ENTRY MULTIVARIABLE CALCULUS

[ENTRY MULTIVARIABLE CALCULUS] Author: Oliver Knill: March 2000 -March 2004 Literature: Standard glossary of multivariable calculus course as taught at the Harvard mathematics department.

acceleration

The [acceleration] of a parametrized curve $r(t) = (x(t), y(t), z(t))$ is defined as the vector $r''(t)$. It is the rate of change of the velocity $r'(t)$. It is significant, because Newtons law relates the acceleration $r''(t)$ of a mass point of mass m with the force F acting on it: $mr''(t) = F(r(t))$. This ordinary differential equation determines completely the motion of the particle.

advection equation

The [advection equation] $u_t = cu_x$ is a linear partial differential equation. Its general solution is u(t,x)=f(x+ct), where f(x)=u(0,x). The advection equation is also called transport equation. In higher dimensions, it generalizes to the gradient flow $u_t = c\text{grad}(u)$.

Archimedes spiral

The [Archimedes spiral] is the plane curve defined in polar coordinates as $r(t) = ct$, where c is a constant. In Euclidean coordinates, it is given by the parametrization $r(t) = (ct\cos(t), ct\sin(t))$.

axis of rotation

The [axis of rotation] of a rotation in Euclidean space is the set of fixed points of that rotation.

Antipodes

Two points on the sphere of radius r are called [Antipodes] (=anti-podal points) if their Euclidean distance is maximal $2r$. If the sphere is centered at the origin, the antipodal point to (x, y, z) is the point $(-x, -y, -z)$.

boundary

The [boundary] of a geometric object. Examples:

- The boundary of an interval $I = \{a \leq x \leq b\}$ is the set with two points $\{a, b\}$. For example, $\{0 \leq x \leq 1$ the boundary $\{0, 1\}$.
- The boundary of a region G in the plane is the union of curves which bound the region. The unit disc has as a boundary the unit circle. The entire plane has an empty boundary.
- The boundary of a surface S in space is the union of curves which bound the surface. For example: A semisphere has as the boundary the equator. The entire sphere has an empty boundary.
- The boundary of a region G in space is the union of surfaces which bound the region. For example, the unit ball has the unit sphere as a boundary. A cube has as a boundary the union of 6 faces.
- The boundary of a curve $r(t), t \in [a, b]$ consists of the two points $r(a), r(b)$.

The boundary can be defined also in higher dimensions where surfaces are also called manifolds. The dimension of the boundary is always one less then the dimension of the object itself. In cases like the half cone, the tip of the cone is not considered a part of the boundary. It is a singular point which belongs to the surface. While the boundary can be defined for far more general objects in a mathematical field called "topology", the boundaries of objects occuring in multivariable calculus are assumed to be of dimension one less than the object itself.

Burger's equation

The [Burger's equation] $u_t = uu_t$ is a nonlinear partial differential equation in one dimension. It is a simple model for the formation of shocks.

Cartesian coordinates

[Cartesian coordinates] in three-dimensional space describe a point P with coordinates x, y and z. Other possible coordinate systems are cylindrical coordinates and spherical coordinates. Going from one coordinate system to an other is called a coordinate change.

Cavalieri principle

[Cavalieri principle] tells that if two solids have equal heights and their sections at equal distances have have areas with a given ratio, then the volumes of the solids have the same ratio.

change of variables

A [change of variables] is defined by a coordinate transformation. Examples are changes between cylindrical coordinates, spherical coordinates or Cartesian coordinates. Often one uses also rotations, allowing to use a convenient coordinate system, like for example, when one puts a coordinate system so that a surface of revolution has as the symmetry axes the z-axes.

circle

A [circle] is a curve in the plane whose distance from a given point is constant. The fixed point is called the center of the circle. The distance is the radius of the circle. One can parametrize a circle by r(t) = (cos(t),sin(t) or given as an implicit equation $g(x, y) = x^2 + y^2 = 1$. The circle is an example of a conic section, the intersection of a cone with a plane to which ellipses, hyperbola and parabolas belong to.

cone

A [cone] in space is the set of points $x^2 + y^2 = z^2$ in space. Also translates, scaled and rotated versions of this set are still called a cone. For example $2x^2 + 3y^2 = 7z^2$ is an elliptical cone.

conic section

A [conic section] is the intersection of a cone with a plane. Hyperbola, ellipses and parabola lines and pairs of intersecting lines are examples of conic sections.

continuity equation

The [continuity equation] is the partial differential equation $\rho_t + div(\rho v) = 0$, where ρ is the density of the fluid and v is the velocity of the fluid. The continuity equation is the consequence of the fact that the negative change of mass in a small ball is equal to the amount of mass which leaves the ball. The later is the flux of the current $j = v\rho$ through the surface and by the divergence theorem the integral of div(j).

cos theorem

The [cos theorem] relates the length of the edges a,b,c in a triangle ABC with one of the angles α: $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ Especially, if $\alpha = \pi/2$, it becomes the theorem of Pythagoras.

critical point

A [critical point] of a function $f(x, y)$ is a point (x_0, y_0), where the gradient $\nabla f(x_0, y_0)$ vanishes. Critical points are also called stationary points. For functions of two variables $f(x, y)$, critical points are typically maxima, minima or saddle points realized by $f(x, y) = -x^2 - y^2, f(x, y) = x^2 + y^2$ or $f(x, y) = x^2 - y^2$.

chain rule

The [chain rule] expresses the derivative of the composition of two functions in terms of the derivatives of the functions. It is $(fg)'(x) = f'(g(x))g'(x)$. For example, if $r(t)$ is a curve in space and F a function in three variables, then $(d/dt)f(r(t)) = \text{grad}(f) \cdot r'(t)$. Example. If T and S are maps on the plane, then $(TS)' = T'(S)S'$, where T' is the Jacobean of T and S' is the Jacobean of S.

change of variables

A [change of variables] on a region R in Euclidean space is given by an invertible map $T : R \to T(R)$. The change of variables formula $\int_{T(R)} f(x)\ dx = \int_R f(Tx) det(T'(x))\ dx$ allows to evaluate integrals of a function f of several variables on a complicated region by integrating on a simple region R. In one dimensions, the change of variable formula is the formula for substitution. Example: (2D polar coordinates) $T(r, \phi) = (x\cos(\theta), y\sin(\theta)$. with det(T')=r maps the rectangle $[0, s] \times [0, 2\pi]$ into the disc. Example of 3D spherical coordinates are $T(r, \theta, \phi) = (r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))$, $det(T') = r^2\sin(\phi)$ maps the rectangular region $(0, s) \times (0, 2\pi 0 \times (0, \pi)$ onto a sphere of radius s.

curl

The [curl] of a vector field $F = (P, Q, R)$ in space is the vector field $(R_y - Q_z, P_z - R_x, Q_x - P_y)$. It measures the amount of circulation = vorticity of the vector field. The curl of a vector field F=(P,Q) in the plane is the scalar field $(Q_x - P_y)$. It measures the vorticity of the vector field in the plane.

curvature

The [curvature] of a parametrized curve r(t) = (x(t),y(t),z(t)) is defined as $k(t) = |r'(t) \times r''(t)|/|r'(t)|^3$. Examples:

- The curvature of a line is zero.
- The curvature of a circle of radius r is 1/r.

curve

A [curve] in space is the image of a map $X : t- > r(t) = (x(t), y(t), z(t))$, where $x(t), y(t), z(t)$ are three piecewise smooth functions. For general continuous maps $x(t), y(t), z(t)$, the length or the velocity of the curve would no more be defined.

cross product

The [cross product] of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is the vector $(v_2w_3 - w_2v_3, v_3w_1 - w_3v_1, v_1w_2 - w_2v_1)$.

curve

A [curve] in three-dimensional space is the image of a map $r(t) = (x(t), y(t), z(t))$, where $x(t), y(t), z(t)$ are three continuous functions. A curve in two-dimensional space is the image of a map $r(t) = (x(t), y(t))$.

cylinder

A [cylinder] is a surface in three dimensional space such that its defining equation f(x,y,z)=0 does not involve one of the variables. For example, $z = 2\sin(y)$ defines a cylinder. A cylinder usually means the surface $x^2 + y^2 = r$ or a translated rotated version of this surface.

derivative

The [derivative] of a function f(x) of one variable at a point x is the rate of change of the function at this point. Formally, it is defined as $lim_{dx \to 0}(f(x + dx) - f(x))/dx$. One writes f'(x) for the derivative of f. The derivative measures the slope of the graph of f(x) at the point. If the derivative exists for all x, the function is called differentiable. Functions like $\sin(x)$ or $\cos(x)$ are differentiable. One has for example $f'(x) = \cos(x)$ if $f(x) = sin(x)$. An example of a function which is not differentiable everywhere is f(x)=—x—. The derivative at 0 is not defined.

cylindrical coordinates

[cylindrical coordinates] in three dimensional space describe a point P by the coordinates $r = (x^2 + y^2 + z^2)^{1/2}$, phi=arctan(y/x),z, where P=(x,y,z) are the Cartesian coordinates of P. Other coordinate systems are Cartesian coordinates or spherical coordinates.

determinant

The [determinant] of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$. The determinant of a matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is $aei + bfg + cdh - ceg - fha - ibd$. The determinant is relevant when changing variables in integration.

directional derivative

The [directional derivative] of f(x,y,z) in the direction v is the dot product of the gradient of f with v. It measures the rate of change of f at a point P when moving trough the point (x,y,z) with velocity v.

distance

The [distance] of two points P=(a,b,c) and Q=(u,v,w) in three dimensional Euclidean space is the square root of $(a - u)^2 + (b - v)^2 + (c - w)^2$. The distance of two points P=(a,b) and $Q = (u, v)$ in the plane is the square root of $(a - u)^2 + (b - v)^2$.

distance

The [distance] between two nonparallel lines in three dimensional Euclidean space is given by the forumla $d = |(v \times w) \cdot u|/|(v \times w)|$, where v and w are arbitrary nonzero vectors in each line and u is an arbitrary vector connecting a point on the first line to a point of the second line.

divergence

The [divergence] of a vector field F=(P,Q,R) is the scalar field $div(F) = P_x + Q_y + R_z$. The value $div(F)(x, y, z)$ measures the amount of expansion of the vector field at the point (x, y, z).

dot product

The [dot product] of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is the scalar $v_1w_1 + v_2w_2 + v_3w_3$.

ellipse

An [ellipse] is the set of points in the plane which satisfy an equation $(x - a)^2/A^2 + (y - b)^2/B^2 = 1$. It is inscribed in a rectangle of length A and width B centered at (a, b). Ellipses can also be defined as the set of points in the plance whose sum of the distances to two points is constants. The two fixed points are called the foci of the ellipse. The line through the foci of a noncircular ellipse is called the focal line, the points where focal axes and a noncircular ellipse cross, are called vertices of the ellipse. The major axis of the ellipse is the line segment connecting the two vertices, the minor axis is the symmetry line of the ellipse which mirrors the two focal points or the two vertices. Ellipses are examples of conic sections, the intersection of a cone with a plane.

ellipsoid

An [ellipsoid] is the set of points in three dimensional Euclidean space, which satisfy an equation $(x - a)^2/A^2 + (y - b)^2/B^2 + (z - c)^2/C^2 = 1$. It is inscribed in a box of length, width and height A,B,C centered at (a, b, c).

equation of motion

The [equation of motion] of a fluid is the partial differential equation $\rho Dv/dt = -grad(p) + F$, where F are external forces like gravity rho g, or magnetic force $j \times B$ and Dv/dt is the total time derivative $Dv/dt = v_t + vgrad(v)$. The term - grad(p) is the pressure force. Together with an incompressibility assumption div(v)=0, these equations of motion are called Navier Stokes equations.

flux integral

The [flux integral] of a vector field F through a surface S=X(R) is defined as the double integral of X(F).n over R, where $n = X_u \times X_v$ is the normal vector of the surface as defined through the parameterization $X(u, v)$.

Fubinis theorem

[Fubinis theorem] tells that $\int_a^b \int_c^d f(x, y)\, dxdy = \int_c^d \int_a^b f(x, y)\, dydx$.

gradient

The [gradient] of a function f at a point P=(x,y,z) is the vector $(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$ where f_x denotes the partial derivative of f with respect to x.

Hamilton equations

The [Hamilton equations] to a function $f(x,y)$ is the system of ordinary differential equations $x'(t) = f_y(x,y), y'(t) = -f_x(x,y)$. which is called Hamilton system. Solution curves of this system are located on level curves $f(x,y) = c$ because by the chain rule one has $d/dt f(x(t), y(t)) = f_x x' + f_y y' = f_x f_y - f_x f_y = 0$. The preservation of f is in physics called energy conservation.

heat equation

The [heat equation] is the Partial differential equation $u_t = mu\Delta(u)$, where mu is a constant, and Δu is the Laplacian of u. The heat equation is also called the diffusion equation.

Hessian

The [Hessian] is the determinant of the Hessian matrix.

Hessian matrix

The [Hessian matrix] of a function f(x,y,z) at a point (u,v,w) is the $3x3$ matrix $f''(u,v,w) = H(u,v,w) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$. The Hessian matrix of a function $f(x,y)$ at a point (u,v) is the 2x2 matrix $f'' = H(u,v,w) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$. The Hessian matrix is useful to classify critical points of $f(x,y)$ using the second derivative test.

hyperbola

A [hyperbola] is a plane curve which can be defined as the level curve $g(x,y) = x^2/a^2 + y^2/b^2 = 1$ or given as a parametrized curve $r(t) = (a\cosh(t), b\sinh(t))$. A hyperbola can geometrically also be defined as the set of points whose distances from two fixed points in the plane is constant. The two fixed points are called the focal points of the hyperbola. The line through the focal points of a hyperbola is called the focal axis. The points, where the focal axis and the hyperbola cross are called vertices. A hyperbola is an example of a conic section, the intersection of a cone with a plane.

hyperboloid

A [hyperboloid] is the set of points in three dimensional Euclidean space, which satisfy an equation $(x-u)^2/a^2-(y-v)^2/b^2-(z-w)^2/c^2=I$, where $I=1$ or $I=-1$. For a=b=c=1, the hyperboloid is obtained by rotating a hyperbola $x^2-y^2=1$ around the x-axes. It is two-sided for I=-1 (the intersection of the plane z=c with the hyperboloid is then empty) and one-sided for I=1.

incompressible

A vector field F is called [incompressible] if its divergence is zero $\mathrm{div}(F)=0$. The notation has its origins from fluid dynamics, where velocity fields F of fluids, gases or plasma often are assumed to be incompressible. If a vector field is incompressible and is a velocity field, then the corresponding flow preserves the volume.

continuity equation

The [continuity equation] $\rho_t+\mathrm{div}(i)=0$ links density ρ and velocity field i. It is an infinitesimal description which is equivalent to the preservation of mass by the theorem of Gauss. The change of mass $M(t)\int\int\int_R \rho\, dV$ inside a region R in space is the minus the flux of mass through the boundary S of R.

interval

An [interval] is a subset of the real line defined by two points a,b. One can write $I=\{a\leq x\leq b\}$ for a closed interval, $I=\{a<x<b\}$ for an open interval and $I=\{a\leq x<b\}, I=\{a<x\leq b\}$ for half open intervals. If $a=-\infty$ and $b=\infty$, then the interval is the entire real line. If $a=0, b-\infty$, then $I=(a,b)$ is the set of positive real numbers. Intervals can be characterized as the connected sets in the real line.

integral

An [integral] of f(x) over an interval I on the line is the limit $(1/n)\sum_{i=1}^n f(i/n)$ for $n\to\infty$ over the integers and the sum is taken over all i such that i/n is in I. An integral of $f(x,y)$ over a region R in the plane is the limit $(1/n^2)\sum_{(i/n,j/n)\in R} f(i/n,j/n)$ for $n\to\infty$. Such an integral is also called double integral. Often, double integrals can be evaluated by iterating two one-dimensional integrals. An integral of f(x,y,z) over a domain R in space is the limit $(1/n^3)\sum_{(i/n,j/n,k/n)\in R} f(i/n,j/n,k/n)$ for n to infinity. Such an integral is also called a triple integral. Often, triple integrals can be evaluated by iterating three one-dimensional integrals.

intercept

An [intercept] is the intersection of a surface with a coordinate axes. Like traces, intercepts are useful for drawing surfaces by hand. For example, the two sheeted hyperboloid $x^2+y^2-z^2=-1$ has the intercepts $x^2-z^2=-1$ and $y^2-z^2=-1$ (hyperbola) and an empty intercept with the z axes.

jerk

The [jerk] of a parametrized curve r(t)=(x(t),y(t),z(t)) is defined as r"'(t). It is the rate of change of the acceleration. By Newtons law, the jerk measures the rate of change of the force acting on the body.

Lagrange multiplier

A [Lagrange multiplier] is an additional variable introduced for solving extremal problems under constraints. To extremize $f(x, y, z)$ on a surface $g(x, y, z) = 0$ then an extremum satisfies the equations $f' = Lg', g = 0$, where L is the Lagrange multiplier. These are four equations for four unknowns x,y,z,l. Additionally, one has to check for solutions of $g'(x, y, z) = 0$.
Example. If we want to extremize $F(x, y, z) = -x \log(x) - y \log(y) - z \log(z)$ under the constraint $G(x, y, z) = x + y + z = 1$, we solve the equations $-1 - \log(x) = \lambda 1$ $-1 - \log(y) = \lambda 1$ $-1 - \log(z) = \lambda 1$ $x + y + z = 1$, the solution of which is $x = y = z = 1/3$.

Lagrange method

The [Lagrange method] to solve extremal problems under constraints:
1) in order that a function f of several variables is extremal on a constraint set $g = c$, we either have $\nabla g = 0$ or the point is a solution to the Lagrange equations $\nabla f = \lambda \nabla g, g = c$.
2) in order to extremize a function f of several variables under the contraint set $g = c, h = d$, we have to solve the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solve $\nabla g = \nabla h = 0$.

Laplacian

The [Laplacian] of a function $f(x, y, z)$ is defined as $\Delta(f) = f_x x + f_y y + f_z z$. One can write it as $\Delta = \mathrm{divgrad}(f)$. Functions for which the Laplacian vanish are called harmonic. Laplacian appear often in PDE's Examples: the Laplace equation $\Delta(f) = 0$, the Poisson equation $\Delta(f) = \rho$, the Heat equation $f_t = \mu\Delta(f)$ or the wave equation $f_t t = c^2 \Delta f$. The [length] of a curve r(t)=(x(t),y(t),z(t)) from t=a to t=b is the integral of the speed —r'(t)— over the interval a,b. Example. the length of the curve r(t)=(cos(t),sin(t)) from t=0 to $t = \pi$ is π because the speed $|r'(t)|$ is 1.

length

The [length] of a vector $v = (a, b, c)$ is the square root of $v \cdot v = a^2 + b^2 + c^2$. An other word for length is norm. If a vector has length 1, it is called normalized or a unit vector.

level curve

A [level curve] of a function $f(x, y)$ of two variables is the set of points which satisfy the equation $f(x, y) = c$. For example, if $f(x, y) = x^2 - y^2$, then its level curves are hyperbola. Level curves are orthogonal to the gradient vector field grad(f).

level surface

A [level surface] of a scalar function $f(x,y,z)$ is the set of points which satisfy $f(x,y,z) = c$. For example, if $f(x,y,z) = x^2 + y^2 + 3z^2$, then its level surfaces are ellipsoids. Level surfaces are orthogonal to the gradient field grad(f).

linear approximation

The [linear approximation] of a function f(x,y,z) at a point (u,v,w) is the linear function $L(x,y,z) = f(u,v,w) + \nabla f(u,v,w) \cdot (x-u, y-v, z-w)$.

line

A [line] in three-dimensional space is a curve in space given by $r(t) = P + tv$, where P is a point in space and v is a vector in space. The representation r(t)=P+tv is called a parameterization of the line. Algebraically, a line can also be given as the intersection of two planes: $ax+by+cz = d$, $ux+vy+wz = q$. The corresponding vector v in the line is the cross product of (a,b,c) and (u,v,w). A point $P = (x,y,z)$ on the line can be obtained by fixing one of the coordinates, say z=0 and solving the system $ax+by = d$, $ux+vy = q$ for the unknowns x and y.

line integral

The [line integral] of a vector field $F(x,y)$ along a curve $C : r(t) = (x(t), y(t)), t \in [a,b]$ in the plane is defined as

$$\int_C F \cdot ds = \int_a^b F(r(t)) \cdot r'(t)\, dt \ ,$$

where $r'(t) = (x'(t), y'(t))$ is the velocity. The definition is similar in three dimensions where $F(x,y,z)$ is a vector field and $C : r(t) = (x(t), y(t), z(t)), t \in [a,b]$ is a curve in space.

Maxwell equations

The [Maxwell equations] are a set of partial differential equations which determine the electric field E and magnetic field B, when the charge density ρ and the current density j are given. There are 4 equations:

$\text{div}(B) = 0$	no magnetic monopoles
$\text{curl}(E) = -B_t/c$	Faradays law, change of magnetic flux produces voltage
$\text{curl}(B) = E_t/c + (4\pi/c)j$	Ampere's law, current or E change produce magnetism
$\text{div}(E) = 4\pi\rho$	Gauss law, electric charges produce an electric field

nabla

[nabla] is a mathematical symbol used when writing the gradient ∇f of a function $f(x,y,z)$. Nabla looks like an upside down Δ. Etymologically, the name has the meaning of an Egyption harp.

nabla calculus

The [nabla calculus] introduces the vector $\nabla = (\partial_x, \partial_y, \partial_z)$. It satisfies $\nabla(f) = \text{grad}(f)$, $\nabla x F = curl(F)$, $\nabla \cdot F = div(F)$. Using basic vector operation rules and differentiation rules like $\nabla(fg) = (\nabla f)g + f(\nabla g)$ one can verify identities: like for example
$\text{div}(\text{curl})F = 0$, $\text{curl}(\text{grad})f = 0$, $\text{curl}(\text{curl}F) = grad(divF) - \Delta(F)$, $\text{div}(E \times F) = F \cdot curl(E) - E \cdot curl(F)$.

nonparallel

Two vectors v and w are called [nonparallel] if they are not parallel. Two vectors in space are parallel if and only if their cross product $v \times w$ is nonzero.

normal vector

A [normal vector] to a parametrized surface $X(u, v) = (x(u, v), y(u, v), z(u, v))$ at a point P=(x,y,z) is the vector $X_u x X_v$. It is orthogonal to the tangent plane spanned by the two tangent vectors X_u and X_v.

normalized

A vector is called [normalized] if its length is equal to 1. For example, the vector $(3/5, 4/5)$ is normalized. The vector $(2, 1)$ is not normalized.

octant

An [octant] is one of the 8 regions when dividing three dimensional space with coordinate planes. It is the analogue of quadrant in two dimensions.

open set

An [open set] R in the plane or in space is a set for which every point P is contained in a small disc U which is still contained in R. The disc $x^2 + y^2 < 1$ is an example of an open set. The set $x^2 + y^2 \leq 1$ is not open because the point $(1, 0)$ for example has no neighborhood disc contained in R.

open

A set is called [open], if it is an open set. It means that every point in the set is contained in a neighborhood which still is in the set. The complement of open sets are called closed.

ordinary differential equation

An [ordinary differential equation] (ODE) is an equation for a function or curve $f(t)$ which relates derivatives f,f',f".... of f. An example is f'=c f which has the solution $f(t) = Ce^(ct)$, where C is a constant. Only derivatives with respect to one variable may appear in an ODE. In most cases, the variable t is associated with time. Examples:

$f' = cf$	population model $c > 0$.
$f' = -cf$	radioactive decay $c > 0$
$f' = cf(1-f)$	logistic equation
$f'' = -cf$	harmonic oscillator
$f'' = F(f)$	general form of Newton equations

By increasing the dimension of the phase space, every ordinary differential equation can be written as a first order autonomous system $x' = F(x)$. For example, $f'' = -f$ can be written with the vector $x = (x_1, x_2) = (f, f')$ as $(x_1', x_2') = (f', f'') = (f', f) = (x_2', -x_2')$. There is a 2×2 matrix such that $x' = Ax$.

orthogonal

Two vectors v and w are called [orthogonal] if $v \cdot w = 0$. An other word for orthogonal is perpendicular. The zero vector 0 is orthogonal to any other vector.

parabola

A [parabola] is a plane curve. It can be defined as the set of points which have the same distance to a line and a point. The line is called the directrix, the point is called the focus of the parabola. One can parametrize a parabola as $r(t) = (t, t^2)$. It is also possible to give a parabola as a level curve $g(x, y) = y - x^2 = 0$ of a function of two variables. A parabola is an example of a conic section, to which also circles, ellipses and hyperbola belong.

parallelogram

A [parallelogram] E can be defined as the image of the unit square under a map $T(s, t) = sv + tw$, where u and v are vectors in the plane. One says, E is spanned by the vectors v and w. The area of a parallelogram is $|v \times w|$.

parallelepiped

A [parallelepiped] E can be defined as the image of the unit cube under a linear map $T(r, s, t) = ru+sv+tv$, where u,v,w are vectors in space. One says, E is spanned by the vectors u, v and w. The volume of a parallelepiped is $|u \cdot (v \times w)|$.

perpendicular

Two vectors v and w are called [perpendicular] if $v \cdot w = 0$. An other word for perpendicular is orthogonal. The zero vector $v = 0$ is perpendicular to any other vector.

quadratic approximation

The [quadratic approximation] of a function f(x,y,z) at a point (u, v, w) is the quadratic function $Q(x, y, z) = L(x, y, z) + [H(u, v, w)(x - u, y - v, z - w)] \cdot (x - u, y - v, z - w)/2$, where $H(u, v, w)$ is the Hessian matrix of f at (u, v, w) and where $L(x, y, z)$ is the linear approximation of $f(x, y, z)$ at (u, v, w). For example, the function $f(x, y) = 3 + \sin(x + y) + \cos(x + 2y)$ has the linear approximation $L(x, y) = 4 + x + y$ and the quadratic approximation $Q(x, y) = 4 + x + y + (x + 2y)^2/2$.

quadrant

A [quadrant] is one of the 4 regions when dividing the two dimensional space using coordinate axes. It is the analogue of octant in three dimensions. For example, the set $\{x > 0, y > 0\}$ is the open upper right quadrant. The set $\{x \geq 0, y \geq 0\}$ is the closed upper right quadrant.

parallel

Two vectors v and w are called [parallel] if there exists a real number λ such that $v = \lambda w$. Two vectors in space are parallel if and only if their cross product $v \times w$ is zero.

parametrized surface

A [parametrized surface] is defined by a map

$$X(u, v) = (x(u, v), y(u, v), z(u, v))$$

from a region R in the uv-plane to xyz-space. Examples

- Sphere: $X(u, v) = (r \cos(u) \sin(v), r \sin(u) \sin(v), r \cos(v))$ $R = [0, 2\pi) \times [0, \pi]$, u and v are called Euler angles.
- Plane $X(u, v) = P + uU + vV$, where P is a point, U,V are vectors and R is the entire plane.
- Surface of revolution is parametrized by $X(u, v) = (f(v) \cos(u), f(v) \sin(u), v)$ where u is an angle measuring the rotation round the z axes and $f(v)$ is a nonnegative function giving the distance to the z-axes at the height v.
- A graph of a function $f(x, y)$ is parametrized by $X(u, v) = (u, v, f(u, v))$.
- A torus is parametrized by $X(u, v) = (a + b \cos(v)) \cos(u), (a + b \cos(v)) \sin(u), sin(v))$ on $R = [0, 2\pi) \times [0, 2\pi)$.

parametrized curve

A [parametrized curve] in space is defined by a map $r(t) = (x(t), y(t), z(t))$ from an interval I to space. Examples are

- Circle in the xy-plane $r(t) = (\cos(t), \sin(t), 0)$ with $t \in [0, 2\pi]$.
- Helix $r(t) = (\cos(t), \sin(t), t)$ with $t \in [a, b]$.
- Line $r(t) = P + tV$, where V is a vector and P is a point and $-\infty < t < \infty$.
- A line segment connecting P with Q $r(t) = P + t(Q - P)$, where $t \in [0, 1]$.

partial derivative

The [partial derivative] of a function of several variables f is the derivative with respect to one variable assuming the other variables are constants. One writes for example $f_y(x, y, z)$ for the partial derivative of f(x,y,z) with respect to y.

partial differential equation

A [partial differential equation] is an equation for a function of several variables in which partial derivatives with respect to different variables appear. Examples:

$u_t = cu_x$	Advection equation
$u_t = \mu\Delta(u)$	Heat equation
$u_t t = c^2\Delta(u)$	Wave equation
$u_t t = c^2\Delta(u) - m^2 u$	Klein Gordon equation
$\Delta(u) = 0$	Laplace equation
$\Delta(u) = \rho$	Poisson equation
$u_t + u_{xxx} + 6uu_x = 0$	KdV equation
$u_t = uu_x$	Burger equation
$\text{div}(B) = \text{div}(E) = 0$ $B_t = -c\text{curl}(E)$ $E_t = c\text{curl}(B)$	Maxwell equation (vacuum)
$ihu_t = h^2/2m\Delta u + Vu$	Schroedinger equation
$\text{curl}(A) = F$	Vector potential equation

plane

A [plane] in three dimensional space is the set of points (x, y, z) which satisfy an equation $ax + by + cz = d$. A parametrization of a plane is given by the map $(s, t) \mapsto X(u, v) = sv + tw$, where v, w are two vectors. If three points P_1, P_2, P_3 are given in space, then $X(s, t) = P_1 + s(P_2 - P_1) + t(P_3 - P_1)$ is a parametrisation of the plane which contains all three points.

polar coordinates

[polar coordinates] in the plane describe a point P=(x,y) with the coordinate (r, t) where $r = (x^2 + y^2)^{1/2}$ is the distance to the origin and t is the angle between the line OP and the x axes. The angle $t = \arctan(y/x) \in (-\pi/2, \pi/2]$ has to be augmented by π if $x < 0$ or $x = 0, y < 0$. The Cartesian coordinates of P are obtained from the Polar coordinates as $x = r\cos(t), y = r\sin(t)$.

potential

A function $U(x, y, z)$ is called a [potential] to a vector field $F(x, y, z)$ if $grad(U) = F$ at all points. The vector field F is then called conservative or a potential field. Not every vector field is conservative. If $curl(F) = 0$ everywhere in space, then F has a potential.

projection

The [projection] of a vector v onto a vector w is the vector $w(v \cdot w)/|w|^2$. The scalar projection is the length of the projection.

right handed

A coordinate system in space is [right handed] if it can be rotated into the situation such that if the z axes points to the observer of the xy plane, then a 90 degree rotation brings the x axes to the y axes. Otherwise the coordinate system is called left handed. If u is a vector on the positive x axes, v is a vector on the positive y axes and w is a vector on the positive z axes, then the coordinate system is right handed if and only if the triple product $u \cdot (v \times w)$ is positive.

second derivative test

The [second derivative test]. If the determinant of the Hessian matrix $\det(f''(x, y)) < 0$ then (x, y) is a saddle point. If $f''(x, y) > 0$ and and $f_{xx}(x, y) < 0$ then (x, y) is a local maximum. If $\det(f''(x, y)) < 0$ and $f_{xx}(x, y) > 0$ then (x, y) is a local minimum.

Space

[Space] is usually used as an abbreviation for three dimensional Euclidean space. In a wider sense, it can mean linear space a vector space in which on can add and scale.

speed

The [speed] of a curve $r(t) = (x(t), y(t), z(t))$ at time t is the length of the velocity vector $r'(t) = (x'(t), y'(t), z'(t))$.

sphere

A [sphere] is the set of points in space, which have a given distance r from a point P=(a,b,c). It is the set $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$. For a=b=c=0,r=1 one obtains the unit sphere: $x^2 + y^2 + z^2 = 1$. Spheres can be define in any dimenesions. A sphere in two dimensions is a circle. A sphere in 1 dimension is the union of two points. The unit sphere in 4 dimensions is the set of points $(x, y, z, w) \in R^4$ which satisfy $x^2 + y^2 + z^2 + w^2 = 1$ Spheres can be defined in any space equiped with a distance like $d((x, y), (u, v)) = |x - u| + |y - v|$ in the plane.

superformula

The [superformula] describes a class of curves with a few parameters m, n_1, n_2, n_3, a, b. It is the polar graph

$$r(t) = (|\cos(mt/4)|^{n_1}/a + |\sin(mt/4)|^{n_2}/b)^{-1/n_3} .$$

It had been proposed by the Belgian Biologist Johan Gielis in 1997.

superposition

The principle of [superposition] tells that the sum of two solutions of a linear partial differential equation (PDE) is again a solution of the PDE. For example, $f(x,y) = \sin(x-t)$ and $g(x,y) = e^{x-t}$ are both solutions to the transport equation $f_t(t,x) + f_x(t,x) = 0$. Therefore also the sum $\sin(x-t) + e^{x-t}$ is a solution. For nonlinear partial differential equations the superposition principle is no more true which is one of the reasons for the difficulty with dealing with nonlinear systems.

surface

A [surface] can either be described as a parametrized surface or implicitely as a level surface $g(x,y,z) = 0$. In the first case, the surface is given as the image of a map $X : (u,v) \mapsto (x(u,v), y(u,v), z(u,v))$ where u,v ranges over a parameter domain R in the plane. In the second case, the surface is determined by a function of three variables. Sometimes, one can describe a surface in both ways like in the following examples:

Sphere:	$X(u,v) = (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v))$, $g(x,y,z) = x^2+y^2+z^2 = r^2$
Graphs:	$X(u,v) = (u, v, f(u,v))$, $g(x,y,z) = z - f(x,y) = 0$
Planes:	$X(u,v) = P + uU + vV$, $g(x,y,z) = ax + by + cz = d$, $(a,b,c) = UxV$.
Surface of revolution:	$X(u,v) = (f(v)\cos(u), f(v)\sin(u), v)$, $g(x,y,z) = f((x^2+y^2)^(1/2)) - z = 0$

surface of revolution

A [surface of revolution] is a surface which is obtained by rotating a curve around a fixed line. If that line is the z-axes, the surface can be given in cylindrical coordinates as r = f(z). A parametrization is $X(t,z) = (f(z)\cos(t), f(z)\sin(t), z)$.

surface area

The [surface area] of surface $S = X(R)$ is defined as the integral of $\int\int_R |X_u \times X_v(u,v)|\, dudv$. For example, for $X(u,v) = (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v))$ on $R = \{0 \leq u \leq 2\pi, 0 \leq v \leq \pi\}$, where $S = X(R)$ is the sphere of radius r, one has $X_u \times X_v = r\sin(v)X$ and $|X_u \times X_v| = \sin(v)r^2$. The surface area is $\int_0^{2\pi}\int_0^{\pi} r^2 \sin(v)\, dudv = 4\pi r^2$.

surface integral

A [surface integral] of a function $f(x,y,z)$ over a surface $S = X(R)$ is defined as the integral of $f(X(u,v))|X_u \times X_v(u,v)|$ over R. In the special case when $f(x,y,z) = 1$, the surface integral is the surface area of the surface S.

tangent plane

The [tangent plane] to an implicitely defined surface $g(x, y, z) = c$ at the point (x_0, y_0, z_0) is the plane $ax + by + cz = d$, where $(a, b, c) = \nabla f(x_0, y_0, z_0)$ is the gradient of g at (x_0, y_0, z_0) and $d = ax_0 + by_0 + cz_0$.

tangent line

The [tangent line] to an implicitely defined curve $g(x, y) = c$ at the point (x_0, y_0) is the line $ax + by = d$, where (a, b) is the gradient of $g(x, y)$ at the point (x_0, y_0) and $d = ax_0 + by_0$.

theorem of Clairot

The [theorem of Clairot] assures that one can interchange the order of differentiation when taking partial derivatives. More precicely, if $f(x, y)$ is a function of two variables for which both $f_{xy} = f_{yx}$ are continuous, then $f_{xy} = f_{yx}$.

theorem of Gauss

The [theorem of Gauss] states that the flux of a vector field F through the boundary S of a solid R in three-dimensional space is the integral of the divergence div(F) of F over the region R:

$$\int\int\int_R \text{div}(F)\ dV = \int\int_S F \cdot dS\ .$$

theorem of Green

The [theorem of Green] states that the integral of the $curl(F) = Q_x - P_y$ of a vector field $F = (P, Q)$ over a region R in the plane is the same as the line integral of F along the boundary C of R.

$$\int\int_R \text{curl}(F)\ dA = \int_C F\ ds\ .$$

The boundary C is traced in such a way that the region is to the left. The boundary has to be piecewise smooth. The theorem of Green can be derived from the theorem of Stokes.

Green's theorem

[Green's theorem] see theorem of Green.

Green's theorem

The determinant of the Jacobean matrix is often called Jacobean or Jacobean determinant.

Jacobean matrix

[Jacobean matrix] If $T(u,v) = (f(u,v), g(u,v))$ is a transformation from a region R to a region S in the plane, the Jacobean matrix dT is defined as $\begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}$. It is the linearization of T near (u,v). Its determinant called the Jacobean determiant measures the area change of a small area element $dA = dudv$ when maped by T. For example, if $T(r,\theta) = (r\cos(\theta), r\sin(\theta)) = (x,y)$ is the coordinate transformation which maps $R = \{r \geq 0, \theta \in [0, 2\pi)\}$ to the plane, then dT is the matrix $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}$ which has determinant r.

theorem of Stokes

The [theorem of Stokes] states that the flux of a vector field F in space through a surface S is equal to the line integral of F along the boundary C of S:

$$\int\int_S \text{curl}(F) \ \cdot dS = \int_C F \ ds \ .$$

three dimensional space

The [three dimensional space] consists of all points (x,y,z) where x,y,z ranges over the set of real numbers. To distinguish it from other three-dimensional spaces, one calls it also Euclidean space.

torus

A [torus] is a surface in space defined as the set of points which have a fixed distance from a circle. It can be parametrized by $X(u,v) = (a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), sin(v))$ on $R = [0, 2\pi) \times [0, 2\pi)$, where a,b are positive constants.

trace

The [trace] of a surface in three dimensional space is the intersection of the surface with one of the coordinate planes x=0 or y=0 or z=0. Traces help to draw a surface when given the task to do so by hand. Other marking points are intercepts, the intersection of the surface with the coordinate axes.

triangle

A [triangle] in the plane or in space is defined by three points P, Q, R. If $v = PQ, w = PR$, then $|v \times w|/2$ is the area of the triangle.

triple product

The [triple product] between three vectors u, v, w in space is defined as the scalar $u \cdot (v \times w)$. The absolute value $|u \cdot (v \times w)|$ is the volume of the paralelepiped spanned by u, v and w.

triple dot product

[triple dot product] (see triple product).

unit sphere

The [unit sphere] is the sphere $x^2 + y^2 + z^2 = 1$. It is an example of a two-dimensional surface in three dimensional space.

unit tangent vector

The [unit tangent vector] to a parametrized curve r(t)=(x(t),y(t),z(t)) is the normalized velocity vector $T(t) = r'(t)/|r'(t)|$. Together with the normal vector $N(t) = T'(t)/|T'(t)|$ and the binormal vector $B(t) = T(t)xN(t)$, it forms a triple of mutually orthogonal vectors.

vector

A [vector] in the plane is defined by two points P, Q. It is the line segment v pointing from P to Q. If $P = (a, b)$ and $Q = (c, d)$ then the coordinates of the vector are $v = (c - a, d - b)$. Points P in the plane can be identified by vectors pointing from 0 to P. A vector in space is defined by two points P,Q in space. If $P = (a, b, c)$ and $Q = (d, e, f)$, then the coordinates of the vector are $v = (d - a, e - b, f - c)$. Points P in space can be identified by vectors pointing from 0 to P. Two vectors which can be translated into each other are considered equal. Remarks.

- One could define vectors more precisely as affine vectors and introduce an equivalence relation among them: two vectors are equivalent if they can be translated into each other. The equivalence classes are the vectors one deals with in calculus. Since the concept of equivalence relation would unnessesarily confuse students, the more fuzzy definition above is prefered.
- One should avoid definitions like "Vectors are objects which have length and direction" given in some Encyclopedias. The zero vector $(0, 0, 0)$ is an example of an object which has length but no direction. It nevertheless is a vector.

vector field

A [vector field] in the plane is a map $F(x, y) = (P(x, y), Q(x, y))$ which assigns to each point (x, y) in the plane a vector $F(x, y)$. An example of a vector field in the plane is $F(x, y) = (-y, x)$. An other example is the gradient field $F(x, y) = \nabla f(x, y)$ where $f(x, y)$ is a function. A vector field in space is a map $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ which assigns to each point (x, y, z) in space a vector $F(x, y, z)$. An example is the vector field $F(x, y, z) = (x^2, yz, x - y)$. An other example is the gradient field $F(x, y, z) = \nabla f(x, y, z)$ of a function $f(x, y, z)$.

velocity

The [velocity] of a parametrized curve r(t)=(x(t),y(t),z(t)) at time t is the vector $r'(t) = (x'(t), y'(t), z'(t))$. It is tangent to the curve at the point $r(t)$.

volume

The [volume] of a body G is defined as the integral of the constant function f(x,y,z)=1 over the body G.

wave equation

The [wave equation] is the partial differential equation $u_{tt} = c^2 \Delta(u)$, where $\Delta(u)$ is the Laplacian of u. Light in vacuum satisfies the wave equation. This can be derived from the Maxwell equations: the identity $\Delta(B) = \text{grad}(\text{div}(B) - \text{curl}(\text{curl}(B))$ gives together with $\text{div}(B) = 0$ and $\text{curl}(B) = E_t/c$ the relation $\Delta(B) = -d/dt\text{curl}(E)/c$ which leads with the Maxwell equation $B_t = -c\text{curl}(E)$ to the wave equation $\Delta B = B_{tt}/c^2$. The equation $E_{tt} = c^2 \Delta E$ is derived in the same way.

zero vector

The [zero vector] is the vector for which all components are zero. In the plane it is $v = (0, 0)$, in space it is $v = (0, 0, 0)$. The zero vector is a vector. It has length 0 and no direction. Definitions like "a vector is a quantity which has both length and direction" are misleading.

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