ENTRY SINGLE VARIABLE CALCULUS I

[ENTRY SINGLE VARIABLE CALCULUS I] Authors: Oliver Knill: 2001 Literature: not yet

Abel's partial summation formula

[Abel's partial summation formula] is a discrete version of the partial integration formula: with $A_n = \sum_{k=1}^n a_k$ one has $\sum_{k=m}^n a_k b_k = \sum_{k=m}^n A_k(b_k - b_{k+1}) + A_n b_{n+1} - A_{m-1} b_m$.

Abel's test

[Abel's test]: if a_n is a bounded monotonic sequence and b_n is a convergent series, then the sum $\sum_n a_n b_n$ converges.

absolute value

The [absolute value] of a real number x is denoted by $|x|$ and defined as the maximum of x and $-x$. We can also write $|x| = +\sqrt{x^2}$. The absolute value of a complex number $z = x + iy$ is defined as $\sqrt{x^2 + y^2}$.

accumulation point

An [accumulation point] of a sequence a_n of real numbers is a point a which the limit of a subsequence a_{n_k} of a_n. A sequence a_n converges if and only if there is exactly one accumulation point. Example: The sequence $a_n = \sin(\pi n)$ has two accumulation points, $a = 1$ and $a = -1$. The sequence $a_n = \sin(\pi n)/n$ has only the accumulation point $a = 0$. It converges.

Achilles paradox

The [Achilles paradox] is one of Zenos paradoxon. It argues that motion can not exist: "set up a race between Achilles A and tortoise T. At the initial time $t_0 = 0$, A is at the spot $s = 0$ while T is at position $s_1 = 1$. Lets assume A runs twice as fast. The reace starts. When A reaches s_1 at time $t_1 = 1$, its opponent T has already advanced to a point $s_2 = 1 + 1/2$. Whenever A reaches a point s_k at time t_k, where T has been at time t_{k-1}, T has already advanced further to location s_{k+1}. Because an infinite number of timesteps is necessary for A to reach T, it is impossible that A overcomes T." The paradox exploits a misunderstanding of the concept of summation of infinite series. At the finite time $t = \sum_{n=1}^\infty (t_n - t_{n-1}) = 2$, both A and T will be at the same spot $s = \lim_{n\to\infty} s_n = 2$.

addition formulas

The [addition formulas] for trigonometric functions are

$$\begin{aligned} \cos(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \sin(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \end{aligned}$$

alternating series

An [alternating series] is a series in which terms are alternatively positive and negative. An example is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n/n = -1 + 1/2 - 1/3 + 1/4 - ....$ An alternating series with $a_n \to 0$ converges by the alternating series test.

alternating series test

Leibniz's [alternating series test] assures that an alternating series $\sum_n a_n$ with $|a_n| \to 0$ is a convergent series.

acute

An angle is [acute], if it is smaller than a right angle. For example $\alpha = \pi/3 = 60^\circ$ is an acute angle. The angle $\alpha = 2\pi/3 = 120^\circ$ is not an acute angle. The right angle $\alpha = \pi/2 = 90^\circ$ does not count as an acute angle. The angle $\alpha = -\pi/6 = -30^\circ$ is an acute angle.

antiderivative

The [antiderivative] of a function f is a function $F(x)$ such that the derivative of F is f that is if $d/dxF(x) = f(x)$. The antiderivative is not unique. For example, every function $F(x) = \cos(x) + C$ is the antiderivative of $f(x) = \sin(x)$. Every function $F(x) = x^{n+1}/(n+1) + C$ is the antiderivative of $f(x) = x^n$.

Arithmetic progression

[Arithmetic progression] A sequence of numbers a_n for which $b_n = a_{n+1} - a_n$ is constant, is called an arithmetic progression. For example, $3, 7, 11, 15, 19, ...$ is an arithmetic progression. The sequence $0, 1, 2, 4, 5, 6, 7$ is not an arithmetic progression.

arrow paradox

The [arrow paradox] is a classical Zeno paradox with conclusion that motion can not exist: "an object occupies at each time a space equal to itself, but something which occupies a space equal to itself can not move. Therefore, the arrow is always at rest."

asymptotic

Two real functions are called [asymptotic] at a point a if $\lim_{x\to a} f(x)/g(x) = 1$. For example, $f(x) = \sin(x)$ and $g(x) = x$ are asymptotic at $a = 0$. The point a can also be infinite: for example, $f(x) = x$ and $g(x) = \sqrt{x^2+1}$ are asymptotic at $a = \infty$.

Bernstein polynomials

The [Bernstein polynomials] of a continuous function f on the unit interval $0 \leq x \leq 1$ are defined as $B_n(x) = \sum_{k=1}^{n} f(k/n)x^k(1-x)^{n-k}n!/(k!(n-k!)$.

Binomial coefficients

[Binomial coefficients] The coefficients $B(n,k)$ of the polynomial $(x+1)^n$ for integer n are called Binomial coefficients. Explicitly one has $B(n,k) = n!/(k!(n-k)!)$, where $k! = k(k-1)!$, $0! = 1$ is the factorial of k. The function $B(n,k)$ can be defined for any real numbers n,k by writing $n! = \Gamma(n+1)$, where Γ is the Gamma function. If p is a positive real number and k is an integer, one has one has $B(p,k) = p(p-1)...(p-k+1)/k!$. For example, $B(1/2,0) = B(1/2,1) = 1/2, B(1/2,2) = -1/8$. Indeed, $(1+x)^{1/2} = 1 + x/2 - x^2/8 + ...$

Binominal theorem

The [Binominal theorem] tells that for a real number $|z| < 1$ and real number p, one has $(1+z)^p = \sum_{k=0}^{\infty} B(p,k)z^k$, where $B(p,k)$ is called the Binomial coefficient. If p is a positive integer, then $(1+z)^p$ is a polynomial. For example:

$$(1+z)^4 = 1 + 4z + 6z^2 + 4z^3 + z^4 \ .$$

If p is a noninteger or negative, then $(1+z)^p$ is an infinite sum. For example

$$(1+z)^{-1/2} = 1 - x/2 + 3x^2/8 - 5x^3/16 + ...$$

bisector

A [bisector] is a straight line that bisects a given angle or a given line segment. For example, the y-axis $x = 0$ in the plane bisects the line segment connecting $(-1,0)$ with $(1,0)$. The line $x = y$ bisects the angle $\angle(CAB)$ where $C = (0,1), A = (0,0), B = (1,0)$ at the point A.

Bolzano's theorem

[Bolzano's theorem] also called intermediate value theorem says that a continuous function on an interval (a,b) takes each value between $f(a)$ and $f(b)$. For example, the function $f(x) = \sin(x)$ takes any value between -1 and 1 because f is continuous and $f(-\pi/2) = -1$ and $f(\pi/2) = 1$.

Fermat principle

The [Fermat principle] tells that if f is a function which is differentiable at z and $f(x) > f(z)$ for all points in an interval $(z-a, z+a)$ with $a > 0$, then $f'(z) = 0$.

fundamental theorem of calculus

The [fundamental theorem of calculus]: if f is a differentiable function on $a \leq x \leq b$ where $a < b$ are real numbers, then $f(b) - f(a) = \int_a^b f'(x)\, dx$.

integration rules

[integration rules]:

- $\int af(x)\, dx = a \int f(x)\, dx$.
- $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$.
- $\int fg\, dx = fG - \int f'G$, where $G' = g$. This is called integration by parts.

intermediate value theorem

The [intermediate value theorem] also called Bolzno theorem assures that a continuous function on an interval $a \leq z \leq b$ takes each each value between $f(a)$ and $f(b)$. For example $f(x) = \cos(x) + \cos(3x) + \cos(5x)$ takes any value between $[-3, 3]$ on $[0, \pi]$ because $f(0) = 3$ and $f(\pi) = -3$.

Cauchy's convergence condition

[Cauchy's convergence condition]: a sequence a_n converges, if and only if it is a Cauchy sequence that is if for every constant $c > 0$, we can find n, such that for all $k > n, m > n$ one has $|a_m - a_k| < c$. For example, the sequence $a_n = 1/\log(n)$ converges because for all $c > 0$, and the integer n closest to $e^{2c} + 1$ one has for $k > n$ and $m > n$ $|a_m - a_k| \leq c$.

Cauchy's convergence test

[Cauchy's convergence test]. Given a series $\sum_{k=1}^\infty a_k$ with positive summands a_k. If $r = \lim_{n \to \infty} a_n^{1/n} < 1$ then the series is a convergent series. If $r > 1$, then the series diverges.

continuous

A function f is called [continuous] at a point x if for every open interval V around $f(x)$ there exists an open interval U around x such that $f(U)$ is a subset of V. A function f is continuous in a set Y if it is continous at every point in Y. This definition is equivalent to: for every sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$. Examples:

- Any polynomial like $x^5 + 5x^3 + 3x$ is continuous on the entire line.
- The sum and product of continuous functions is continuous.
- the composition of two continuous functions is continuous.

Discontinuities can happen in different ways: the function can become infinite like $f(x) = 1/x$ at 0 or $\tan(x)$ at $x = \pi/2$, the function can jump like $f(x) = sign(x)$ which is 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$. A function can also become too oscillatory at a point like $f(x) = \sin(1/x)$ at $x = 0$. Note that $f(x) = x\sin(1/x)$ is continuous on the entire real line. There are functions which are discontinuous at every point. An example is $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational.
Note that be restricting the domain of a function, one an make it continuous. For example: $f(x) = 1/x$ is continuous on the positive real axes.

converges

A function $f(x)$ [converges] to a value z at x if the function g which agrees with f away from x and satisfies $g(x) = z$ is continouous at x. The value z is called the limit of f at x. For example the function $f(x) = (1 - x^2)/(1 - x)$ has the limit $z = 2$ at $x = 1$. The function $g(x)$ which is defined bo be $f(x)$ for $x \neq 1$ and $g(1) = 2$ is indeed continuous. One writes $z = \lim_{y \to x} f(y)$. One has

- $\lim_{x\to z}(f(x) + g(x)) = \lim_{x\to z} f(x) + \lim_{x\to z} g(x)$.
- $\lim_{x\to z}(f(x)g(x)) = \lim_{x\to z} f(x) + g(x)$.
- $\lim_{x\to z} f(g(x)) = f(\lim_{x\to z} g(x))$.

A series a_n is a [convergent series], if the partial sum sequence $b_n = \sum_{k=1}^{n} a_k$ converges to a finite limit a.

absolutely convergent series

A series $\sum_n a_n$ is called an [absolutely convergent series] if $\sum_n |a_n|$ is a convergent series.

series

Summing up a sequence is called a [series]. An important example is the geometric series $1 + 1/2 + 1/4 + 1/8 + 1/16 + ...$, which sums up to 2. An other example is the harmonic series $1 + 1/2 + 1/3 + 1/4 + 1/5 + ...$ which has no finite limit.

change of variables

The [change of variables] in integration theory is the formula $\int f(x)dx = \int f(g(u))g'(u)du$ if $x = g(u)$. For example, $\int \sqrt{1 - x^2}\, dx$ becomes with $x = g(u) = \sin(u)$ and $dx = g'(u)du = \cos(u)du$ the integral $\int \cos 2(u)\, du$.

differentiable

A function f is called [differentiable] at z if there exists a function g which is continuous at z such that $f(x) = f(z) + (x - z)g(x)$. The derivative of f at z is $g(z)$ and also denoted $f'(x)$. By solving for $g(x)$ and letting $x \to z$ one can write $g(z) = \lim_{x\to z}(f(x) - f(z))/(x - z)$. The quotient is called the differential quotient.

- The sum of two at z differentiable functions is differentiable at z and $(f + g)'(z) = f'(z) + g'(z)$. This is called the sum rule.
- The prduct of two at z differentiable functions is differentiable at z and $(fg)' = f'g + fg'$. This is called the product rule.
- The composition of two differentiable functions is differentiable and $(f \circ g)' = (f' \circ g)g'$. This is called the chain rule.

Functions can be continuous without being differentiable. For example $f(x) = |x|$ is continuous at 0 but not differentiable at 0. There are functions which are continuous everywhere but not differentiable at most points. An example is the Weierstrass function $f(x) = \sum_{k=1}^{\infty} \cos(k^2 x)/k^2$.

Extended mean value theorem

[Extended mean value theorem]. If $f(x)$ and $g(x)$ are differentiable on the interval (a, b) and are continuous on the closed interval $I = \{a \le x \le b\}$ then there exists a point $x \in I$ for which

$$f'(x)/g'(x) = (f(b) - f(a))/(g(b) - g(a)) .$$

Proof. Otherwise one would have one of the following two possibilities:

$$\begin{array}{ll} f'(x)(g(b) - g(a)) < g'(x)(f(b) - f(a)) & \text{for all } x \text{ in } (a, b) \text{ or} \\ f'(x)(g(b) - g(a)) < g'(x)(f(b) - f(a)) & \text{for all } x \text{ in } (a, b). \end{array}$$

Integration of these expressions using the fundamental theorem of calculus gives

$$\begin{array}{ll} (f(b) - f(a))(g(b) - g(a)) < (g(b) - g(a))(f(b) - f(a)) & \text{or} \\ (f(b) - f(a))(g(b) - g(a)) > (g(b) - g(a))(f(b) - f(a)) & \text{which both are not possible.} \end{array}$$

The special case $g(x) = x$ is called the mean value theorem.

factorial

The [factorial] of a positive integer n is defined recursively by $n! = n(n - 1)!$ and $0! = 1$. For example, $5! = 120$. The factorial function can be extended to the real line and is then called the Γ function: $n! = \Gamma(n + 1)$, where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt .$$

which is finite everywhere except at $z = 0, -1, -2, ...$

limit

The [limit] of a sequence of numbers a_n is a number a such that a_n converges to a in the following sense: for every $c > 0$ there exists an integer m such that $|a_n - a| < c$ for $n > m$. Limits can be defined in any metric space and more generally in any topological space.

maximum-value theorem

The [maximum-value theorem] assures that a continuous function on an interval $a < z < b$ has a maximum on that interval.

parabola

A [parabola] is the graph of the function $f(x) = x^2$. More general parabolas can be obtained as graphs of $f(x) = a(x-b)^2 + c$ where a, b, c are constants or curves obtained by rotating such a curve in the plane. For example the set of points in the plane satisfying $x = y^2$ form a parabola. Parabolas are examples of conic sections, intersections of a plane with a cone.

L'Hopital rule

The [L'Hopital rule] tells that if f and g are differentiable functions at x and $f(x) = g(x)$ and $g'(x) \neq 0$, then $\lim_{x \to z} f(x)/g(x) = \lim_{x \to z} f'(x)/g'(x)$.
For example, $\lim_{x \to 0} \sin(3x)/x = \lim_{x \to 0} 3\cos(3x) = 3$. The rule essentially tells that one can replace the functions by their linear approximation near a point to find the limit. The proof follows immediatly from the definition of differentiability: there exist continuous functions F, G such that $f(x) = f(z) + (x-z)F(x)$ and $g(x) = g(z) + (x-z)G(x)$. Because $G(z) \neq 0$, the function $F(x)/G(x)$ is continuous at z with value $F(z)/G(z)$. Now: $\lim_{x \to z} f(x)/g(x) = \lim_{x \to z}(f(z) + (x-z)F(x))/(g(z) + (x-z)G(x)) == \lim_{x \to z} f(z)/g(z)$.

Hopital rule

[Hopital rule] see L'Hopital rule.

hyperbola

A [hyperbola] is curve in the plane which can be described as the graph of the function $f(x) = 1/x$. Also translated, scaled and rotated versions of this curve is called a hyperbola. For example, the set of points (x, y) in the plane which satisfy $(x-1)^2 - (y-2)^2 = 5$ is a hyperbola.

Mean value theorem

[Mean value theorem]. If $f(x)$ is a continuous function on an interval $I = \{a \leq x \leq b\}$ which is differentiable on the open interval (a, b), then there exists a point $x \in I$ for which $f'(x) = C = (f(b) - f(a))/(b-a)$. Proof. Otherwise, $f'(x) < C$ on (a, b) or $f'(x) > C$ on (a, b). Integration gives using the fundamental theorem of calculus

$$\begin{array}{ll} f(x) - f(a) = \int_a^x f'(t)dt < C(x-a) & \text{or} \\ f(x) - f(a) = \int_a^x f'(t)dt > C(x-a) & \end{array}$$

especially

$$\begin{array}{ll} f(x) - f(a) = \int_a^b f'(t)dt < C(b-a) & \text{or} \\ f(x) - f(a) = \int_a^b f'(t)dt > C(b-a) & \text{which is a contradiction} \end{array}$$

The mean value theorem is a special case of the extended mean value theorem.

Rolle's theorem

[Rolle's theorem] If $f(x)$ is a continuous function on the interval $I = \{a \leq x \leq b\}$ which is differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists a point $x \in (a, b)$, for which $f'(x) = 0$.
Proof. f takes both its maximum and minimum on I. If the maximum is equal to the minimum, then $f(x)$ is constant on I, otherwise, either the minium or the maximum is a point x in (a, b). At that point $f'(x) = 0$. qed. Rolle's theorem is a special case of the mean value theorem

rule of three

The [rule of three] ia a rough rule of thumb when solving calculus problems or teaching calculus:

> Look at a calculus problem graphically, numerically and analytically.

In other words, one should try to understand a calculus problem geometrically, algebraically and computationally. For example, the notion of the derivative of a function of one variable can be understood geometrically as a slope, can be understood through algebraic manipulations like $(x^n)' = nx^{n-1}$ or computationally by plugging in numbers or doing things on a computer.

Weierstrass function

A [Weierstrass function] is an example of a function which is continuous but almost nowhere differentiable. An example is $f(x) = \sum_{k=1}^{\infty} \cos(k^2 x)/k^2$.

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