutions: there is a row $(0...0\|1)$ in rref$(A\|b)$.

logarithm

The [logarithm] $\log(z)$ of a complex number $z = x + iy \neq 0$ is defined as $\log|z| + i\arg(z)$, where $\arg(z)$ is the argument of z. The imaginary part of the logarithm is only defined up to a multiple of 2π.

Markov matrix

A [Markov matrix] is a square matrix, where all entries are nonnegative and the sum of each column is 1. One of the eigenvalues of a Markov matrix is 1 because A^T has the eigenvector $(1, 1, ..., 1)$. If all entries of a Markov matrix are positive, then $A^k v$ converges to the eigenvector v with eigenvalue 1. This vector is called the "steady state" vector.

matrix

A [matrix] is a rectangular array of numbers. The following 3×4 matrix for example consists of three rows and four columns: $A = \begin{pmatrix} 2 & 8 & 4 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 2 \end{pmatrix}$. The first index addresses the row, the second the column of the matrix. A $n \times m$ matrix maps the m-dimensional space to the n-dimensional space.

Matrix multiplication

[Matrix multiplication] is an operation defined between a $(n \times m)$ matrix A and a $(m \times p)$ matrix B. $(AB)_{ij}$ is the dot product between the i'th row of A with the j'th column of B. Example: $(n = 2, m = 3, p = 4)$
$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 & 2 \\ 3 & 4 & 1 & 1 \end{pmatrix}$.

minor

A [minor] of a matrix A is a matrix $A(i,j)$ which is obtained from A by deleting row i and column j.

nilpotent matrix

A [nilpotent matrix] is a matrix A for which some power A^k is the zero matrix. A nilpotent matrix has only zero eigenvalues. he matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ for example satisfies $A^3 = 0$ and is therefore nilpotent.

non-leading coefficient

A [non-leading coefficient] is an entry in the row reduced echelon form of a matrix A which is nonzero and which comes after the leading 1. The relevance of this definition comes from the fact that the number of columns with non-leading coefficients is the dimension of the kernel of the map.

normal equation

The [normal equation] to the linear equation Ax=b is the consistent system $A^T Ax = A^T b$.

normal matrix

A matrix A is called a [normal matrix], if $AA^T = A^T A$. A normal matrix has orthonormal possibly complex eigenvectors.

null space

The [null space] of a matrix A is the same as the kernal of A. It is spanned by the solutions $Av = 0$. The dimension of the null space is $n - r$, where n is the number of columns of A and r is the rank of A.

ordinary differential equation

An [ordinary differential equation] is a differential equation, where derivatives appear only with respect to one variable. By adding new variables if necessary (for example for t, or derivatives u_t, u_{tt} etc. one can write such an equation always in the form $x_t = f(x)$. An ordinary differential equation defines a continuous dynamical system. The initial condition $x(0)$ determines the trajectories $x(t)$. An ordinary differential equation of the form $u_t = Au$, where A is a matrix is called a linear ordinary differential equation.

orthogonal

Two vectors v,w are [orthogonal] if their dot product $v \cdot w$ vanishes.

orthogonal basis

An [orthogonal basis] is a basis such that all vectors in the basis are orthogonal.

orthonormal basis

An [orthonormal basis] is a basis such that all vectors are orthogonal and normed.

orthonormal complement

The [orthonormal complement] of a linear subspace V in R^n is the set of vectors which are orthogonal to V.

orthogonal projection

The [orthogonal projection] onto a linear space V is $\mathrm{proj}_V(x) = (x \cdot v_1)v_1 + ... + (x \cdot v_n)v_n$, where the v_j form an orthonormal basis in V. Despite the name, an orthogonal projection is not an orthogonal transformation. It has a kernel. In an eigenbasis, a projection has the form $(x, y) \to (x, 0)$.

Euclidean space

[Euclidean space] is the linear space of all vectors = $1 \times n$ matrices. R^0 is the space 0. The space R^1 is the real linear space of all real numbers, the R^2 is the plane, the R^3 the Euclidean three dimensional space.

parallel

Two vectors v and w are called [parallel] if v both are nonzero and one is a multiple of the other.

parallelepiped

A set in R^n is a [parallelepiped] E if it is the linear image $A(Q)$ of the unit cube Q. The volume of a n-dimensional parallelepiped E in R^n satisfies $\mathrm{vol}(E) = |\det(A)|$, in general, $\mathrm{vol}(E) = (\det(A^T A))^{1/2}$.

partial differential equation

A [partial differential equation] (PDE) is an equation for a multi-variable function which involves partial derivatives. It is called linear if $(u+v)$ and rv are solutions whenever u and v are solutions. Examples of linear PDEs:

$u_t = cu_x$	transport equation
$u_t = bu_{xx}$	heat equation
$u_t t = au_{xx}$	wave equation
$u_x x + u_y y = 0$	Laplace equation
$u_x x + u_y y = f(x,y)$	Poisson equation
$ihu_t = -u_x x + V(x)$	Schroedinger equation

Examples of nonlinear PDE:

$u_t + uu_x = au_{xx}$	Burger equation
$u_t + uu_x = -u_{xxx}$	Korteweg de Vries equation
$u_{tt} - u_{xx} = \sin(x)$	Sine Gordon equation
$u_{tt} - u_{xx} = f(x)$	Nonlinear wave equation
$ihu_t = -u_x x - \|x\|^2 x$	Nonlinear Schroedinger equation
$u_t + u_x(x,t)^2/2 + V(x) = 0$	Hamilton Jacobi equation

permutation matrix

A [permutation matrix] A is a square matrix with entries $A_{ij} = I_{i\pi(j)}$ where π is a permutation of $1, ..., n$ and where I is the identiy matrix. There are $n!$ permutation matrices. Example: for $n = 3$ the permutation $\pi(1,2,3) = (2,1,3)$ defines the permutation matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

pivot

A [pivot] d is the first nonzero diagonal entry when a row is used in Gaussian elimination.

pivot column

A column of a matrix is called a [pivot column] if the corresponding column of $\text{rref}(A)$ contains a leading one. The pivot columns are important because they form a basis for the image of A.

polar decomposition

The [polar decomposition] of a matrix A is $A = OB$, where O is orthogonal and where B is positive semidefinit.

positive definite matrix

A [positive definite matrix] is a symmetric matrix which satisfies $v \cdot Av > 0$, for every nonzero vector v.

power

The n'th [power] of a matrix A is defined as $A^n = AA^(n-1) = AA....A$. The eigenvalues of A^n are λ_i^n, where λ_i are the eigenvalues of A.

QR decomposition

The [QR decomposition] of a matrix A is obtained during the Gramm-Schmitt orthogonalization process. It is $A = QR$, where Q is an orthogonal matrix and where R is an upper triangular matrix.

rank

The [rank] of a linear matrix A is the set of leading 1's in the matrix rref(A).

orientation

The [orientation] of n vectors $v_1, ..., v_n$ in the n-dimensional Euclidean space is defined as the sign of $\det(A)$, where A is the matrix with v_i in the columns.

orthogonal

A square matrix A is [orthogonal] if it preserves length: $||Av|| = ||v||$ for all vectors v.

perpendicular

Two vectors v and w are called [perpendicular] if their dot product vanishes: $v \cdot w = 0$. A synonym of perpendicular is orthogonal.

projection matrix

A [projection matrix] is a matrix P which satisfies $P^2 = P = P^T$. It has eigenvalues 1 or 0. The image is a linear subspace S. The vectors in S are eigenvectors to the eigenvalues 1. The vectors in the orthogonal complement of S are eigenvectors to the eigenvalue 0. If A is the matrix which contains the basis of S as the columns, then $P = A(A^TA)^{-1}A^T$ is the projection onto S.

pseudoinverse

The [pseudoinverse] of a $(m \times n)$ matrix A is the $(n \times m)$ matrix A^+ that maps the image of A to the image of A^T. The kernel of A^+ is the kernel of A^T and the rank of A^+ is equal to the rank of A. A^+A is the projection on the image of A^T and AA^+ is the projection on the image of A. Especially, if A is an invertible $(n \times n)$ matrix, then A^+ is the inverse of A. The pseudoinverse is also called Moore-Penrose inverse.

rank

The [rank] of a matrix A is the dimension of the image of A.

Rayleigh quotient

The [Rayleigh quotient] of a symmetric matrix A is defined as the function $q(v) = (v \cdot Av)/(v \cdot v)$. The maximal value of $q(v)$ is the maximal eigenvalue of A and the minimal value of $q(v)$ is the minimal eigenvalue of A.

rotation

A [rotation] is a linear transformation which preserves the angle between two vectors as well as their lengths. A rotation in three dimensional space is determined by the axis of rotation as well as the rotation angle.

rotation matrix

A [rotation matrix] is the matrix belonging to a rotation. A rotation matrix is an example of an orthogonal matrix. Example: In two dimensions, a rotation matrix has the form $A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$.

rotation-dilation matrix

A [rotation-dilation matrix] is a 2x2 matrix of the form $A = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$. It has the eigenvalues $p \pm iq$. The action of A represents the complex multiplication with the complex number p+iq in the complex plane.

row

A [row] of a matrix is formed by horizontal lines $A_{1j}, j = 1, ..n$ of a $m \times n$ matrix A.

shear

A [shear] is a linear transformation in the plane which has in a suitable basis the form $T(x,y) = (x, y + ax)$. More generally, in n dimensions, one can define as shear along a m-dimensional plane. If a basis is chosen so that the plane has the from $(x, 0)$ then a shear is $T(x,y) = (x, y + ax)$. Shears have determinant 1 and preserve therefore volume.

singular

A square matrix A is called [singular] if it has no inverse. A matrix A is singular if and only if $\det(A) = 0$.

singular value decomposition

The [singular value decomposition] (SVD) of a matrix writes a matrix A in the form $A = UDV^T$, where U, V are orthogonal and D is diagonal. The first r columns of U form an orthonormal basis of the image of A and the first r columns of V form an orthonormal basis of the image of A^T. The last columns of U form an orthonormal basis of the kernel of A^T and the last columns of V form a basis of the kernel of A.

skew symmetric

A matrix A is [skew symmetric] if it is minus its transpose that is if $A^T = -A$. The eigenvalues of a skew-symmetric matrix are purely imaginary. The eigenvectors are orthogonal. If A is skew symmetric, then $B = \exp(At)$ is an orthogonal matrix, because $B^T B = \exp(-At) exp(At) = 1$. For example $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ gives the rotation matrix $exp(At)$.

span

The [span] of a set of vectors $v_1, ...v_n$ is the set of all linear combinations of $v_1, ..., v_n$.

spectral theorem

The [spectral theorem] tells that a real symmetric matrix A can be diagonalized $A = UDU^T$, where U is an orthonormal matrix containing an orthonormal eigenbasis in the columns and where D is a diagonal matrix $D = \text{diag}(x_1, ..., x_n)$, where x_i are the eigenvalues of A.

square matrix

A [square matrix] is a matrix which has the same number of rows than columns.

asymptotically stable

A linear dynamical system is called [asymptotically stable] if $A^n x \to 0$ for all initial values x, where A^n is the n'th power of the matrix A. This is equivalent to the fact that all eigenvalues λ of A satisfy $|\lambda| < 1$.

Stability triangle

[Stability triangle]. A discrete dynamical system in the plane is asymptotically stable if and only if the trace and determinant are in the stability triangle $|\text{tr}(A)| - 1 < \det(A) < 1$. A rotation-dilation A is asymptotically stable if and only if $\det(A) < 1$.

reduced row echelon form

The [reduced row echelon form] rref(A) of a m x n matrix A is the end product of Gauss-Jordan elimination. The matrix rref(A) has the following properties:

- if a row has nonzero entries, then the first nonzero entry is 1, called leading 1.
- if a columns contains a leading 1, then all other entries in that column are 0.
- if a row contains a leading 1, then every row above contains a leading 1 further left.

The algorithm to produce rref(A) from A is obtained by putting the cursor to the upper left corner and repeating the following steps until nothing changes anymore

1. if the cursor entry is zero swap the cursor row with the first row below that has a nonzero entry in that column
2. divide the cursor row by the cursor entry to make the cursor entry = 1
3. eliminate all other entries in cursor column by subtracting suitable multiples of the cursor row from the other row
4. move the cursor down one row and and to the right one column. If the cursor entry is zero and all entries below are zero, move the cursor to the next column.
5. repeat 4 if as long as necessary and move then to 1

reflection

A [reflection] is a linear transformation T different from the identity transformation which satisfies $T^2 = 1$. The eigenvalues of T are -1 or 1. In an eigenbasis, the reflection has the form $T(x, y) = (x, -y)$. The determinant of a reflection is 1 if and only if the dimension of the eigenspace to -1 is even. For example, a reflection at a line in the plane has the matrix $A = \begin{pmatrix} \cos(2x) & sin(2x) \\ \sin(2x) & -cos(2x) \end{pmatrix}$ which has determinant -1. A reflection at the origin in the plane is $-I$ with determinant 1.

root

A [root] of a polynomial $p(x)$ is a complex value z such that $p(z) = 0$. According to the fundamental theorem of algebra, a polynomial of degree n has exactly n roots.

symmetric

A matrix A is [symmetric] if it is equal to its transpose. The spectral theorem for symmetric matrices tells that they have real eigenvalues and symmetric matrices can always be diagonalized with an orthogonal matrix S.

span

The [span] of a finite set of vectors $v_1, ..., v_n$ is the set of all possible linear combinations $c_1 v_1 + c_2 v_2 + ... + c_n v_n$ where c_i are real numbers. For example, if $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$, then the span of v_1, v_2 in three dimensional space is the xy-plane. The span is a linear space.

spectral theorem

The [spectral theorem] for a symmetric matrix A assures that A can be diagonalized: there exists an orthogonal matrix S such that $A^(-1)AS$ is diagonal and contains the eigenvalues of A in the diagonal.

standard basis

The [standard basis] of the n-dimensional Euclidean space consists of the columns of the identity matrix I.

symmetric

A matrix A is called [symmetric] if $A^T = A$. A symmetric matrix has to be a square matrix. Real symmetric matrices can be diagonalized.

Toepitz matrix

A [Toepitz matrix] is a square matrix A, where the entries are constant along the diagonal. In other ways A_{ij} depends only on $j - i$. Example: a 3×3 Toeplitz matrix is of the form $A = \begin{pmatrix} c & d & e \\ b & c & d \\ a & b & c \end{pmatrix}$.

trace

The [trace] of a matrix A is the sum of the diagonal entries of A. The trace is independent of the basis and is equal to the sum of the eigenvalues of A.

transpose

The [transpose] A^T of a matrix A is the matrix with entries A_{ij} if A has the entries A_{ji}. The rank of A^T is equal to the rank of A. For square matrices, the eigenvalues of A^T and A agree because A and A^T have the same eigenvalues. Transposition satisfies $(A^T)^T = A$, $(AB)^T = B^T A^T$ and $(A^(-1))^T = (A^T)^(-1)$.

triangle inequality

The [triangle inequality] tells that in a linear space, $||v + w|| \leq ||v|| + ||w||$. One has equality if and only if the vectors v and w are orthogonal.

eigenvalues of a two times two matrix

The [eigenvalues of a two times two matrix] $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are $\lambda_1 = \mathrm{tr}(A)/2 + ((\mathrm{tr}(A)/2)^2 - \det(A))^{1/2}$ and $\lambda_2 = \mathrm{tr}(A)/2 - ((\mathrm{tr}(A)/2)^2 - \det(A))^{1/2}$. The eigenvectors are $v_i = \begin{pmatrix} \lambda_i - d \\ c \end{pmatrix}$ if $c \neq 0$ or $v_i = \begin{pmatrix} b \\ \lambda_i - a \end{pmatrix}$ if $b \neq 0$. (If $b = 0, c = 0$, then the standard vectors are eigenvector.)

unit vector

A [unit vector] is a vector of length 1. A given nonzero vector can be made a unit vector by scaling: $v/||v||$ is a unit vector.

Vandermonde Matrix

A [Vandermonde Matrix] is a square matrix with entries $A_{ij} = x_i^{j-1}$, where $x_1, ..., x_n$ are some real numbers. The determinant of a Vandermonde Matrix is $\prod_{j>i}(x_j - x_i)$. Example: $x_1 = 2, x_2 = 3, x_3 = -1$ defines the Vandermonde Matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & -1 & 1 \end{pmatrix}$ which has the determinant $\det(A) = (3-2)(-1-2)(-1-3) = 12$.

vector

A [vector] is a matrix with one column. The entries of the vector are called coefficients.

vector space

A [vector space] X is a set equipped with addition and scalar multiplication. A vector space is also called a linear space. The addition operation is a group:

$f + g = g + f$	Commutativity
$(f + g) + h = f + (g + h)$	Associativity
$f + 0 = 0$	Existence of a neutral element
$f + x = 0$	Existence of a unique inverse

The scalar multiplication satisfies:

$r(f + g) = rf + rg$ Distributivity
$(r + s)f = rf + sf$ Distributivity
$r(sf) = (rs)f$ Associativity
$1f = f$ One element

wave equation

The [wave equation] is the linear partial differential equation $u_{tt} = c^2 u_{xx}$ where c is a constant. The wave equation on a finite interval $0 < x < 1$ with boundary conditions $u(0,t) = 0$, $u(\pi,t) = 0$ and initial conditions $u(x,0) = f(x), u_t(x,0) = g(x)$ can be solved with the Fourier series: $u(x,t) = \sum_n > 0 a_n \sin(nx)\cos(nct) + b_n \sin(nx)\sin(nct)$ where $a_n = (2/\pi)\int_0^\pi f(x)\sin(nx)\ dx$, and $b_n = (2/\pi) int_0^\pi g(x)\sin(nx)\ dx/(cn)$ are Fourier coefficients.

zero matrix

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