Oliver Knill, UT **Talk of December 17, 1999**

Classical and Open Problems about Billiard Dynamics

BIRKHOFF BILLIARDS:
The motion of a free particle in a bounded region reflecting elastically at the boundary is called a **billiard**. Convex two-dimensional convex regions in the plane define **Birkhoff billiards** (Birthplace: Harvard!)

EXAMPLES:
1) **Elliptic billiard**: $x^2/a^2 + y^2/b^2 = 1$.
2) **Polygonal billiards**: eg. triangle
3) **Polygons with rounded corners**: eg. stadium
4) **Tables of equal width**: eg. Ruleaux triangle

BILLIARD PIONEERS:

Boltzmann (1844-1906) hard sphere gas
Artin (1898-1962) in 1924, billiard in hyperbolic plane
Hadamard (1865-1963)-Hedlund-Hopf, geodesic flow
Birkhoff (1884-1944) in 1927, model for 3-body problem
Poritski in 1950, integrability question

WHY STUDY BILLIARDS?
A beautiful and simple dynamical system featuring complexities of Hamiltonian systems in general. Limiting case of geodesic flow. Illustrates theorems in topology, geometry or ergodic theory. Related to Dirichlet problem $\Delta u = \lambda u$. Open problem "can one hear the shape of a **smooth** drum". Relation of quantum mechanics to classical mechanics.

THE BILLIARD MAP:
$s \in \mathbf{T} = [0,1]$(mod 1): point on the boundary.
$\phi \in [0,\pi]$: angle of velocity vector to the tangent at s. $u = \cos(\phi) \in [-1,1]$. s', new intersection with boundary. $u' = \cos(\phi')$ from impact angle ϕ'. **Billiard map** on annulus $X = \mathbf{T} \times [-1,1]$:

$$T : (s,u) \mapsto (s',u')$$

The boundaries $\mathbf{T} \times \{-1\}$, $\mathbf{T} \times \{1\}$ consist of **fixed points**.

$T : X \to X$ AREA-PRESERVING.
Equivalent (by change of variable formula): Jacobian

$$DT(s,u) = \begin{pmatrix} \frac{\partial}{\partial s} s'(s,u) & \frac{\partial}{\partial u} s'(s,u) \\ \frac{\partial}{\partial s} u'(s,u) & \frac{\partial}{\partial u} u'(s,u) \end{pmatrix}$$

satisfies $\det DT(s,u) = 1$. Exercise: calculate directly or follow Birkhoff, 1927. (There is an elegant proof using more advanced calculus.)

THEOREM (BIRKHOFF): **Elliptic billiards are integrable**.
PROOF. $L(s,u)$ line connecting s_1 and s_2, if $T(s_1,u_1) = (s_2,u_2)$. $d^{\pm}(s_i,u_i)$ oriented distances of $L(s_i,u_i)$ to focal points $F^{\pm}$ of ellipse. The function $f(s,u) = d_1(s,u)d_2(s,u)$ is a real-analytic integral of the elliptic billiard.

ELLIPSE PROPERTY: $f_1 = f_2$ if $f_i = d^+(s_i,u_i)d^-(s_i,u_i)$.
PROOF: Ellipse $t \mapsto s(t) = (a\cos(t), b\sin(t))$ has focal points $F^{\pm} = \pm\sqrt{|a^2-b^2|}$, normal $n(t) = (b\cos(t), a\sin(t))$ and tangent $v(t) = (-a\sin(t), b\cos(t))$. Set $B(t,r) = s(t) + n(t) + rv(t)$. If point in incoming ray is $B(t,r)$), then point in outgoing ray is $B(t,-r)$). Let $d^{\pm}(t,r)$ be the distance of $F^{\pm}$ from A to $B(t,r)$. Computation shows $d^+(t,r)d^-(t,r) = [b^2(1+2r^2) - a^2 + (a^2-b^2)\cos(2t)]/2(1+r^2)$. This is even in r so that $d^+(t,r)d^-(t,r) = d^+(t,-r)d^-(t,-r)$.

INTEGRABILITY:
$T : X \to X$ is **integrable** if there exists a piecewise continuous $f : X \to \mathbf{R}$ such that each set $\{f(s,u) = c\}$ is a union of points or lines.
WARNING. Different definitions of "integrability" exist.

BIRKHOFF-PORITSKI CONJECTURE:
Any integrable smooth, convex billiard is an ellipse.
Formulated by Poritski (work at Harvard under Birkhoff $\sim$ 1927, published only in 1950).

PERIODIC ORBITS. A **n-periodic point** of the billiard map is a point $x = (s,u) \in X$ such that $T^n(x) = x$.

BIRKHOFF: a differentiable strictly convex billiard has for $n \geq 2$ and $r \leq n/2$, a n-periodic orbit rotating r times around the table. The proof uses that periodic points extremize the length of the billiard trajectory and that any differentiable function on the torus has a maximum.

POLYGONAL BILLIARDS. The billiard in a convex polygon is called a **polygonal billiard**. If all angles of the table are π-rational, the billiard is called a **rational billiard**.

EXAMPLE: **Square billiard**. The billiard is rational and integrable with integral $f(s,u) = u/\sqrt{1-u^2} =: \alpha$ $s \in [0,1/4] \cup [1/2,3/4]$ and $f(s,u) = 1/\alpha$ else. The dynamics on $\{f = \alpha\}$ is conjugated to an irrational rotation $s \mapsto s + \alpha$ with angle α. A point $(s,u) \in X$ is periodic if and only if $\alpha = u/\sqrt{1-u^2}$ is rational.

PROBLEM FOR POLYGONAL BILLIARDS. Does every polygonal billiard have a periodic orbit? Not known even for triangles. (Yes, for right triangles). Can a smooth billiard have an open set of p-periodic points?

RATIONAL BILLIARDS HAVE PERIODIC ORBITS. Proof. Take an initial condition $(s,0)$. Area preservation and Poincare recurrence: $\exists\ T^n(s,0) = (s_n,u_n)$ with arbitrarily small $|u_n|$ and with s_n on same side as s. Rational billiard: $u_n = 0$ for large n. This orbit is periodic.

PERIODIC POINTS FOR SMOOTH BILLIARDS. Is the set of p-periodic orbits nowhere dense?

VARIATIONAL CONSTRUCTION: if $s = (s_1, \dots, s_n, s_{n+1} = s_1)$ are points on **T**, let $\mathcal{L}(s_1, \dots, s_n) = \sum_{k=1}^n \mathcal{L}(s_k, s_{k+1})$, where $\mathcal{L}(s, s') = |P(s) - P(s')|$. A periodic orbit $T^k(s, u) = (s_k, u_k)$ corresponds to a critical point of $\mathcal{L}$, a point, where the gradient $\nabla \mathcal{L}$ vanishes. There exists a maximum and so a critical point.

CAUSTICS:
Curves for which tangent billiard trajectories remains tangent after successive reflections.

EXAMPLES:
1) Ellipses have **conformal conics** $x^2/(a^2+\lambda)+y^2/(b^2+\lambda) = 1$ as caustics, hyperbola for $-b^2 < \lambda < -a^2$, ellipses for $-a^2 < \lambda$.
2) γ convex curve, the **string construction** leads to table with γ as caustic.
3) **Curves of equal width** have caustics which agree with the evolute of the table.

INVARIANT CURVES-CAUSTICS.
$\Gamma : s \mapsto (s, \cos(\theta(s)))$ invariant curve under T, $v(s)$ unit vector in trajectory direction, $P(s)$ point on table, $\kappa(s)$ curvature at s. Caustic: $\gamma(s) = P(s) + \sin(\theta(s))/(\kappa(s) + \Gamma'(s))v(s)$

PROOF. $T(s, u) = (s_1, u_1)$. $\delta(s)$, angle of trajectory $L(s, s_1)$ to x-axes. $\gamma(s) = P(s) + b(s)u(s)$, where $b(s) = \sin(\theta)(d\delta/ds)^{-1}$. $\alpha(s)$: angle of tangent at $T(s)$, then $\delta(s) = \theta(s) + \alpha(s)$. Furthermore $\delta'(s) = \theta'(s) + \alpha'(s) = \kappa(s) + \Gamma'(s)$, so that $b(s) = \sin(\theta)(\kappa(s) + \Gamma'(s))^{-1}$.

CURVES OF EQUAL WIDTH.
α angle of tangent. $\rho(\alpha) = 1/\kappa(\alpha) > 0$ **radius of curvature** at α. $P(\alpha) = P(0) + \int_0^\alpha \rho(\beta)e^{i\beta}\ d\beta$ defines a closed convex curve if $\int_0^{2\pi} \rho(\beta)e^{i\beta}\ d\beta = 0$. If additionally $\rho(\alpha) + \rho(\alpha + \pi) = \text{const}$, this is a **table of equal width**.

CAUSTIC AND EVOLUTE:
$\gamma(\alpha) = P(\alpha) + \rho(\alpha)ie^{i\alpha}$ defines caustic to invariant curve $\{u = 0\} \subset X$. This is the **evolute** of the curve. $\gamma'(\alpha) = \rho'(\alpha)ie^{i\alpha}$ shows ρ and γ have identical critical points. **Vertices** of table $\Leftrightarrow$ **cusps** of γ.

ARE THERE FRACTAL CAUSTICS?
Are there caustics which are fractals? Are there fractal evolutes of convex curves of equal width?

DEFINITION OF FRACTALS.
Z subset of an Euclidean space. For $\epsilon > 0, s > 0$, define $h^s_\epsilon(Z) = \inf_{\{U_j\}} \sum_{U \in \{U_j\}} |U|^s$, where $\{U_j\}$ runs over all open ϵ-covers of Z. ($|U_j| < \epsilon$, U_j open and $Z \subset \bigcup_j U_j$). The limit $h^s(Z) = \lim_{\epsilon \to 0} h^s_\epsilon(Z)$ is called **s-dimensional Hausdorff measure** of Z. It exists in $[0, \infty]$ because $\epsilon \mapsto h^s_\epsilon(Z)$ increases for $\epsilon \to 0$. If $h^s(Z) < \infty$, then $h^t(Z) = 0$ for all $t > s$. Define the **Hausdorff dimension** $\dim_H(Z) \geq 0$ by $s < \dim_H(Z) \Rightarrow h^s(Z) = \infty$, $s > \dim_H(Z) \Rightarrow h^s(Z) = 0$. **Fractals** are sets Z with non-integer $\dim_H(Z)$.

LYAPUNOV EXPONENT.
$\lambda(s, u) = \limsup_{n \to \infty} n^{-1} \log ||dT^n(s, u)||$ is called the **Lyapunov exponent** of T at (s, u). It measures **sensitive dependence on initial conditions**.

CHAOS. A billiard is **chaotic** if the **Pesin set** $\{\lambda(s, u) > 0\}$ has positive Lebesgue measure. No smooth, convex chaotic billiard is known!

SUMMARY: OPEN PROBLEMS.
1) **Birkhoff-Poritski**: a smooth, integrable Birkhoff billiard is an ellipse.
2) **Chaotic billiards**: there exist smooth Birkhoff billiards with positive Lyapunov exponents on a set of positive Lebesgue measure.
3) **Periodic points for polygons**: every polygonal billiard has a periodic orbit.
4) **Fractal caustic problem**: there exists a Birkhoff billiard with fractal caustic.
5) **Algebraic return map problem**: if a Birkhoff billiard map T is conjugated to an algebraic map, then the table is an ellipse.
6) **Guillemin problem**: T_1, T_2 smooth Birkhoff billiard maps. If $T_1 = ST_1S^{-1}$ with a homeomorphism S, then the tables are similar.
7) **Size of periodic orbits**: the set of n-periodic orbits of a smooth strictly convex Birkhoff billiard is nowhere dense for all n.

FURTHER READING:
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