ENTRY FUNCTIONAL ANALYSIS

[ENTRY FUNCTIONAL ANALYSIS] Authors: Oliver Knill: 2002 Literature: various notes

adjoint

The [adjoint] of a bounded linear operator A on a Hilbert space is the unique operator B which satisfies $(Ax, y) = (x, By)$ for all $x, y \in H$. One calls the adjoint A^*. An bounded linear operator is selfadjoint, if $A = A^*$.

Alaoglu's theorem

[Alaoglu's theorem] (=Banach-Alaoglu theorem): the closed unit ball in a Banach space is weak-* compact.

angle

The [angle] ϕ between two vectors v and w of a Hilbert space is a solution ϕ of the equation $\cos(\phi)||v||||w|| = (v, w)$, usually the smaller of the two solutions.

balanced

A subset Y of a vector space X is called [balanced] if tx is in Y whenever x is in Y and $t < 1$.

B*-algebra

A [B*-algebra] is a Banach algebra with a conjugate-linear anti-automorphic involution $*$ satisfying $||xx^*|| = ||x||^2$.

Banach algebra

A [Banach algebra] is an algebra X over the real numbers or complex numbers which is also a Banach space such that $||xy|| \leq ||x||||y||$ for all $x, y \in X$.

Banach limit

A [Banach limit] is a translation-invariant functional f on the Banach space of all bounded sequence such that $f(c) = c_1$ for constant sequences.

Banach space

A [Banach space] is a complete normed space.

barrel

A [barrel] is a closed, convex, absorbing, balanced subset of a topological vector space.

basis

A [basis] (= Schauder basis) of a separable normed space is a sequence of vectors x_j such that every vector x can uniquely be written as $y = \sum_j y_j x_j$.

basis

A [basis] in a vector space is a linearly independent subset that generates the space.

baralled space

A [baralled space] is a topological vector space in which every barrel contains a neighborhood of the origin.

biorthogonal

Two sequences a_n and b_n in a Hilbert space are called [biorthogonal] if $A_{nm} = (a_n, b_m)$ is an unitary operator.

Bergman space

The [Bergman space] for an open subset G of the complex plane C is the collection of all anlytic function f on G for which $\int \int_G |f(x+iy)|^2 \, dxdy$ is finite. It is an example of a Hilbert space.

Buniakovsky inequality

The [Buniakovsky inequality] (=Cauchy-Schwarz inequality) in a Hilbert space tells that $|(a,b)| \leq ||a|| \, ||b||$.

Cauchy-Schwartz inequality

The [Cauchy-Schwartz inequality] in a Hilbert space H states that $|(f,g)| \leq ||f|| \, ||g||$. It is also called Buniakovsky inequality or CBS inequality.

compact operator

A [compact operator] is a bounded linear operator A on a Hilbert space, which has the property that the image $A(B)$ of the unit ball B has compact closure in H.

compact operator

A bounded operator A on a separable Hilbert space is called [diagonalizable] if there exists a basis in H such that $Hv_i = \lambda_i v_i$ for every basis vector v_i. Compact normal operators are diagonalizable.

dimension

The [dimension] of a Hilbert space H is the cardinality of a basis of H. A Hilbert space is called seperable, if the cardinality of the basis is the cardinality of the integers.

Egorov's theorem

[Egorov's theorem] Let (X, S, m) be a measure space, where $m(S)$ has finite measure. If a sequence f_n of measurable functions converges to f almost everywhere, then for every d¿0, there is a set $E_d \subset X$ such that $f_n \to f$ uniformly on $E \setminus E_d$ and $m(E_d) < d$.

finite rank operator

A bounded linear operator A on a Hilbert space H is called a [finite rank operator] if the rank of A is finite dimensional. Finite rank operators are examples of compact operators.

Hilbert space

A [Hilbert space] H is a vector space equiped with an inner product (x, y) for which the corresponding metric $d(x, y) = ||x - y|| = (x - y, x - y)$ makes (H, d) into a complete metric space. Examples:

- $l^2(N)$ is the collection of sequences a_n such that $\sum_n |a_n| < \infty$ is a Hilbert space with inner product $(a, b) = \sum_n a_n b_n$.
- $L^2(G)$ the space of all analytic functions on an open subset of the complex plane which are also in $L^2(G, \mu)$, where μ is the Lebesgue measure on G.
- All vectors of a finite dimensional vector space, where the inner product is the usual dot product.
- All square integrable functions $L^2(X, \mu)$ on a measure space (X, S, μ).

idempotent

A bounded linear operator A on a Hilbert space is called [idempotent] if $A^2 = A$. Projections P are examples of idempotent operators.

Lusin's theorem

[Lusin's theorem] If (X, S, m) is a measure space and f is a measureable function on S. For every $d > 0$, there is a set E_d with $m(E_d) < d$ and a measurable function g such that g is continuous on E_d.

linear operator

A [linear operator] is a linear map between two Hilbert spaces or two Banach spaces. Important examples are bounded linear operators, linear operators which also continuous maps. Linear operators are also called linear transformations.

norm

The [norm] of a bounded linear operator A on a Hilbert space H is defined as $||A|| = \sup_{||x|| \leq 1, x \in H} ||Ax||$.

normal

A bounded linear operator A is called [normal] if $AA^* = A^*A$, where A^* is the adjoint of A. Examples:

- selfadjoint operators are normal.
- unitary operators are normal

Open mapping theorem

[Open mapping theorem] If a map A from X to Y is a surjective continuous linear operator between two Banach spaces X and Y, and U is an open set in X, then $A(U)$ is open in Y.
The proof of the theorem which is also called the Banach-Schauder theorem uses the Baire category theorem. implications:

- A bijective continuous linear operator between the Banach spaces X and Y has a continuous inverse.
- Closed graph theorem: if for every sequence $x_n \in X$ with $x_n \to 0$ and $Ax_n \to y$ follows $y = 0$, then A is continuous.

Riesz representation theorem

[Riesz representation theorem] If f is a bounded linear functional on a Hilbert space H, then there exists a vector $y \in H$ such that $f(x) = (x, y)$ for all $x \in H$.

Sturm Liouville operator

A [Sturm Liouville operator] L is an unbounded operator on the Hilbert space $L^2[a, b]$ defined by $L(f) = -f'' + gf$, where g is a continuous function on $[a, b]$.

unitary

A bounded linear operator A on a Hilbert space is called [unitary] if $AA^* = A^*A = 1$ if 1 is the identity operator $1(x) = x$.

unit ball

The [unit ball] B in a Hilbert space H is the set of all points $x \in H$ satisfiying $||x| \leq 1$.

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