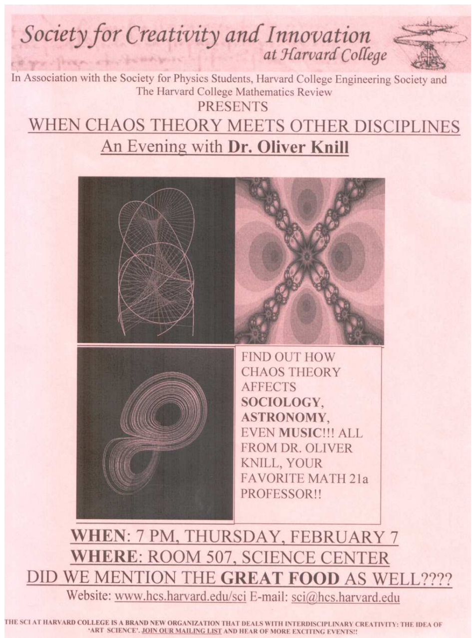
On the lecture: "When chaos theory meets other disciplines"

Harvard February 7, 2008, Oliver Knill

Here are some annotations about the lecture.

The abstract

"The mathematics of low dimensional chaos theory is now over 100 years old. The subject got a popularity boost about 30 years ago when personal computers became widely available. While the field has lost some of its fancy, it has inspired and influenced other disciplines both in mathematics and beyond. This took place not only because of the beautiful results which were achieved but also because it became a prototype where a fresh point of view invigorated a mathematical field, excited the general public and inspired other scientific areas. In this highly illustrated and accessible talk, we take a pedestrian walk through some examples and see how the subject has inspired other disciplines. We will see examples from mathematical fields like geometry or number theory, wave motion physics, celestial dynamics in astronomy and mathematical areas like population dynamics in sociology, financial predictions in economics, music, visual art or film. The story is an example on how crossing scientific boundaries has boosted the creativity on both sides."



Geometry: About chaos

The theory of dynamical systems traditionally aims to predict the outcome of future events. As Henry Poincaré already realized early in the 20'th century, this goal is often futile, even for low dimensional systems: there are systems for which the computation to a given precision needs an effort which grows exponentially with time. He made his case for the three body problem already. Many definitions of chaos are known. For Hamiltonian systems, chaos can be defined as the positivity of Lyapunov exponents on a set of positive measure or the positivity of metric entropy. In a Birmingham conference, I had suggested a spectral definition for interpolating between "integrable" and chaotic. Integrable means point spectrum, chaotic means absolutely continuous spectrum. The crux is to investigate this for all invariant measures.

The lecture illustrates chaos with an experiment, which I first showed with Mathematica in 1994 in a class at Caltech: the two maps $T(x) = 4x(1-x)$ and $S(x) = 4x - 4x^2$ do not produce the same orbit, even when done with a computer algebra system to high accuracy. One can rephrase this as the fact that computers do not obey the associativity law. The reason are **rounding errors**, which happen in a different way. These rounding errors propagate in time and become of order 1 afte n steps where $e^{\lambda n} 10^{-17} = 1$ which means we expect them after $n = \log(10^{17})/\log(2) = 57$ steps. This is pretty close to what we see in the experiments.

Chaos: low dimensional maps

The fact that it is difficult to understand even the simplest two dimensional maps fascinated me early on and I worked myself for more than a decade on the problem to show that the map $T(x, y) = (2x + c\sin(x) - y, x)$ is chaotic in the sense that $DT^n(x, y)$ grows exponentially for a set of positive measure. The problem is still open. I myself have tried geometric, variational methods, complex analysis and spectral methods to solve this. I produced the phase space pictures with a C program "chirikov" I had written 10 years ago. It is pretty fast but still it takes a while to compute a reasonable picture of the phase space because the return time to a set increases like $1/area$. The movie about the Henon map with the deformed parameter was done with Mathematica.

Geometry: The pedal map

The pedal map is a dynamical system on the set of triangles. A triangle is maped into its **pedal triangle**. The system was introduced in 1988 by Kingston and Synge [**?**]. Peter Lax noticed in 1990 that this map is chaotic [4]. The shape of a triangle is determined by its three angles α, β, γ. It can be represented by a point in an equiliateral triangle, the barycentric coordinates of which are the angles. This map is conjugated to a Bernoulli shift. It is completely chaotic and of the same type then the logistic map with $c = 4$.

Sociology: Evolution of evil

Can moral principles be explained dynamically? Is there an evolutionary process in which certain moral standards emerge naturally? A primitive model to investigate this question is to take a group of "people" and let them interact. Divide them up into "good" people and "bad" people. Good people cooperate and trade fairly leaving both moderately happy. Bad people cheat and take advantage of good people. What will win? If people interact with neighbors and adjust their moral principles to the neighbor who does best in the current situation. The cellular automaton shown in the lecture is the May-Nowak automaton. I programmed this in Mathematica using the same colors as used in Nowak's book [6]. Mathematica is an ideal tool to investigate such things. The actual computation of the movies took a few hours during the night. It was more time than the actual programming.

Mechanics: the falling stick

I had implemented this system in javascript and in flash a few years ago as a web entertainment. The simulated had been done there as a two particle system, which are glued together with a well potential which keeps the particles at a constant distance. The particles bounce at the floor as usual. Using finite particle dynamics works pretty well for simulations with dissipation. The advantage is that one has a very simple system. The Mathematica code for the simulation for this talk took me several days of programming. There is no momentum or angular momentum conservation which helps to evolve the system. There is energy conservation however. What happens at the impact? The trick to evolve the system is to use different pictures. There is the particle picture where two particles have momentum and velocity. Then there is the rigid body picture, where the situation is described by the height of the center of mass and the angle. The code uses both systems. In the free fall case, without interaction, the rigid body picture is used. For the interaction, the particle system is used. One of the difficulties is to detect when one of the particles hits the ground. One has to find accurate roots of transcendental functions, which is geometrically the intersection of an accelerated epicycle with a line. This system is very natural because if we throw a coin, we use it as a random number generator. A higher dimensional version is throwing a dice. In the later case, we have a free falling rigid body which reflects at a plane.

Crystallography: nature of crystals

The word **crystal** is traditionally used for periodic point arrangements, often lattices. Around 1994, crystals with icosahedral symmetry were found and an entire new branch of solid state physics started. These crystals were no more periodic, they were named quasicrystals [8]. A mathematical theorem assures tells a fife fold rotational symmetry are impossible for periodic crystals in the plane or in space: We have periodic crystals, quasicrystals for which one has no periodicity but points spectrum turbulent crystals, where there is no point spectrum and finally chaotic crystal, crystals with absolutely continuous spectrum. Again, also here, for systems with higher dimensional time, the spectral point of view is perfect. I had been interested in crystals because they are examples of dynamical systems in higher dimensional time and because I had used these systems to look for dense sphere packings [3].

Number theory: The Catalan system

These slides were taken from a talk on perfect numbers given at the Math circle. Define $T(x) = \sigma(x) - x$, the sum of all proper divisors of x. Because we have $T(1) = 0$ and $\sigma(0)$ is not defined, we could define $T(0) = 0$ and add so an other fixed point. Because $T^{-1}(1)$ is a prime, the entire set of prime numbers is ending up in 0. Positive fixed points x of T are **perfect numbers**. **Amicable numbers** are periodic orbits of period 2 of T. Periodic cycles of T are called **aliquot cycles**. Periodic cycles of period larger than 2 are called **sociable numbers**. Periodic cycles of length 4, 5, 6, 8, 9 and 28 are known. It is conjectured that every orbit of the dynamical system T is bounded. **Eugene Catalan** first studied this system the year 1888. Experimentally, one sees that about 1 percent of all numbers, for which no end is known. I used this system for a math circle talk. I think, it was the first time that the system has been illustrated on the Ulam spiral.

Complex Dynamics: Mandelbrot set

Even so this is a theme beaten to death, it is still fascinating. Also after having seen it for the 1000'th time. Producing the movie of the Mandelbrot and Newton method zooms were done with the help of the program "xaos". This program allows to make movies while you zoom by hand but unfortunately, there are bugs and the movies shake and rattle when you watch them. But the program uses an external scripting language and I could use Mathematica to write the script for xaos and then ran it. Xaos is a fantastic Mandelbrot zoomer. I know it since more than 10 years and even on my first PC, a pentium 2, it rocked. The author of the program wrote also an addictive game "xkoules", which I still play. This guy knows how to program! I wish more in linux would be written like this. There is a tendency to use scripting languages like Python for much of the unix administration system.

Astronomy: The Sitnikov system

This system is covered in Moser's book "stable and random motion" [5] I like the system because it is possible to demonstrate here without too much technicalities that a three body system is chaotic. It is a restricted three body problem in the sense that the third body is assumed not to influence the two suns. I do not even know whether the two suns stay bounded if one adds some mass to the third body. In my lecture notes on n-body problems in 2005, you find more about it.

Economics: market prediction

The movie "The Bank" suggests that dynamical systems theory and complex dynamics in particular allows to predict the stock market. This is of course rubbish but there are some serious approaches to this. The "fractal market analysis" approach [7] replaces the stochastic differential equations approaches and suggests to model the financial data with fractals. Terms from Gaussian approaches like standard deviation are replaced by fractal dimensions. The need for this could be that the standard deviations are infinite. The risks are larger.

Art: generating art

Using mathematics to motivate or generate art is old. Mathematics appears in tiles like in Eschers work. Fractals are an other source. One of the most amazing project is "Electric Sheep", a collaborative art project by thousands of people all over the world. to collectively share the work of creating morphing abstract animations known as "sheep". This collective "android dream" is an homage to Philip K. Dick's novel "Do Androids Dream of Electric Sheep?"

Music: generating random music

In college, I built from scratch a sequencer for my Atari computer attached by Midi to my electronic piano. The sequencer was programmed in Pascal. Despite the fact that this machine was much less powerful than PC's today, it was able to analyze what I play and in real time produce random

Markovian music. Using random number generators and more generally Markov chains to generate music is old. Randomness is a source for creativity. The theme has been explored by many artists. It appears for example in the book [2]. An example of an article about the dynamical systems perspective in music is [1].

Choreography: generating motion

The motion of the legs of the spider are indeed computed by the Lorentz system. The evolution of the 8 systems of differential equations was done entirely with Povray. Evenso Povray is a ray tracer, it is a Turing complete programming language. If Povray computes a movie, it forgets about internal parameters, the only thing which changes from frame to frame is an internal clock variable. To integrate differential equations, one has after every frame computation store the most recent variables in an external file.

Credit

- CAS: mathematica
- Raytracing: povray
- Sider uses Povray code by Rafael Ghiglia, 2001
- Fractal zoomer with additional scripts
- Choreography sequence: bradley and Stuart (Colorado)
- Trading movie sequence: Movie: Trading Placess
- Chaos sequence: Movie: Jurassic Park
- Movie scene from the movie "the bank"
- Crysis game scene demo, viewable on YouTube
- Movie scene from "Grease" with John Travolta
- Primes on Ulam spiral: movie Code conspiracy
- Cellular automaton accord: wolfram research
- Black hole merger: NASA
- Poem by William Shakespeare
- Poem generator A.D.A.M. by Nandy Millar
- Fractal paintings: Kelly Dietrich
- 3 body problem picture by Mathew Holman and Joe Christy in forword by D. Goroff to Poincare. Nouvelle Methods

References

[1] D. Burrows. The dynamical systems perspective on music. *The Journal of Musicology*, 15:529–545, 1997.

[2] L. Harkleroad. *The Math behind the Music.* MAA, 2006.

[3] O. Knill. Maximizing the packing density on a class of almost periodic sphere packings. *Expo. Math.*, 14:227–246, 1996.

[4] P. D. Lax. The ergodic character of sequences of pedal triangles. *American Mathematical Monthly*, 97:377–381, 1990.

[5] J. Moser. *Stable and random Motion in dynamical systems.* Princeton University Press, Princeton, 1973.

[6] Martin Nowak. *Evolutionary Dynamics: Exploring the Equations of Life.* Harvard University Press, 2006.

[7] E. E. Peters. *Fractal Market Analysis.* A. Wiley, 1994.

[8] M. Senechal. *Quasicrystals and geometry.* Cambridge University Press, 1995.