Fancy stuff related to two connected sticks.

Math Circle, Northeastern University, May 9, 2004, Oliver Knill

FASHION IN MATH. Mathematics does not lack fashion. Not at all: every century, every field of mathematics has its fancy topics. If you would ask a mathematician, what is fashion today or what was fashion during a certain period, you would get different answers. Here is a list, I would write down out of the blue.

Greeks:	planar geometry, logic, nature of numbers
16'th century:	Diophantine geometry, infinitesimals
17'th century:	planetary motion, calculus of variations, packing problems
18'th century:	polynomial equations, differential equations, algebraic surfaces
19'th century:	infinities, fractal theory, set theory, complex analysis, transcendental numbers, n-body problem, geometry of surfaces
20'th century:	topology, relativity, algorithms, complexity, catastrophe theory, chaos, nonstandard analysis, solitons, category theory, perturbation theory, artificial intelligence, game theory, computational number theory
21'th century:	quantummathematics (quantum calculus, noncommutative geometry topology measure theory, probability etc, quantum computing) partial differential equations in geometry, complexity theory

With fashion I would mean keywords, which ring a bell with a general audience, topics which are picked up by authors in novels or enter movies, topics which top in numbers of publication keywords or nowadays lead the ranking list in search engines. Fashionable subjects are not necessarily the most important subjects in mathematics but typically, they are appealing for the general audience and the interest in them comes and fades.

One indication, whether a subject will remain important is the number of open problems in the field. An other indication for "fancyness" are prizes set out to solve the problem. Examples are Hilbert's list of problems from 1900, the King Oscars prize set out for the question in the field of celestial mechanics, or nowadays the seven "1 Million dollar problems" of the Clay Math institute.

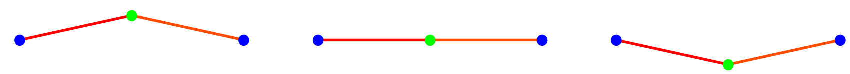
Today, we visit a few fields in mathematics which were once fashionable but which still are fascinating subjects and will remain so for some time. The story is woven around two connected sticks.

BOAT STABILITY AND SWITCHES.

Yesterday, on May 8'th, CNN run the news that a small boat carrying 6 people capsized in the Taunton river. A week before, on May 3, an accident was reported from Texas: on Lake Travis, a party boat capsized and dumped all 60 people into Lake Travis. Wanting to catch a glimpse of nude sunbathers, they crowded on one side of the boat. The result was a catastrophe.

Catastrophe theory was a fancy mathematical field in the 60'ies. People were excited and thought that it would be a universal tool to solve many problems. The hype faded and after much criticism, the subject became somehow "anti fashion". It is often the case that fashionable things turn out to be the laughing stock after some time.

Nowadays, from some distance, one recognizes catastrophe theory as a beautiful piece of mathematics which will not disappear. Similar to fractal theory or chaos theory, (fields which were hyped and faded too), the subject has matured to a standard "classical field" in mathematics. We make a few experiments here in the math circle. Catastrophes also appear with buckling sticks. This instability is implemented in essentially all mechanical switches.

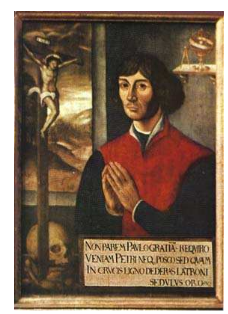


FROM ARISTOTELES AND PTOLEMY TO COPERNICUS.

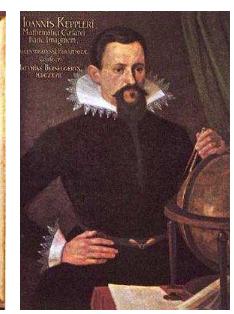
The struggle to find the correct picture for planetary motion can almost be seen as a topic of catastrophe theory itself. There were two sudden changes (catastrophes): the change from a **geocentric** to a **heliocentric** picture, then the change from an **epicycle** picture to a description by **ellipses**.

Aristoteles

Ptolemy



Copernicus

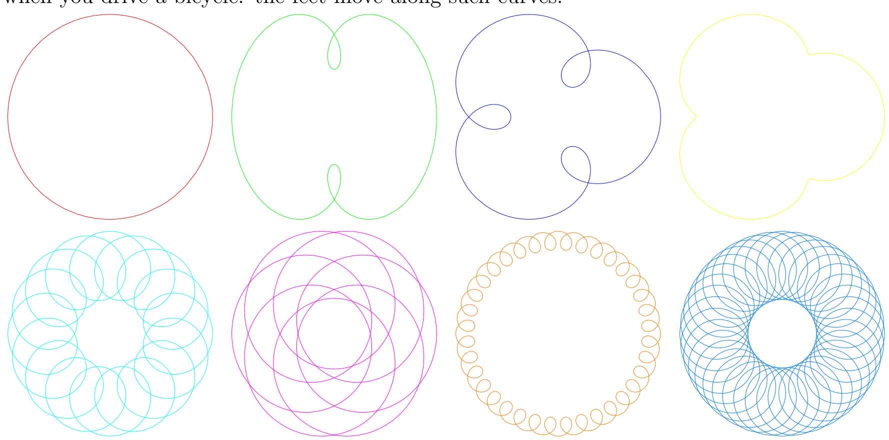


Kepler

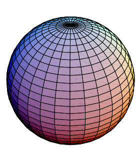
The Greeks loved circles. Circles and spheres were definitely in fashion at that time. **Archimedes** even died for a circle with the words "Don't disturb my circles". So, the motion of planets had to be explained using these objects. **Aristoteles** thought that the planets as well the sun would circle the earth on perfect circles.

Also the **Ptolemy** (85-165) model of planetary motion put the earth into the center and let all the planets as well as the sun rotate around it. To accommodate discrepancies, planets were put on circles circling circles. The corresponding curves are called **epicycles**.
Copernicus (1473-1543) changed the system in that he would put the sun into the center and let the planets rotate around it. Again, there were discrepancies which Copernicus incorporated by borrowing the still fancy epicycles.
Only **Kepler**, using data from Tycho Brahe realized that ellipses were the right thing to consider. The Kepler theory had the advantage that it could be derived from basic mathematical principles like the inverse square law for the attraction of bodies as Hooke and Newton have seen.

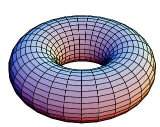
FUN WITH EPICYCLES.
Lets experiment a bit with epicycles. What kind of curves can you obtain? Epicycles appear when you drive a bicycle: the feet move along such curves.



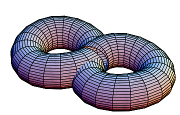
DOUGHNUTS, MOEBIUS STRIPS AND KLEIN BOTTLES.
The sphere was the fanciest surface for the greeks and remained so during the middle ages. Its symmetry is appealing. In the nineteen'th century, one began to be interested in other surfaces.



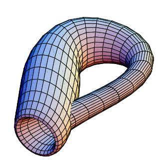
Sphere



Torus



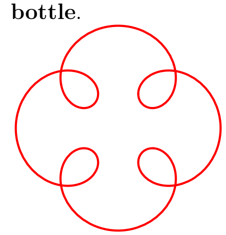
Brezel



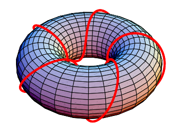
Klein bottle

If you take a circle and move it along a circle, you obtain a **torus** or doughnut. Every point on the doughnut is characterized by two angles. This is the same with the position of the two sticks. One can say, the set of all the possible configurations of the two sticks forms a doughnut. If we look at the epicycle curve, we can translate the curve onto the doughnut. How does it look like there?

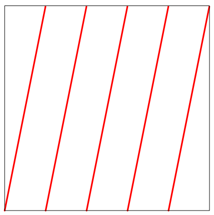
Lets flatten our doughnut. You certainly have played "packman" or other games, where it is possible to leave on one side of the screen and reenter on the opposite side? One of these games is the shooting game "asteroids". The space in which one plays this game is actually a doughnut. One can demonstrate this with a piece of paper. The epicycle curve is a straight line on the flattend torus.
What happens if we glue two sides the opposite way? We obtain a surface called the **Klein bottle**.



The epicycle

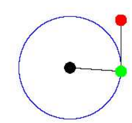


The corresponding toruscurve

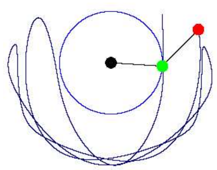


Flattend out

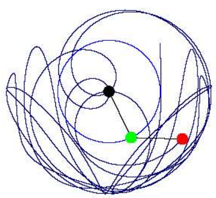
CHAOS AND INTEGRABILITY.
Let us experiment with the motion of the double pendulum, where the two sticks are moving and subject to gravity. By experimenting with it, we see that the motion can be tame, we can get motion which produces epicycles. We can also see examples, where the motion is chaotic. What is chaos? There are zillions of definitions and not all are equivalent. One of the commonly accepted signs of chaos is a "sensitive dependence" on initial conditions. A brother of chaos is "turbulence" which we can see in air. See demonstration.



The double pendulum



A short orbit

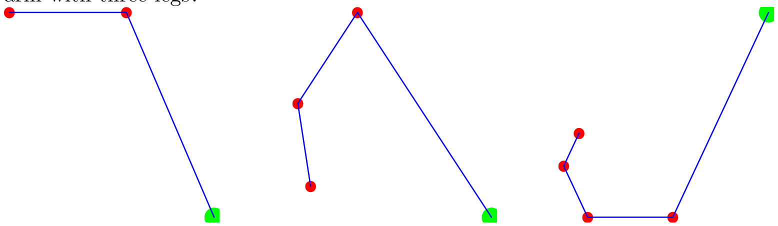


A bit longer orbit

The antipode of "chaotic" is called "integrable". Integrable systems are the ones which one can solve explicitly. An example is the simple pendulum which moves so regularly that one has used it to build clocks. The study of integrability was fashion too. The key word is "solitons", these are solutions to systems which are unexpectedly integrable and which interact in a nonlinear but predictable way.

ROBOT ARMS.

If the double stick is a robot arm, we would like to know from the angle, which position we are or reverse, find the possible angles to reach a certain point. Which points can we reach with exactly one pair of angles? Which points can we reach with a finite set of pairs of numbers. Are there points to which infinitely many angle pairs belong? Do you see a relation with a doughnut projected onto a plane. What would be the geometric "surface" describing a robot arm with three legs?

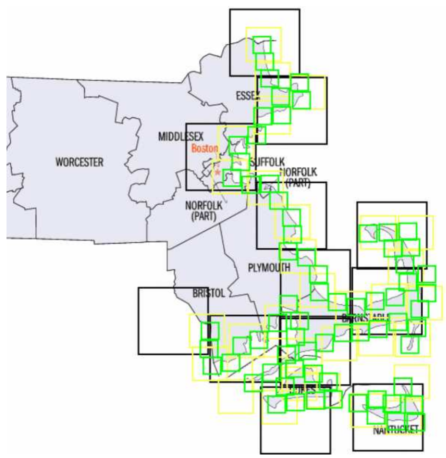


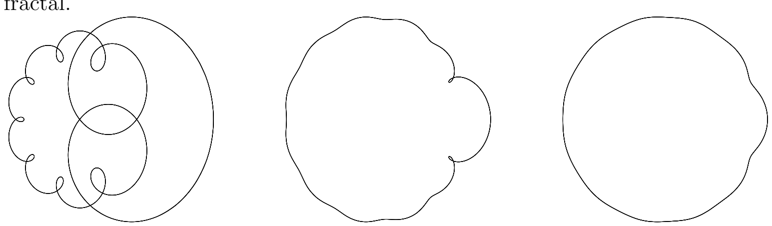
Two legs   Three legs   Four legs

FOURIER SERIES AND FRACTALS.

What about three arms? Things become more complicated. Actually, by allowing an arbitrary number of arms (the sum of the lenghts should be finite), one can model any curve. This is an other fancy subject called **Fourier theory**, where for every t, we write the position of the curve with the following formulas:
$x(t) = a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t)...$
$y(t) = a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t)...$

One can even get **fractals**, curves with dimension bigger than one. The Massachusets coast line is an example of a fractal.





AREA AND COMPUTER ALGEBRA SYSTEMS.

Before computers were available, many computations or measurements would be done mechanically. The earliest analogue machine probably the abacus, later, the slide rule was used. An instrument to to compute the area of regions is called the **planimeter**. It is an example of an **analogue computer**.

The proof that this instrument really measures the area, needs some fancier calculus and is actually quite painful to do by hand, even for Harvard students. It is more convenient to prove the things using a computer algebra system.

The use of technology in mathematics is one thing one could probably not call a "fashion". It appeared and will steadily increase in use. We are now in a stage, where computers can solve many calculus problems on an undergraduate level. It will take a while but I have no doubt that this time will come, when we can use computers to help us explore mathematical facts and even find proofs of theorems. This is not scary. We will use it to progress faster and progress in areas where it would be impossible to continue without that help.

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(* Mathematica code to check a claim about the Planimeter vector field    *)

Simplify[Solve[{ (x-a)^2+(y-b)^2==1, a^2+b^2==1 }, {a,b}]]

(* leads to the following expressions for (a,b)                           *)

a[x_,y_]:=(x^3+x*y^2+Sqrt[4*x^2*y^2-x^4*y^2+4*y^4-2*x^2*y^4-y^6])/(2*(x^2+y^2));
b[x_,y_]:=(x^2*y+y^3-Sqrt[4*x^4-x^6+4*x^2*y^2-2*x^4*y^2-x^2*y^4])/(2*(x^2+y^2));

(* We compute now the curl                                                 *)

curl=D[a[x,y],x]+D[b[x,y],y];
Simplify[curl]

(*  when staring at the result, you see that it is  equal to  1            *)
(*  Mathematica does not "see" it                                          *)
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