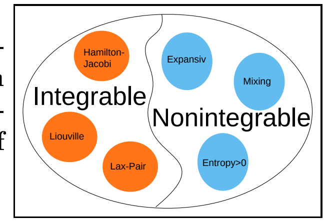
THEME. Where do we draw the line between integrability and non-integrability? In this talk, we look at one possible notion of integrability which applies to a large class of systems. Whether it is a useful notion of integrability depends on open mathematical problems. For example, we want classes of dynamical systems with certain non-integrable behavior to be non-integrable.



NOTIONS OF INTEGRABILITY.

Liouville integrable
Separation of variables
Extended Liouville integrable
Poisson integrable
Birkhoff integrable
Open sets of foliations

NOTIONS OF NONINTEGRABILITY.

Embedded subshift of finite type
Positive entropy
Mixing properties
Sensitive dependence on initial conditions
Expansiveness
Transitivity and dense periodic set
Decay of correlations

SIGNS FOR INTEGRABILITY.

Lax pair
Zero curvature equations
Backlund transformations
Painleve property for ODE's
Bihamiltonian structure
Explicit solution formulas
Fast periodic approximation

SIGNS FOR NONINTEGRABILITY.

Horse shoe
Complexity
Noncompressibility
Positive Lyapunov exponent
Any mixing property
Nonalgebraic special solutions
No analytic continuation

DYNAMICAL SYSTEMS. R semigroup acting by continuous maps on X compact Hausdorff, **dynamical system** (R, X).

Examples:
$T : X \mapsto X$ continuous map, $R = \mathbf{Z}^+$ action.
$T : X \mapsto X$ homeomorphism on compact space, $R = \mathbf{Z}$ action.
Flow on manifold. $R = \mathbf{R}$ action
Semiflow $R = \mathbf{R}^+$ action $T, S : X \mapsto X$ commuting continuous maps, $\mathbf{Z}^2$ action.
X tiling systems, $R_{t,s}(x) = x + (t, s)$. $\mathbf{R}^2$ action.

DS-INTEGRABILITY. (R, X) is DS-integrable, if for every R-invariant ergodic Borel probability measure μ, every map $T_r : X \mapsto X, r \neq 0$, (X, T_r, μ) has discrete spectrum.

DS-integrability is a notion for topological dynamical systems. No compactness, no invariant measures are necessary. (A system with no finite invariant measure like $x \mapsto x + 1$ in $\mathbf{R}$ or a linear map in Euclidean space are by definition DS-integrable. For finite R, DS-integrability. Ergodicity requirement is necessary: $(x, y) \mapsto (x + y, y)$ is DS-integrable but has Lebesgue spectrum for invariant Lebesgue measure.

DS-INTEGRABLE SYSTEMS AND UNIFORM RECURRENCE.

For every ergodic measure μ, the correlation coefficients are almost periodic.
For every measure ergodic μ and subset U there exists a time sequence along which U and $T^{t_k}(U)$ approach.

DS-INTEGRABILITY IMPLIES COMPUTABILITY.

For every pair of open sets U, V, and every ergodic μ there exists an algorithm which computes $\mu(T^n(U) \cap V)$ up to an error $\epsilon > 0$ in $\log(n)$ steps.

Many systems (i.e. system contining a Bernoulli shift) can simulate any computation because Turing machines can be realized as subshifts and universal TM can do any computation. Unknown: Can one realize a universal Turing machine as a subsystem of any non DS-integrable system?

PRESERVE INTEGRABILITY.

Factor system
Subsystem
Product
Iteration

NOT PRESERVE INTEGRABILITY.

Skew product
Time change
Induced system
Integral extension
Suspension

DS-integrability is fragile. Need strong topology on dynamical systems and narrow class of systems in order to have integrability preserved under perturbations. For Hamiltonian systems, integrability can survive on subsets (KAM). Flows on tori show that DS-integrability is fragile. For smooth systems, open sets of integrable systems with finitely many periodic attractors.

CRITERIA FOR NONINTEGRABILITY.

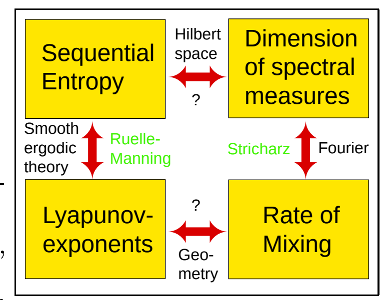
- Numerical: positive entropy.
- No total recurrence.
- arbitrary long wandering intervals.
- no reversibility: $\exists \mu$, T is not conjugated to T^{-1}.
- lower bound on the speed of approximation.
- some sequential entropy is positive
- $\exists u \in L^2(X,\mu)$, $\{U^n u\}$ not totally bounded.

CRITERIA FOR INTEGRABILITY.

- Smooth systems, bound on $||DT^n||_{L^1}$ (Furman).
- Establish system as factor of group translation.
- Use structural stability:
(conjugate the system to a DS-integrable system).
- minimality and no topological weak mixing.
- Sequential entropies are all zero (Kushnirenko).
- $\forall u \in L^2(X,\mu)$, $\{U^n u\}$ totally bounded

DISTANCE FROM INTEGRABILITY.

I) Speed of Periodic approximation
II) Sequential entropy
III) Dimension of spectral measures
IV) Sequential Lyapunov exponents.

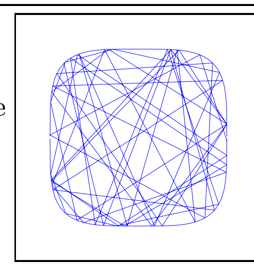


Relation: Speed of approximation with dimension of spectral measures (Hof-Knill, Iwanik, Strichartz).
Relation: Sequential entropy with sequential Lyapunov exponents (Furman, Ruelle-Margulis)
Relations: Sequential entropy with dimension of spectral measures or sequential Lyapunov exponents with dimension of spectral measures. (Work in progress).

EXAMPLE: FLOW ON 2-TORUS.

$(\dot{x}, \dot{y}) = (1/F, 1/(\lambda F))$ is DS-nonintegrable for dense set of (F, λ) and DS-integrable for dense set of (F, λ) (Hof-Knill). Zero dimensional spectrum for most systems (Hof-Knill, recently generalized to arbitrary dimensions by Fayad). (Not known, whether 0-dim spectrum for all.)

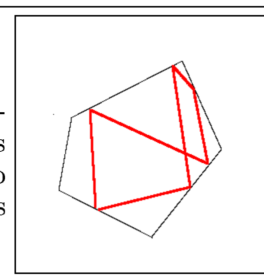
EXAMPLE: SMOOTH CONVEX BILLIARDS.



Ellipses are DS-integrable. Probably all other systems are not DS-integrable (A version of the Birkhoff-Poritski conjecture).

Other integrability notions: open annular set with invariant foliation.

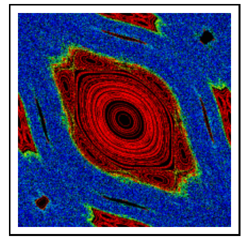
EXAMPLE: POLYGONAL BILLIARDS.



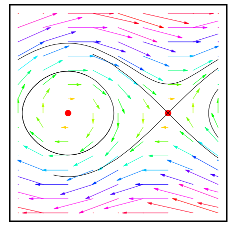
Class of systems with weak chaos: zero entropy, no mixing, generically transitive, directional flow usually weakly mixing. Questions about periodic orbits difficult: not even known whether every triangle has a periodic orbit. No weakly mixing polygonal billiards is known. Subtle question, which polygons are DS-integrable.

EXAMPLE: VOLUME PRESERVING DYNAMICAL SYSTEMS ON TORUS.

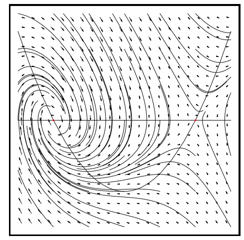
A Bair generic volume-preserving homeomorphism is ergodic, weakly mixing but not strongly mixing and therefore not DS-integrable.



EXAMPLE: HAMILTONIAN SYSTEMS.
DS-integrable: examples: free top in $\mathbf{R}^n$, Lagrange or Lobatchevsky top in $\mathbf{R}^3$, two central forces, Toda systems, KdV or Sine-gordon in certain cases, Calogero-Moser or Sutherland-Moser systems (infinite dim Vlasov version in certain cases), geodesic flow on ellipsoid, 1D Hamiltonian flows. DS-notintegrable: Whenever finite-dimensional invariant KAM torus is present (PDE or ODE), perturbation can render it weakly mixing and so nonintegrable (Knill,1999). Existence of homoclinic points and so horse shoe: 3body problem, Störmer problem, 4 vortex problem. More general: homoclinic tangencies.

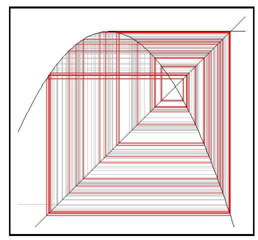


EXAMPLE: LOW DIMENSONAL ODE'S.
DS-integrability in dimensions ≤ 2. In general, there is no DS-integrability in dimensions > 2. Examples: Lorentz, Rössler, driven 1D systems. Hard to distinguish large periodic orbits from chaotic attractors.
Open sets of integrable ODS with stable hyperbolic orbits.

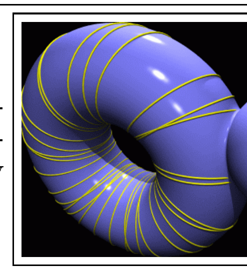


EXAMPLE: 1D MAPS.

Circle diffeomorphisms: DS-integrable even in Denjoy case.
"Period 3 $\Rightarrow$ chaos" by Li Yorke. Probably means not DS-integrable.
Piecewise expanding C^2 maps on an interval are not integrable.
Stable attractor periodic integrable.



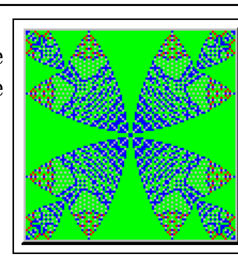
EXAMPLE: GEODESIC FLOWS.
DS-integrable: Flat tori, surface of revolution (Clairot), Paternain-Spazier constructons, Because DS-integrability implies $h_{top} = 0$, integrability imposes restrictions on the topology of the manifold. Perturbations in general destroy DS-integrability.



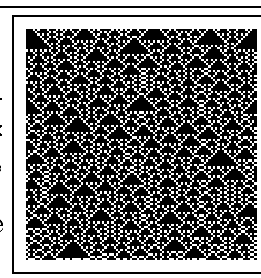
EXAMPLE: GROUP ACTIONS. R separable locally compact group.
DS-integrability: for every invariant measure of a group action on X, the representation $g \mapsto U_{T_g}$ is a direct sum of finite dimensional irreducible representations of G.

DS-nonintegrability. Not even clear, what type of representations can appear.

DS-integrability can be defined for discrete Borel equivalence relations.

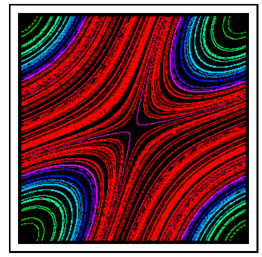


EXAMPLE: SUBSHIFTS. $X \subset A^{\mathbf{Z}}$ closed. Usually strict ergodicity assumed.
Sturmian sequences etc.: DS-integrable.
Substitution sequences: Can be DS-integrable or not. example of DS-integrable case: Pisot $0 \mapsto 01100, 1 \mapsto 01$, example of DS-nonintegrable case: Michel: $0 \mapsto 01, 1 \mapsto 1100$ (mixed spectrum) **Attractors of CA**: X subshift, ϕ CA, $\phi(X)$ topological factor. Attractor $\bigcap \phi^n(X_0)$ is subshift.
Autocorrelation $\hat{\mu}_n = (x_0 x_m)$ almost periodic in DS-integrable case (converse not always true).



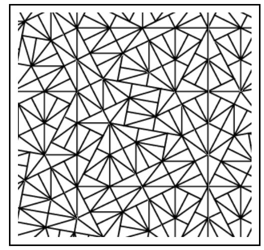
EXAMPLE: AREA PRESERVING MAPS. $S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x) - y \\ x \end{pmatrix}$

EXAMPLE OF DS-INTEGRABLE MAPS:
$f(x) = 2kx/(1 + x^2)$ **McMillan map** on plane
$f(x) = 2x + 4 \cdot \arg(1 + k \cdot e^{-ix})$ **SBKP map** integrable on torus



EXAMPLE: CRYSTALS OR TILINGS.

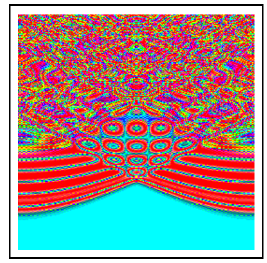
Crystals are Deloney point configurations with strictly ergodic R^n action. DS-integrable crystals: quasi-crystals: have discrete refraction spectrum. examples are tilings obtained by the projection method.
DS-nonintegrable crystals: turbulent (singular continuous spectrum) or chaotic (absolutely continuous spectrum).



EXAMPLE: COUPLED MAP LATTICES.

An infinite product of DS-integrable systems is DS-integrable. Nearest neighbor interaction produces nonintegrability. Already linear interaction is nontrivial. (Examples: quantum dynamics).

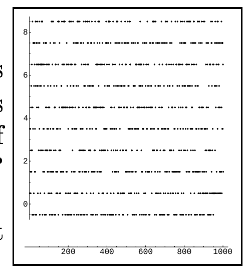
A propos quantum mechanics: The dynamics of a unitary operator U on the weakly compact unit ball is DS-integrable.



EXAMPLE: DETERMINISTIC SEQUENCES.

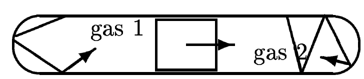
Sequences a_n like the decimal expansion of $\pi, \sqrt{2}$ in $\{0, ..., 9\}^{\mathbf{N}}$, the Moebius function $\mu(n)$ in $\{-1, 0, 1\}^{\mathbf{N}}$ or $\sqrt{n^5}$ mod1 in $[0, 1]^{\mathbf{N}}$ define dynamical systems (the shift T on the closure X of all translates in the product topology). If a sequence is DS-integrable, then a_n can be computed fast. Is the converse true?

The dynamics of a dynamical system defined by decimal expansions of most irrational numbers is unknown. (Which real numbers are DS-integrable?)



EXAMPLE: VLASOV DYNAMICS. Continuum limit of n-body problem has new features, convergence to equilibrium is possible (i.e. gas in Bunimovich stadium). The Vlasov system is then integrable (every invariant measure is on equilibrium measure fixed point).

Open problem: convergence to equilibrium in case of additional moving macroscopic boundaries.



QUESTIONS.

DS-integrability compatible with topological mixing or with topological weak mixing?
DS-integrability compatible with Devaney chaos (transitivity and dense set of periodic orbits)?
DS-integrability compatible with Li-Yorke chaos ($\underline{\lim}_{n\to\infty} d(T^n x, T^n y) = 0$ and $\overline{\lim}_{n\to\infty} d(T^n x, T^n y) \geq \delta$)?
Does topological weak mixing imply DS-nonintegrability? (DS-nonintegr. + minimal. $\Rightarrow$ topol. weak mixing).
$\exists$ DS-integrable systems which are not factors of group translations?
Is the irrational Toda flow DS-integrable?
$\exists$ finite dim DS-integrable systems with almost periodic, not quasi-periodic spectral measure?
Is every DS-integrable smooth convex billiard an ellipse?
What are the DS-integrable polygonal billiards, CA, substitution systems etc?
Class of minimal systems on the torus: Is the dimension of the spectrum always 0?
Can one compute for a DS-integrable system $T^n(x)$ up to accuracy ϵ in $\log(n)$ steps?
What is the closure of the set of DS-integrable systems in $C(X, X)$?