

NATURAL ORBITAL NETWORKS

OLIVER KNILL

ABSTRACT. Given a finite set T of maps on a finite ring R , we look at the finite simple graph $G = (V, E)$ with vertex set $V = R$ and edge set $E = \{(a, b) \mid \exists t \in T, b = t(a), \text{ and } b \neq a\}$. An example is when $R = \mathbb{Z}_n$ and T consists of a finite set of quadratic maps $T_i(x) = x^2 + a_i$. Graphs defined like that have a surprisingly rich structure. This holds especially in an algebraic setup when T is generated by polynomials on \mathbb{Z}_n . The characteristic path length μ and the mean clustering coefficient ν are interlinked by global-local quantity $\lambda = -\mu/\log(\nu)$ which often appears to have a limit for $n \rightarrow \infty$ like for two quadratic maps on a finite field \mathbb{Z}_p . We see that for one quadratic map x^2+a , the probability to have connectedness goes to zero and for two quadratic maps, the probability goes to 1, for three different quadratic maps $x^2 + a, x^2 + b, x^2 + c$ on \mathbb{Z}_p , we always appear to get a connected graph for all primes.

In Memory of Oscar Lanford III.

1. INTRODUCTION

The interest in applied graph theory has exploded during the last decade [19, 6, 18, 11, 28, 26, 40, 21, 41, 5, 20, 38, 28, 15, 12, 34]. The field is interdisciplinary and has relations with other subjects like computer, biological and social sciences and has been made accessible to a larger audience in books like [42, 7, 2, 37, 10].

We look here at generalizations of Cayley graphs which can be used to model networks having statistical features similar to known random networks but which have an appearance of real networks. We have mentioned some connections with elementary number theory already in [23]. The graphs are constructed using **transformation monoids** analogously as **Cayley graphs** are defined by a group acting on a group. The deterministic construction produces graphs which resemble empirical networks seen in synapses, chemical or biological networks,

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the web or social networks, and especially in peer to peer networks.

A motivation for our construction is the iteration of quadratic maps in the complex plane which features an amazing variety of complexity and mathematical content. If the field of complex numbers C is replaced by a finite ring R like Z_n , we deal with arithmetic dynamical systems in a number theoretical setup. This is natural because a computer simulates differential equations on a finite set. It is still not understood well, why and how finite approximations of continuum systems share so many features with the continuum. Examples of such investigations are [39, 31, 30]. The fact that polynomial maps produce arithmetic chaos is exploited by pseudo random number generators like the Pollard rho method for integer factorization [32].

A second motivation is the construction of deterministic realistic networks. Important quantities for networks are the characteristic path length μ and the mean cluster coefficient ν . Both are averages: the first one is the average of $\mu(x)$, where $\mu(x)$ is the average over all distances $d(x, y)$ with $y \in V$, the second is the average of the fraction of the number of edges in the sphere graph $S(x)$ of a vertex x divided by the number of edges of the complete graph with vertices in $S(x)$. Nothing changes when replace μ with the **median path length** and ν with the **global cluster coefficient** which is the frequency of oriented triangles $3v_2/t_2$ within all oriented connected vertex triples.

A third point of view is more algebraic. As we learned from [9], orbital networks relate to **automata**, edge colored directed graphs with possible self loops and multiple loops. An automaton can encode the monoid acting on the finite set. But the graph of the monoid T is obtained by connecting two vertices x, y if there is no element T with $T(x) = T(y)$. This graph is the null graph if M is synchronizing (there is f which has a single point as an image) and the complete group if M is a permutation subgroup of the full permutation group [8]. An example of a problem in that field is the task to compute the probability that two random endofunctions generate a synchronizing monoid. This indicates that also the mathematics of graphs generated by finitely many transformations can be tricky. We became also aware of [36], who developed a theory of finite transformation monoids. In that terminology, the graph generated by T is called an **orbital digraph**. This prompted us to address the finite simple graphs under consideration as **orbital networks**.

That the subject has some number theoretical flavour has been indicated already [23]. It shows that elementary number theory matters when trying to understand connectivity properties of the graphs. We would not be surprised to see many other connections.

Modeling graphs algebraically could have practical values. Recall that in computer vision, various algorithms are known to represent objects. The highest entropy version is to give a triangularization of a solid or to give a bitmap of a picture. Low entropy realizations on the other hand store the object using mathematical equations like inequalities or polynomials in several variables. It is an AI task to generate low entropy descriptions. This can mean to produce vector graphics representations of a given bitmap, or to use Bézier curves and Nurb surfaces to build objects. Fractal encoding algorithms have been used to encode and compress pictures: iterated function systems encode similar parts of the picture [4]. Similarly, an application of orbital networks could be to realize parts of networks with low entropy descriptions in such a way that relevant statistical properties agree with the actual network.

In the context of computer science, monoids are important to describe languages as **synthactic monoids**. An analogue of Cayley's theorem tells that every group is a subgroup of a transformation group, every finite monoid can be realized as a transformation monoid on a finite set X . When seen from an information theoretical point of view, an orbital network describes a language, where the vertices are the alphabet and the transformations describe the rules. Every finite path in the graph now describes a possible word in the language.

Finally, there is a geometric point of view, which actually is our main interest. Graphs share remarkably many parallels with Riemannian manifolds. Key results on Riemannian manifolds or more general metric spaces with cohomology have direct analogues for finite simple graphs. While functionals like the Hilbert action on Riemannian manifolds are difficult to study, finite simple graphs provide a laboratory to do geometry, where one can experiment with a modest amount of effort. We will look elsewhere at the relation of various functionals on graphs, like the Euler characteristic, characteristic path length, Hilbert action. It turns out that some of the notions known to graph theory only can be pushed to Riemannian manifolds. Functionals studied in graph theory can thus be studied also in Riemannian geometry.

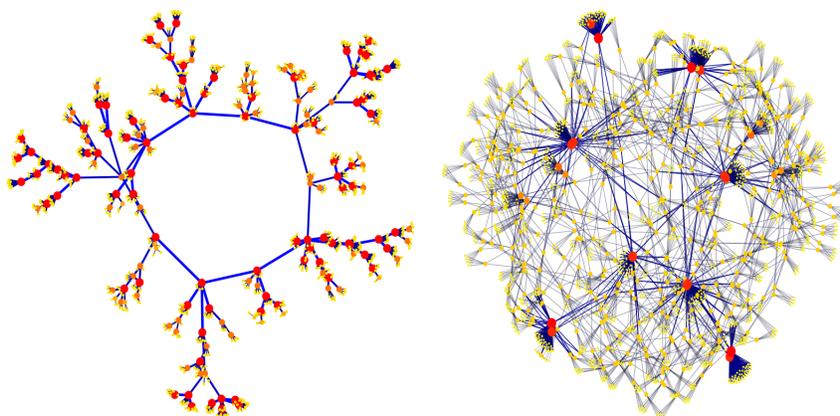


FIGURE 1. The first graph G_1 is obtained from a single quadratic map $f(x) = x^2 + 226$ on Z_{1001} . It has diameter 14, average degree $\delta = 2$, characteristic path length $\mu = 8.6$ and $\nu = 0$. It resembles a typical friendship graph [21]. The second graph is generated by $f(x) = x^2 + 1, g(x) = x^2 + 2$ on Z_{2000} . It has diameter 9, $\delta = 3.99, \mu = 5.8, \nu = 0.0024$ and $\lambda = 0.964$.

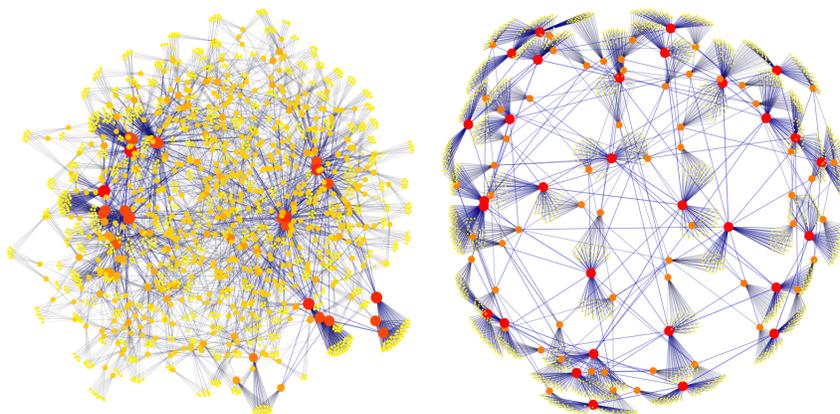


FIGURE 2. We see the graph generated by $f(x) = x^2 + 1, g(x) = x^2 + 31, h(x) = x^2 + 51$ on Z_{2000} . It has diameter 8, average degree 6, $\mu = 4.7, \nu = 0.0084$ and $\lambda = 0.986$. The second graph is generated by $T(x) = 2^x + 11$ and $S(x) = 3^x + 5$ on Z_{2002} . It has diameter 7, $\mu = 4.6, \nu = 0.00097$ and $\lambda = 0.66$.

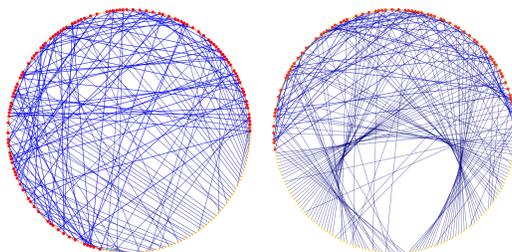


FIGURE 3. The graph on Z_{229} generated by $T(x) = x^2 + 57$ and a graph on Z_{229} generated by $T(x) = 2^x + 13$ are shown with a circular embedding.

Acknowledgement. The graph construction described here emerged during a few meetings in September and October 2013 with Montasser Ghachem who deserves equal credit for its discovery [17], but who decided not to be a coauthor of this paper.

2. CONSTRUCTION

Given a fixed ring R like $R = Z_n$ and a finite set T of maps $R \rightarrow R$, a digraph is obtained by taking the set of vertices $V = R$ and edges $(x, T_i(x))$, where T_i are in T . By ignoring self loops, multiple connections as well as directions, we obtain a finite simple graph, which we call the **dynamical graph** or **orbital network** generated by the system (R, T) . The name has been used before and distinguishes from other classes of dynamically generate graphs: either by random aggregation or by applying deterministic or random transformation rules to a given graph.

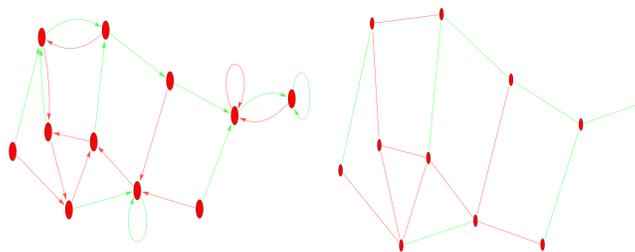


FIGURE 4. The two maps $T_1(x) = x^2 + 3$ and $T_2(x) = x^2 + 2$ on Z_{11} define an **orbital digraph** with edges $x \rightarrow T_i(x)$. Ignoring the direction, multiple connections and self loops we get a finite simple graph which we call an **orbital network**.

Orbital networks have been used since a long time. If a group given by a finite set of generators T acts on itself, the visualization is called the **Cayley graph** of the group. It is a directed graph but leaving away the direction produces a finite simple graph which is often used as a visualization for the group. These graphs are by definition vertex transitive and especially, the vertex degree is constant. For one-dimensional dynamical systems given by maps defined on finite sets, the graph visualization has been used at least since Collatz in 1928 [44]. It appears also in demonstrations [45, 46] or work [22] on cellular automata. The graphs are sometimes called **automata networks** [33]. The iteration digraph of the quadratic map has been studied in [35]: As has been noted first by Szalay in 1992, as has been a symmetry if $n = 2$ modulo 8 or $n = 4$ modulo 8. Szalay also noted that the number of fixed points of $x \rightarrow x^2$ is $2^{\omega(n)}$ where $\omega(n)$ is the number of distinct primes dividing n . The miniature theorem on Fermat primes rediscovered in [23] has already been known to Szalay in 1992 and Rogers in 1996. Somer and Krizek show that there exists a closed loop of length t if and only if $t = \text{ord}_n(2)$ for some odd positive divisor d of the Carmichael lambda function $\lambda(n)$ and where $\text{ord}_n(g)$ is the multiplicative order of g modulo n .

When looking at classes of maps, we get a probability space of graphs. For example, if we consider two quadratic maps $T_i(x) = x^2 + c_i$ with $c_i \in R$. we get a probability space R^2 of parameters (c_1, c_2) . With k quadratic maps, the probability space R^k presents itself.

An other example is to take the ring $R = Z_n \times Z_n$ and to consider the Hénon type maps $T_i(x, y) = (x^2 + c_i - y, b_i x)$, where c_i are parameters, where again with k maps the probability space is R^k . If $b_i = 1$, the maps are invertible and generate a subgroup of the permutation group. An interesting class of graphs obtained with $T_i(x) = [x^\alpha + c_i]$ on Z_n , where $[x]$ is the floor function will be studied separately. We focus first on maps which are defined in an arithmetic way. An other example is by replacing polynomials with exponential maps like with $T_i(x) = 2^x + c_i$ on Z_n or linear matrix maps like $T_i(x, y) = A_i x$ on Z_n^k or higher degree polynomial maps like $T_i(x, y) = ax^2 + by^2 + cxy + c_i$. An example of algebro-geometric type is to look at the ring $R = K[x]/I$, where $K[x]$ is the polynomial ring over a finite field K and I an ideal, then consider the two maps $T_1(f) = f^2 + a, T_2(f) = f^2 + b$, where a, b are fixed polynomials.

Observation 1. Every finite simple graph is an orbital network with d generators, where d is the maximal degree of the graph.

Proof. If the graph has n vertices, take $R = Z_n$. Now label the elements in each sphere $S(x)$ with $y_1, \dots, y_{k(x)} \leq d$. Now define $T_i(x) = y_i$ for $i \leq k(x)$ and $T_i(x) = x$ for $i > k(x)$. \square

We often can use less generators. Of course our interest is not for general graphs, but rather for cases, where T_i are given by simple arithmetic formulas, where we have a natural probability space of graphs. For example, if T is generated by d polynomials T_i on Z_n , then Z_n^d equipped with a counting measure is a natural probability space.

An other natural example is if T_i are random permutations on Z_n . This is a random model but the statistics often mirrors what we see in the case of arithmetic transformations, only that in the arithmetic case, the n dependence can fluctuate, as it depends on the factorization of n .

Observation 2. If the monoid generated by T can be extended to a group, then a connected orbital network is a factor of a Cayley graph.

Proof. If the monoid generated by T is a group U , then all T_i are permutations. Take a vertex $x_0 \in V$. Since the graph is connected, Ux_0 is the entire vertex set. The graph homomorphism $\phi(f) = fx_0$ shows that the graph G is a factor of the Cayley graph of G generated by T . \square

Remark. Embedding the monoid in a group using a Grothendieck type construction is not always possible.

3. COMPARISON WITH REAL NETWORKS

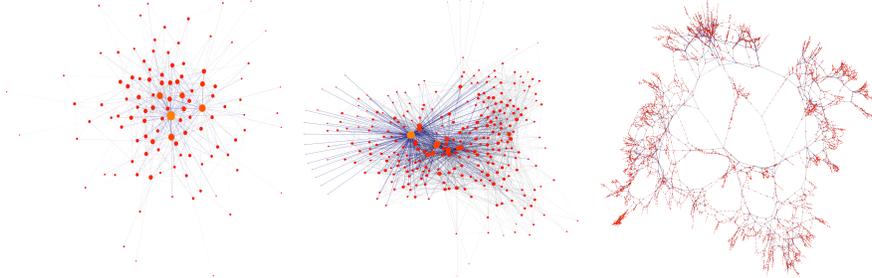


FIGURE 5. Examples of publicly available networks. The first is the word adjacency graph of common adjectives and nouns in "David Copperfield" [27], the second is a neural network of *Caenorhabditis Elegans* [43], the third graph shows the Western states power grid [43]. We have $\lambda(\text{adjnoun}) = 1.36915$, $\lambda(\text{celegans}) = 1.43514$ and $\lambda(\text{powergrid}) = 8.35961$,

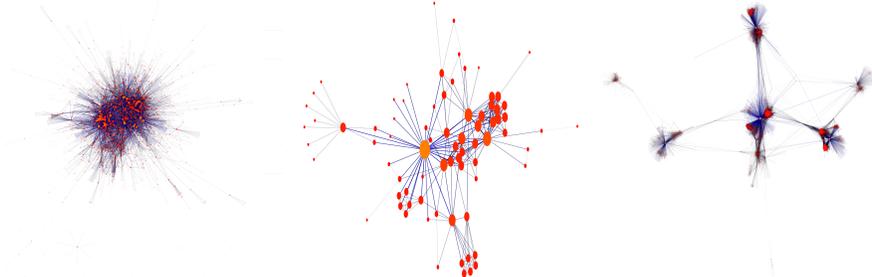


FIGURE 6. The yeast protein-protein network [43], the characters in the novel *Les Misérables* [25, 14] and finally part of facebook from the Stanford database. We measure the length-cluster coefficients: $\lambda(\text{yeast}) = 1.82013$, $\lambda(\text{miserables}) = 3.79865$ and $\lambda(\text{facebook}) = 5.63298$,

4. LENGTH-CLUSTER COEFFICIENT

Various graph quantities can be measured when studying graphs. Even so we have a deterministic construction, we can look at graph quantities as **random variables** over a probability space of a class of generators T . Examples are the global degree average $2|E|/|V|$, the Euler characteristic $\chi(G) = \sum_{k=0} (-1)^k c_k$, where c_k is the number of K_{k+1} subgraphs of G , the inductive dimension $\dim(G)$ [24], the number of cliques

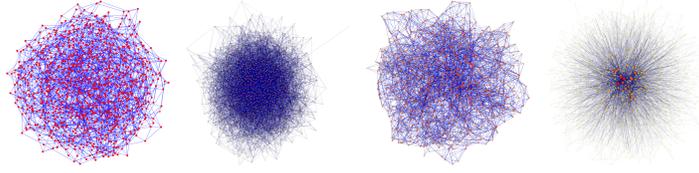


FIGURE 7. Examples of random graphs with 1001 vertices. Their shape is in general more amorphic. The first is a **random permutation graph** T, S on Z_{1001} instead of arithmetic maps. Its average degree is very close to 4, the characteristic path length is 5.6, the global cluster coefficient 0.001 and the length cluster coefficient is 0.81. The second figure shows a **random Erdős-Renyi graph** [13] with 1001 vertices. It has average degree 9.8, characteristic path length 3.3, global cluster coefficient 0.0078 and $\lambda = 0.67$. The third figure is a **Watts-Strogatz graph** [43] with 1001 nodes with an 8-regular graph, rewiring probability of 0.2, characteristic path length 4.3, global cluster coefficient 0.32 and $\lambda = 3.9$. The fourth figure shows a **random Barabasi-Albert graph** [3] with $k = 4$. The average degree is close to 8, the characteristic path length 3.2 and the global clustering coefficient 0.025 and $\lambda = 0.866$.

$v_k(G)$ of dimension k in G , the average degree $2c_1/c_0 = 2|E|/|V|$, the Betti numbers $b_k(G)$, the diameter $\text{diam}(G)$, the characteristic path length L , the global clustering coefficient C which is the average occupancy density in spheres [43] or the variance of the degree distribution or curvature $K(x) = \sum_{k=0}^{\infty} (-1)^k V_{k-1}(x)$, where $V_i(x)$ is the number of subgraphs K_{k+1} in the unit sphere $S(x)$ of a vertex.

An interesting quantity to study in our context is the **length-clustering coefficient**

$$(1) \quad \lambda = E\left[-\frac{\mu}{\log(\nu)}\right],$$

where μ is the average characteristic length and ν the mean cluster coefficient. We are not aware that it has been studied already.

The largest accumulation point limsup and the minimal accumulation point liminf can be called the **upper and lower Length-Cluster coefficient**. For most dynamical graphs, we see that the liminf and limsup exists for $n \rightarrow \infty$ and that the limit exists along primes. It

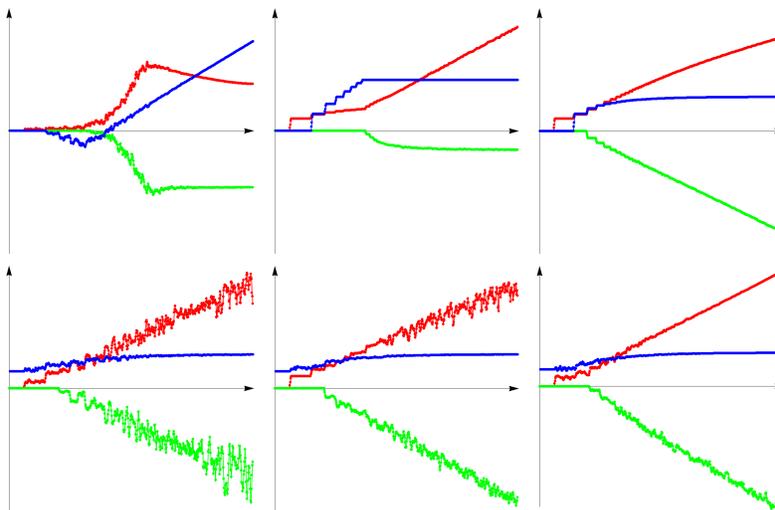


FIGURE 8. The characteristic path length μ (red), the log of the cluster coefficient $\log(\nu)$ (green) and the average vertex degree δ (blue) are shown together as a function of n with logarithmic scale in n ; logarithmic growth therefore is shown linearly. First for Erdos-Renyi (where each edge is turned on with probability $p = 0.1$), then for Watts-Strogatz for $k = 4, p = 0.1$ and then for Barabasi-Albert. In the second row we see first the case of two quadratic maps, then two exponential maps, and finally two random permutations.

exists also for Erdos-Renyi graphs, where μ and ν have limits themselves also for fixed degree random graphs, where we see λ to converge. The limit is infinite for Watts-Strogatz networks in the limit $n \rightarrow \infty$ due to large clustering and zero for Barabasi-Albert networks in the limit $n \rightarrow \infty$. For graphs defined by random permutations, we see λ converges to 1.

It has become custom in the literature to use a logarithmic scale in n so that one gets linear dependence of μ and $\log(\nu)$. Despite the fact that both clustering coefficient and characteristic length are widely used, the relation between these two seems not have been considered already. While the clustering coefficient at a vertex is a local property which often can be accessed, the characteristic length is harder to study theoretically. For random graphs we measure clear convergence of λ in the limit:

In the first case, the probability space is the set of all pairs of permutations on Z_n . It is a set with $(n!)^2$ elements. Here are some questions.

Length-Cluster convergence conjecture I: for random permutation graphs with 2 or more generators, the length-clustering coefficient (1) has a finite limit for $n \rightarrow \infty$.

In the next case, we take for $d = 2$ the probability space Z_n^2 of all pairs (a, b) leading to pairs of transformations $T(x) = x^2 + a$, $S(x) = x^2 + b$. Also for the following question there is strong evidence:

Length-Cluster convergence conjecture II: Graphs defined by $d \geq 2$ quadratic maps on Z_p with prime p , the expectation of the limit λ exists in the limit $p \rightarrow \infty$.

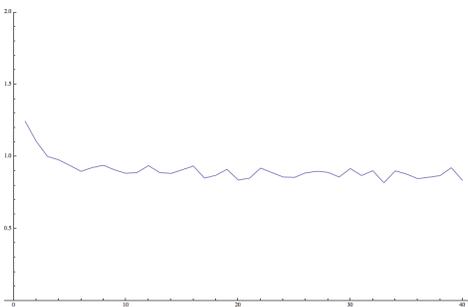


FIGURE 9. This figure shows the expectation of the length-cluster coefficient for the graph generated by two random permutations T, S on Z_n as a function of n . For every n , we have computed the average of 500 random permutations. Unlike for arithmetic maps, number theoretic considerations play a lesser role for random permutations.

This remarkable relation between the global clustering coefficient ν which is the average of a local property and the characteristic length μ , which involves the average length of geodesics in the graph and is not an average of local properties. Intuitively, such a relation is to be expected because a larger ν will allow for shorter paths. If the limit exists then $\mu = -\lambda \log(\nu)$ which would be useful to know because the characteristic length is difficult to compute while the clustering coefficient ν is easier to determine. To allow an analogy from differential geometry, we can compare the local clustering coefficient $\nu(x)$ with curvature, because a metric space large curvature has a small average

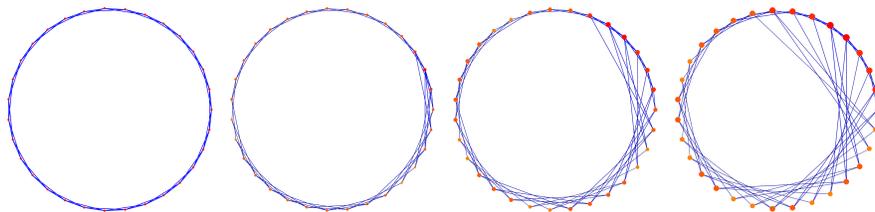


FIGURE 10. The alpha graphs start like the Watts-Strogatz graphs. The rewiring is deterministic. See [16].

distance between two points.

Relations between local properties of vertices and global characteristic length are not new. In [29], a heuristic estimate $mu = 1 + \log(n/d)/\log(d_2/d)$ is derived, where d is the average degree and d_2 the average 2-nearest neighbors. Note that $d = 2e/v$ so that $\log(n/d)$ can be interpreted as the logarithm of the average edge density on n vertices and that $\log(d_2/d)$ can be seen as a **scalar curvature**. Unlike the Newman Strogatz Watts formula, which uses global edge density we take the average of the edge density in spheres. To take an analogy of differential geometry again, we could look at graphs with a given edge density and minimize the average path length between two points. This can be seen as a path integral. We will look at the relation of various functionals elsewhere.

We have studied in [16] graphs generated by finitely many maps $T_i(x) = [x^{\alpha_i} + a_i] \bmod n$, where a_i, n are positive integers and where α is a real parameter. We see that for α larger than 1 and not too close to 1 and α not an integer, the graphs are essentially random, while for $\alpha < 1$ or α close to 1, there are geometric patterns. We are obviously interested in the α dependence. The reason for the interest is that for $a_i = i$ we get Watts-Strogatz initial conditions for $p = 0$. In [16], we especially took maps of the form $T_1(x) = x + 1, T_i(x) = [x^{1+p} + i] \bmod n, i \geq 2$ on Z_n , where $[x]$ denotes the floor function giving the largest integer smaller or equal to x . These are deterministic graphs which produce statistical properties as the Watts-Strogatz models.

5. CONNECTIVITY FOR QUADRATIC ORBITAL GRAPHS

We have looked already at special connectivity questions in [23]. One challenge for affine maps is to find necessary and sufficient conditions that the map $T(x) = ax + b$ leads to a connected graph on Z_n . An other **network Mandelbrot challenge** is to that the graph on Z_n

generated by $T(x) = 3x + 1, S(x) = 2x$ is connected for all n . We look at other examples here, which are more of probabilistic nature.

Given a probability space of graphs, we can look at the probability that a graph is connected. Here are three challenges for quadratic maps along prime n : we see in one dimensions that connectivity gets rarer, in two dimensions that it becomes more frequent and in three or higher dimensions that connectivity is the rule.

In the following, we look at maps $T(x) = x^2 + a$ generated on Z_n , where n is prime. We denote by $C(n)$ the probability that the graph is connected, where the probability space is the set of d different maps $T(x) = x^2 + a$. Denote by p_k the k 'th prime.

Quadratic graph connectivity A) With one quadratic map, the connectivity probability is

$$C(p_k) = O\left(p_k \frac{\log(k)}{k^2}\right).$$

For random permutation graphs, we have $C(n) = 1/n$ because there are $(n - 1)!$ cycles in a group of $n!$ transformations. Dixon's theorem tells that the probability that two random transformations generate a transitive subgroup is $1 - 1/n + O(1/n^2)$ [9].

Quadratic graph connectivity B) If we have two quadratic maps, the connectivity probability is

$$(1 - C(p_k)) = O\left(\left(\frac{\log(p_k)}{k}\right)^2\right)$$

Quadratic graph connectivity C) With three different quadratic maps on Z_p , then all graphs are connected.

We have checked C) until prime $p = 229$.

6. SYMMETRIES IN ARITHMETIC GRAPHS

If the maps under consideration preserve some symmetry, then also the graphs can share this symmetry. We illustrate this with a simple example:

Proposition 1 (Miniature: symmetry). *Assume n is even and $T_i(x) = x^2 + c_i$ on Z_n and c_i are even, then the graph G is the union of two disconnected graphs G_1, G_2 . If n is not a multiple of 8, the two graphs*

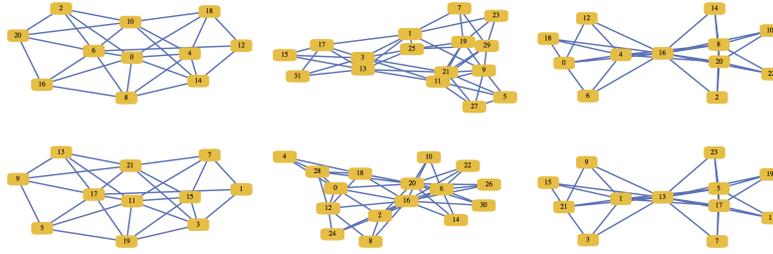


FIGURE 11. The first figure shows the graph on Z_{22} generated by $f(x) = x^2 + 2, g(x) = x^2 + 6, h(x) = x^2 + 16$, where we have an isomorphism between the two components. The second figure shows $n = 32, f(x) = x^2 + 2, g(x) = x^2 + 12, h(x) = x^2 + 16$, where we have no isomorphism. The third example shows $n = 24, f(x) = x^2 + 4, g(x) = x^2 + 12, h(x) = x^2 + 16$, one of the rare cases, where we have an isomorphism, even so n is a multiple of 8.

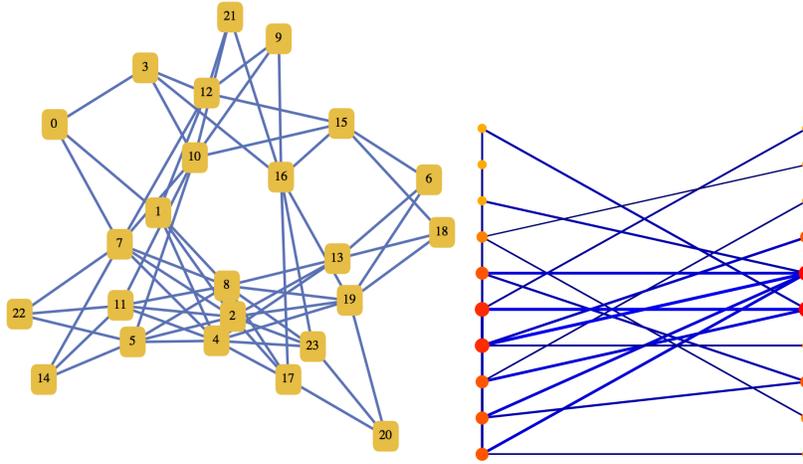


FIGURE 12. The first figure shows the bipartite graph defined on Z_{24} with generators $x^2 + 1, x^2 + 3, x^2 + 7$. The second graph shows it organized as a bipartite graph.

G_i are isomorphic.

Proof. All maps leave the subsets of even and odd nodes invariant. This shows that the graph is the union $G_1 \cup G_2$ of two graphs which have no connection. To see the isomorphism, we look at three cases, $n = 8k + 2, n = 8k + 4$ and $n = 8kj + 6$. In all cases, we construct an

isomorphism which satisfies $\phi(T(x)) = T(\phi(x))$.

(i) If $n = 8k + 2$, then the isomorphism is $\phi(x) = x + n/2$. It maps even nodes to odd nodes and vice versa. We check that

$$\phi(T(x)) - T(\phi(x)) = (8k + 2)(2k + x)$$

showing that the left hand side is zero modulo $n = 8k + 2$. indeed, the left hand side is $(-4 * k - 16 * k^2 - 2 * x - 8 * k * x)$ which agrees with the right hand side.

(ii) If $n = 8k + 4$, then the isomorphism is $\phi(x) = x + n/4$. Also this map gets from G_1 to G_2 and G_2 to G_1 . Again, we check that

$$\phi(T(x)) - T(\phi(x)) = (8k + 4)(k + x)/2 .$$

Now, if k is even, we apply this to the even nodes, if k is odd we apply this map to odd nodes. In both cases, it is invertible.

(iii) Finally, if $n = 8k + 6$, then the isomorphism is again $\phi(x) = x + n/2$. Again we see that $\phi(T(x)) - T(\phi(x))$ is a multiple of n for any map T . \square

Proposition 2 (Miniature: bipartite). *If $T_i(x) = x^2 + c_i$ on Z_n where n is even and c_i are odd, then the graph is bipartite and does not have any odd cycles. It in particular does not have any triangles, so that the global clustering coefficient is zero in that case.*

Proof. We can split the vertex set into an odd and even component $V = V_o + V_e$. Each of the maps T_i forms a connection from V_o to V_e . \square

Remarks.

1) We see experimentally that if n a multiple of 8, then for most cases, there is no isomorphism of the two component. We expect the probability of such events goes to zero for $n \rightarrow \infty$. Statistically, it is unlikely that two random graphs are isomorphic, so that we just might have rare events.

2) If n is even and not divisible by 4 and all generators are polynomials which preserve odd or even numbers, then the graph G is the disjoint union of two graphs. If all generators switch from odd to even numbers, then the graph is bipartite. If $\phi(x) = x + n/2$ and $n = 4k + 2$ and $\phi(T(x)) - T(\phi(x))$ is a multiple of n then ϕ is a conjugation. In any case, if all T_i switch from even to odd, then the two parts of the bipartite graph are the even and odd numbers.

7. REMARKS

Here are some additional remarks:

1) Motivated by the Mandelbrot set which is defined as the set the parameters $c = a + ib$ for which the Julia set J_c is not connected, we can look at all the parameter a, b for which the arithmetic graph generated by $x^2 + a, x^2 + b$ acting on $R = \mathbb{Z}_n$ is connected. This is encoded in the matrix A_{ab} which gives the number of components of the graph. It depends very much on number theoretical properties.

2) How many quadratic generators are necessary on \mathbb{Z}_n to reach a certain edge density? Certainly, n generators $T_i(x) = x^2 + i$ suffice to generate the complete graph K_n . The edge density $|E|/|V|$ is half of the average degree d . By adding more generators, we increase d and C and decrease L . It is in general an interesting **modeling question** to find a set of polynomial maps which produce a graph similar to a given network.

3) On $R = \mathbb{Z}_n$ with d generators of the form $x^2 + a_i$, we call $r_d(n)$ the smallest diameter of a graph which can be achieved. How fast does $r_d(n)$ grow for $n \rightarrow \infty$? This is already interesting for $d = 1$, where we see not all minimal diameters realized. For $n = 90$ and $d = 1$ for example, all graphs are disconnected so that $r_1(90) = \infty$. For $n = 466$, the minimal diameter is 55, the next record is $r_1(486) = 85$, then $r_1(1082) = 89, r_1(1454) = 93$ and $r_1(1458) = 247$. We did not find any larger minimal diameter for $n < 2000$. For $d = 2$, the diameters are smaller. For $n = 2$ the minimal diameter is 1, for $n = 4$, it is 2, for $n = 9$ it is 3, for $n = 17$ it is 4, for $n = 30$ it is 5 for $n = 67$ it is 6. For $n = 131$ the minimal diameter is 7. For $n = 233$ we reach 8. We still have to find an n where the minimal diameter is larger than 8. For $d = 3$, where we average over n^3 graphs, we see $r_3(16) = 3$ and $r_3(41) = 4$ and $r_3(97) = 5$. We did not find n yet for which $r_3(n) > 5$.

4) Let $c_d(k)$ denote the largest clique size which can be achieved by d quadratic maps. If the monoid R is generated by two generators, then the graph can have K_4 subgraphs but it is very unlikely. For three generators, it happens quite often. For example, for $n = 40$ and $f(x) = x^2 + 4, g(x) = x^2 + 29, h(x) = x^2 + 24$ the graph G has 9 cliques K_4 . The question is already interesting for $d = 1$, where we the maximal cliques are triangles. Triangles are rare but they occur. The graph generated on \mathbb{Z}_{57} with $f(x) = x^2 + 30$ has two triangles. This is equivalent to the fact that the Diophantine equation $f(f(f(x))) = x$ of degree 8 has a solution modulo 57.

5) Many graphs on a ring R constructed with maps in T have symmetries. If all elements T are invertible, then we have a group action on R and this group is a subgroup of the automorphism group of the graph. We can for example take n prime and $T = \{f, g\}$ acting on \mathbb{Z}_n^2 by two Henon type maps $f(x, y) = (x^2 - y, x), g(x, y) = (y, y^2 - x)$. The

subgroup of permutations of R generated by these two permutations is a subgroup of the automorphism group of G .

6) We can compute the global clustering coefficient in the graph as an expectation $C(n)$, when looking at the probability space of all pairs of quadratic maps $\{f(x) = x^2 + a, g(x) = x^2 + b\}$. We measure the average to decay like $3/n$. This means that we expect in 3 of n vertices to have a triangle. A triangle means either that $T^2(x) = S(x)$ or $T(x) = S^2(x)$. Counting the number of solutions to the Diophantine equations $(x^2 + a)^2 + a = x^2 + b$ which is $x^4 + x^2(2a - 1) + (a + a^2 + b) = 0$ modulo n . The measurements show that when taking a, b random, we have 3 solutions in average. We have 4 solutions in general but if n factors, then there can be more. On the other hand, multiple solutions brings the average down. For three maps $d = 3$, we see that the clustering coefficients decays like $4/n$ and the average degree to be close to 6. When comparing with random graphs, where the average degree is p , we see that the clustering coefficient is proportional to p .

7) The average degree of a graph $G = (V, E)$ is $a_2(n) = 2|E|/|V|$ by the Euler handshaking lemma. In our case, the average vertex degree for two quadratic polynomials fluctuates but converges to $2d$, where d is the number of generators. We have computed it for all polynomial pairs on Z_n for $n = 1$ to $n = 200$ and $d = 2$. The average vertex degree is exactly 4 if all $T_1(x) \neq T_2(x)$. The difference $a_2(n) - 4$ depends on the number of solution pairs $x^2 + a = y^2 + b$ and is of number theoretical nature. The local maxima are obtained if n is prime.

8) The characteristic path length is defined as the average distance between two vertices in the graph. There are not many analytical results available (see [1]). The networks generated by two quadratic maps have a characteristic path length which grows logarithmically, similar than Watts-Strogatz. We can slow it down and behave like Barabasi-Albert if we take generators f, g, fg, gf which naturally also brings the clustering coefficient up. An other possibility is to add an affine map $T(x) = x + 1$ (see [?]).

9) The vertex distribution can depend very much on arithmetic properties. If n is prime and T consists of two different quadratic maps $x^2 + a$ and $x^2 + b$ then only vertex degrees 2, 3, 4, 5, 6 can appear, half have degree 2 or 6 and half have degree 4. We can get smoother vertex degree distributions by taking maps like $f, g, f(g), g(f), f(f), g(g)$.

10) If T is generated by a single map, the dimension is ≤ 2 and the Euler characteristic of the graph is nonnegative. The reason is that there are no K_4 subgraphs and so no K_5 subgraphs. We see no $K_{3,3}$ subgraphs indicating that all these graphs are planar. In any case, the Euler characteristic is $\chi(G) = c_0 - c_1 + c_2$ because $c_n = 0$ for $n \geq 3$.

By the Euler Poincaré formula it is $\chi(G) = b_0 - b_1$, where b_0 is the number of components and b_1 is the number of cycles. Because every attractor is homotopic to a cycle because every orbit eventually loops on a cycle, the number of components is larger or equal than the number of cycles. More generally, the dimension of an arithmetic graph generated by d transformations has dimension $\leq d + 1$. It would be interesting to get bounds on the Euler characteristic.

11) Instead of Z_n we can take rings like a finite ring $Z_n[i]$ of Gaussian integers. One could ask, for which n the graph on the ring of $Z_n[i]$ generated by x^2, x^3, x^5 is connected.

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DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA, 02138