THE HERMANY PROJECT: MATH PUZZLES, ACTIVITIES, AND GAMES

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Abstract: This is part of a discussion diary when advising Paul Hermany thesis project. "Developing a Taxonomy for Math Puzzles, Activities, and Games and an Online Tool to Improve Content Curation and Discovery". The thesis was successfully submitted in May 2017.

Strategy

Paul built a hierarchical term management engine to organize mathematical problems, puzzles, activities or games.

Taxonomies parameters

Paul was contemplating a lot of possible Taxonomies up in the context of puzzles:

1	Subject:	Examples: Logic, Geometry, Game theory, Recursion, CS
2	Difficulty:	The amount of time needed, to solve it.
3	Complexity:	The amount of work needed to solve it.
4	Maturity:	The level of mathematics needed to solve it.
5	Generality:	A concrete problem or a general problem.
6	Grade:	In which level in school the problem adequate?
7	Subject:	Which subjects are involved?

Type of problems

One can also distinguish problem types:

1	Activities	Just horsing around but no formal rules
2	Games	Several person games Nim, Chess, Go
3	Puzzles	Solitaire games, Rubik, 15 problem, mazes, cross words
4	Theorems	Relations for mathematically defined quantities (Euler line)
5	Problems	Open research problems
6	Practice	Template problems without much conceptional parts (Drill)
7	Illusions	Geometric fallacies
8	Paradoxa	Problems leading to contradictions

As Paul experienced during work, adding detailed **meta data** to a problem is a rather time consuming task. A simplified system could be to assign to each puzzle a vector (D, C, A) encoding difficulty (D), complexity (C) and applicability (A). Every mathematical problem can then be placed as a point in a three dimensional parameter space. All three parameters are complementary. A problem can be difficult for various reasons. It can require a sophisticated mathematics background for example. The "Good will hunting problem" is such a problem. A simple problem can be complex. Factoring a large number is in principle a simple problem, but it is complex. One

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Key words and phrases. Mathematics education, Algebra.

needs to go through a lot of searching or testing to find the solution. Applicability finally again is measuring something rather complementary. It is the pure mathematics - applied mathematics part.

Example problems

Paul's list of example problems is not covered here. Here are some problems he had contemplated or which we have discussed over the time.

The following problem is an example of a puzzle which needs some experimentation. Its difficulty is low, its applicability is low, but the complexity is rather high in the sense that it is well be solved just by computer search.

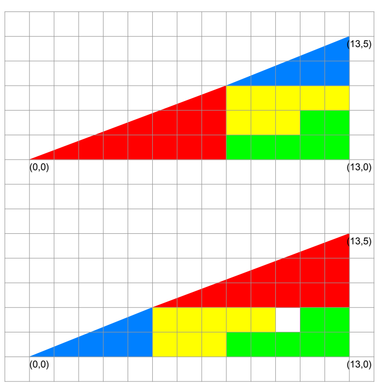
Problem: Millers Puzzle Find a permutation $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ of the 9 digits 1-9 so that $x_1 \cdot x_2x_3 = x_4x_5x_6 = x_7x_8 \cdot x_9$. How many solutions are there?

here is Mathematica code which solves the puzzle brute force:

```
s=Permutations[Range[9]];
f[a_]:=(a[[1]]*(a[[2]]*10+a[[3]]) == a[[4]]*100+a[[5]]*10+a[[6]]) &&
         (a[[4]]*100+a[[5]]*10+a[[6]]) ==(a[[7]]*10+a[[8]])*a[[9]];
Do[If[f[s[[k]]],Print[s[[k]]]],{k,Length[s]}]
```

The next problem is an example of a paradox. There are many versions of this. One can variants in the form of chocolates (which changes the applicability). In this case, the complexity of the problem is low, but the difficulty can be high if one does not know what to look for.

Problem: A tragic Pythagorean paradox: The odd equals the even:



The following is a visual paradox. Also this is geometric. It is difficult if you don't know the solution, has low complexity but quite decent applicability as it is relevant in computer vision, where a machine needs to understand a geometric structure from two dimensional pictures.

Problem: Realize the Penrose triangle geometrically.

This is a classic "pub problem", something you can ask your friends while waiting for the food. It is a difficult problem, has low complexity. There is some applicability as it solves a concrete construction problem.

Problem: Given 6 matches, use them to build 4 equilateral triangles which are pairwise congruent.

Here is an example of a historical problem. It is a problem in Diophantine equations which is formulated in a form which has some applications.

Problem: Archimedes cattle problem. $W = 5B/6+Y, B = 9D/20+Y, D = 13W/42+Y$, $w = 7(B + b)/12, b = 9(D + d)/20, d = 11(Y + y)/30, y = 13(W + w)/42$.

Also the following historical problem is formulated as an application. It has some complexity as one needs to search though a few cases:

Problem: The weight problem of Meziriac is A merchant broke a 40 point weight into 4 pieces, each having an integer pound allowing to measure every integer between 1 and 40.

Problem: Newton's problem of Fields and Cows from Newton Arithmetica universalis)

One can essentially take any theorem in mathematics and geometry in particular and make a puzzle out of it: here is an advanced problem in planimetry. The following example is a bit extreme as it is quite difficult. Its applicability is relatively low.

Problem: Euler line. Centroid, orthocenter, circumcenter and center of Feuerbach circle are on a line.

Problem: Cramer-Castillon problem triangles in triangle hitting 3 points outside the triangle.

The following problem needs some mathematical training. We can find curves arbitrary close to the direct line for which the distance is uniformly larger.

Problem: Staircase Paradox. The taxi metric is not the Euclidean metric.

While this is rather obvious, there is a more difficult variant dealing with area. While every smooth curve can be rectified (one can compute its length by approximating it with a polygon, then take the limit of the lengths of the polygons, this is not possible for smooth surfaces. (Schwartz examples)

Problem: Steiner Ellipse problem : Which ellipse circumscribed about a given triangle, which has the smallest or largest area? It is the Steiner ellipse. The focal points are the Bickart points.

Problem: Counting primes (IMO 1969): Prove that there infinitely many positive integers m for which $n^4 + m$ is not prime for any positive integer n. (if m is a multiple of 5, then take m=5k. If m is not a multiple of 4, then m=5k+4 works.)

Problem: Triangles (IMO 1968 A1): Find all triangles for which the side lengths are three consecutive integers and one angle is twice the other. The only solution is (5,4,5).

Then there are problems which are easy when known:

Problem: You are in a room full of people and everyone shakes hands, how many total handshakes are there?

Problem: Multiplicity problem (Annual Colorado Spring Olympiad 1986, Soifer) Given n integers, prove that either one of them is a multiple of n, or some subset of them adds up to a multiple of n. Solution: Look at the sums s1=a1, s2=a1+a2, ... sn=a1+a2+...+an. By the Pigeon hole principle, two give the same reminder modulo n. The difference is a sum.

Problem: Forty one rooks (Soifer and Slobodik 1973) Forty-one rooks are placed on a 10x10 chessboard. Prove that you can choose five that do not attack each other. (Pigeon hole principle, 41 pigeons = 4*10 + 1. There exists at least one row with at least 5 rooks.)

Problem: Adding odd integers (Soifer) Verify that each square n^2 is the sum of odd consecutive numbers. It is true for $n = 1$. Assume it is true for n, now add 2n+1 to n^2. This is $(n + 1)^2$

Problem: Icosahedron paths (Modified 2016 AIME Problem) Fix a point A and its antipode B on an icosahedron (the unique point of distance 3 from A). How many different paths are there from A to B? Solution: Start at A, there are 5 possible directions to go. Now, there are two possible directions. We reach now a point in distance 1 of B and have no choice any more. The answer is 5*2=10.

Problem: Rain on Saturday or Sunday (Modified from 2016 AIME Problem) There is a 40 percent chance of rain on Saturday and a 30 percent chance of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. What is the probability that it rains at least one day? Solution: rains on Saturday and B the event that it rans on Sunday. We know $P[A], P[B] \cdot P[A \cup B] = P[A] + P[B] - P[A \cap B]$. And $P[A \cap B]/P[A] = P[B|A] = 2(1 - P[B])$. So, $P[A \cap B] = 2P[A](1 - P[B])$.

The following is a famous Martin Garder problem

Problem: Dave has two children. One is a boy, what are the odds that the other is also a boy. The answer is 2/3.

Here is a Pythagoras problem. Easy if seen:

Problem: Chain problem: There is a 10 feet chain nailed to points A,B on the wall. The center dips is 5 feet below A,B. What is the distance between A and B Solution: $x^2/4 + 25 = 25$. Implies x=0.

Problem: x,y,z are positive integers and xyz=x+y+z. Find x,y,z. Solution: 1,2,3

Problem: A girl was ten on her last birthday, and will be twelve on her next birthday. How is this possible? (http://www.doriddles.com/Riddles/Math) Solution: lap day.

Here is a trick problem:

Problem: When is 1500 plus 20 and 1600 minus 40 the same thing? (http://www.doriddles.com/Riddles/Math) Solution: time

Here is a problem with trial and error

Problem: How can you add eight 8's to get the number 1000? Solution: 888+88+8+8+8=1000

Problem: You have have 3 zeros and 3 ones. What is the largest number you can build?

There are computational problems

Problem: Finding the factors of a concrete number like the Fermat number $F_{14} = 2^{(}2^{1}4)+1$, a 4932 digit number which is known to be composite but has unknown factors. A general problem is the question whether there exists an even Gaussian prime which is not the sum of two primes with positive entries.

Added June 10, 2017: Here is a problem which appeared in https://www.nytimes.com/2017/05/15/us/math
counts-national-competition.html

Problem: 100 chickens sit in a circle. Each chicken pecks randomly one of its neighbors to the left or right. What is the expected number of chicks which were not pecked?

The problem got into the news because it can be done quickly The answer is 25. Because there are four cases: either picked from the left, picked from the right or picked from both or not pecked. Each has the same probability.

Cross referencing

Puzzles can be cross referenced with common core standards. Here is a rough overview of the CCP taxonomy:

- Kindergarden: Counting and Cardinality, Measurement and Data, Geometry
- Grade 1: Operations and Algebraic Thinking, Measurement and Data, Represent and interpret data, Reason with shapes and their attributes.
- Grade 2: Operations and Algebraic Thinking, Number and Operations in Base Ten, Measurement and Data, Represent and interpret data, Geometry
- Grade 3: Operations and Algebraic Thinking, Number and Operations in Base Ten, Number and Operations/Fractions, Measurement and Data, Geometry
- Grade 4: Operations and Algebraic Thinking, Measurement and Data, Geometry
- Grade 5: Operations and Algebraic Thinking, Number and Operations and Fractions, Measurement and Data, Geometry of two dimensional figures
- Grade 6: The Number System, Expressions and Equations, Geometry: area, surface area volume, Statistics and Probability
- Grade 7: Ratios and Proportional Relationships, Expressions and Equations, Word problems, Geometry: construct, slice transform, Statistics and Probability
- Grade 8: The Number System, Functions, Graph Geometry, translations, rotations, Pythagoras Statistics and Probability: bivariate data

One problem is that the CCP guidelines not yet organized. The parts are sometimes detailed, sometimes rough.

A database test

- Students on a bench
  Stickels, Terry. Challenging Math Problems. Mineola, N.Y: Dover Publications, 2015.
  If a teacher can place his students eight to a bench, he will have only three students on the final bench. If he decides to place nine students on a bench, he'll have only four students

on the final bench. What is the smallest number of students this class could have?
67. This is a Chinese remainder theorem problem.

- Fold Envelope to tetrahedron
  Gardner, Martin. 2. book of mathematical puzzles
  diversions. Chicago:1987.
  Cut and fold a sealed envelope into a tetrahedron.
  Solution: There are 4 ways. An example: draw an equilateral triangle on both sides of one end of an envelope. Then cut through both layers of the envelope as indicated by the broken line and discard the right-hand piece. By creasing the paper along the sides of the front and back triangles, points A and B are brought together to form the tetrahedron.

- Fox, goose, beans problem
  https://en.wikipedia.org/wiki/Fox,_*goose_and_bag_of_beans_puzzle* Kordemsky, B. A. The Moscow puzzles : 359 mathematical recreations. New York: Dover Publications, 199.
  A farmer must take a fox, a goose, and a bag of beans across a river. His rowboat has only enough room to hold himself an either the fox, the goose, or the bag of beans. If he takes the bag of beans, the fox will eat the goose. If he takes the fox, the goose will eat the beans. Only when the farmer is present are the fox, the goose, and the bag of beans safe from being eaten. Nevertheless, the farmer carries the fox, the goose, and the bag of beans across the river. How?
  Easy by try and error
  Solution: The farmer takes the goose to the other side of the river and leaves it there. The farmer then takes the bag of beans to the other side and returns with the goose. The farmer then takes the fox to the other side and leaves it with the bag of beans. Lastly, the farmer takes the goose to the other side.
- The Clerk of Oxenford's Puzzle
  Dudeney, Henry E. The Canterbury puzzles. Mineola, NY: Dover, 2002. Print.
  The silent and thoughtful clerk of Oxenford, of whom it is recorded that "Every farthing that his friends e'er lent, in books and learning was it always spent," was prevailed upon to give his companions a puzzle. He said, "Ofttimes of late have I given much thought to the study of those strange talismans to ward off the plague and such evils, that are yclept magic squares, and the secret of such things is very deep and the number of such squares truly great. But the small riddle that I did make yester eve for the purpose of this company is not so hard that any may not find it out with a little patience." He then produced the square shown in the illustration and said that it was desired so to cut it into four pieces (by cuts along the lines) that they would fit together again and form a perfect magic square, in which the four columns, the four rows, and the two long diagonals should add up 34. It will be found that this is a just sufficiently easy puzzle for most people's tastes.
  Solution: The illustration shows how the square is to be cut into four pieces and how these pieces are to be put together again to make a magic square. It will be found that the four columns, four rows, and two long diagonals now add up to 34 in every case.
- The Pardoner's Puzzle
  The gentle Pardoner, "that straight was come from the court of Rome", begged to be excused, but the company would not spare him. "Friends and fellow pilgrims," said he, "of a truth the riddle that I have made is but a poor thing, but it is the best that I have been able to devise. Blame my lack of knowledge of such matters if it be not to your liking." But his invention was very well received. He produced the accompanying plan and said that it represented sixty-four towns though which he had to pass during some of his pilgrimages, and the lines connecting them were roads. He explained that the puzzle was to start from the large black town and visit all the other towns once, and once only, in fifteen straight pilgrimages. Try to trace the route in fifteen straight lines with your pencil. You may end where you like, but note that the apparent omission of a little road at the bottom is

intentional, as it seems that it was impossible to go that way.
The diagram will show how the Pardoner started from the large black town and visited all the other towns once, and once only, in fifteen straight pilgrimages. ¡img src="pardoners2.png"¿

- The 3 threes problem
  https://justpuzzles.wordpress.com/category/mathematics/arithmetic/
  Use 3 3s and any arithmetic operations to make exactly 20.
  Hard if not given any hint
  Solution (3+3/.3) = 20
- The 6 match problem
  Classic, Gardner
  Use 6 matches to build three equilateral triangles
  Difficult if never seen, as it requires an out of the box thinking.
  Solution: Build a tetrahedron.
- Stickels problem 46
  Challenging Math problems by Terry Stickels
  Using three 2s and any math symbols and/or operations you chose, can you create 25?
  Difficult as it needs some creativity, one difficulty is not only to put the puzzle together but to "invent the puzzle pieces"!
  Solutions: The book solution is $\sqrt{.2^{(}-2)}^2$. Leo Hentschker (Harvard Math undergrad) found: $(2*2)!+Gamma[2]$
- Stickels problem 45
  From Challenging Math problems by Terry Stickels
  Four circles with a radius of 2 are each tangent to two sides of a square and externally tangent to a circle with a radius of 4. What is the area of the square?
  Easy if drawn, but one needs the idea of projecting. One is seduced to add up areas of circles for example. Without a picture, it is almost impossible to do.
  Solution: Project everything on one side and add up the lengths of the projections of radius plus an integer. $4(4/\sqrt{2}+2/\sqrt{2}+2)^21 = 155.882 = 188+48\sqrt{2}$.
- Millers Puzzle
  http://www.transum.org/software/SW/Starter$_of_the_day/students/hot/MillersPuzzle.asp$
  Find a permutation $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ of the 9 digits 1-9 so that $x_1 \cdot x_2x_3 = x_4x_5x_6 = x_7x_8 \cdot x_9$. More difficult: How many solutions are there?
  basic arithmetic tedious by trial
  Solution. Brute force search
- A tragic Pythagorean paradox.
  http://www.instructables.com/id/Pythagorean-Paradox/
  Two triangles are are partitioned into congruent pieces, but one has a piece missing.
  Easy if seen
  Solution. Not all lines drawn are really lines.
- Realize the Penrose triangle
  Penrose
  Euclidean geometry
  hard if not known

- Archimedes cattle problem
  Wikipedia Problem: solve W=5B/6+Y,B=9D/20+Y,D=13W/42+Y,w=7(B+b)/12,b=9(D+d)/20
  Diophantine equations
  difficult
  System of equations with smallest solution.
- Weight problem of Meziriac
  http://www2.washjeff.edu/users/mwoltermann/Dorrie/2.pdf
  A merchant broke a 40 point weight into 4 pieces, each having an integer pound allowing

to measure every integer between 1 and 40.
additive number theory
easy by trial
1,3,9,27

- Newton's problem of Fields and Cows
Newton Arithmetica universalis http://www2.washjeff.edu/users/mwoltermann/Dorrie/3.pdf
a cows graze b fields in c days, a' cows graze b' fields in c' days, a" cows grace b" fields in c" days. Find relations between a,b,c,...,a",b",c".
linear algebra
difficult
b" c c'(a b' - b a') + c" b" (a'bc' - a b'c) = c" a" b b' (c'-c)
- Euler line theorem
Wikipedia
Centroid, orthocenter, circumcenter and center of Feuerbach circle are on a line
planimetry
difficult, as it requires proof

- Staircase Paradox
https://www.youtube.com/watch?v=LWPOlZBXtD8
The taxi metric between two points gives a different distance between two points evenso the path can be chosen arbitrarily close to a line.
Planimetry
easy. The metric is a different metric.

- Gauss two altitude problem
100 great problems of Elementary mathematics
From the altitudes of two known stars, determine the time and position.
Spherical trigonometry
difficult

- Steiner Ellipse problem
199 great problems in elementary mathematics
Which ellipse circumscribed about a given triangle, which has the smallest or largest area?
calculus
difficult
The Steiner ellipse. The focal points are the Bickart points.
- Counting primes
IMO 1969
Prove that there infinitely many positive integers m for which $n^4 + m$ is not prime for any positive integer n.
number theory
difficult
if m is a multiple of 5, then take m=5k. If m is not a multiple of 4, then m=5k+4 works.
- Triangles
IMO 1968 A1
Find all triangles for which the side lengths are three consecutive integers and one angle is twice the other.
geometry
difficult
$(a, b, c) = (a, a-1, a-2)$. Now $a^2+b^2+abcos(gamma) = c^2, c/sin(gamma) = b/sin(2gamma)$ gives $a^2 + (a-1)^2 + a = (a-2)^2$ gives $a^2 - 5a + 3 = 0$. The same with $c/sin(gamma) = a/sin(2gamma)$ gives $a^2 + (a-2)^2 + a = (a-1)^2$ which gives $a^2 - 7a + 6 = 0$ so that $a = 6$. The only solution is (5,4,5).

- Handshake problem
  classic
  You are in a room full of people and everyone shakes hands, how many total handshakes are there?
  combinatorics
  easy when known

  n(n-1)/2 = B(n,2) Binomial coefficient.
- Multiplicity problem
  Annual Colorado Spring Olympiad 1986, Soifer
  Given n integers, prove that either one of them is a multiple of n , or some subset of them adds up to a multiple of n
  Modular arithmetic
  hard
  Look at the sums s1=a1, s2=a1+a2, ... sn=a1+a2+...+an. By the Pigeon hole principle, two give the same reminder modulo n. The difference is a sum.
- Forty one rooks
  Soifer and Slobodik 1973
  Forty-one rooks are placed on a 1010 chessboard. Prove that you can choose five that do not attack each other.
  chess, combinatorics
  difficult without pigeon hole
  Pigeon hole principle, 41 pigeons = 4*10 + 1. There exists at least one row with at least 5 rooks.
- Adding odd integers
  Soifer
  Verify that each square n^2 is the sum of odd consecutive numbers.
  algebra, recursion
  easy
  It is true for $n = 1$. Assume it is true for n, now add $2n + 1$ to n^2. This is $(n + 1)^2$.
- Icosahedron paths
  Modified 2016 AIME Problem
  Fix a point A and its antipode B on an icosahedron (the unique point of distance 3 from A). How many different paths are there from A to B?
  geometry, combinatorics
  easy
  Start at A, there are 5 possible directions to go. Now, there are two possible directions. We reach now a point in distance 1 of B and have no choice any more. The answer is 5*2=10.
- Rain on Saturday or Sunday
  Modified from 2016 AIME Problem
  There is a 40 percent chance of rain on Saturday and a 30 percent chance of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. What is the probability that it rains at least one day?
  probability
  medium
  Bayesian analysis: Let A be the event that it rains on Saturday and B the event that it rains on Sunday. We know P[A],P[B]. P[A cup B] = P[A]+P[B] - P[A cap B]. And P[A cap B]/P[A] = P[B—A] = 2 (1-P[B]). So, P[A cap B] = 2 P[A] (1-P[B]).
- Dave problem
  Classic Gardner problem
  Dave has two children. One is a boy, what are the odds that the other is also a boy

conditional probability easy
2/3

- Chain problem
There is a 10 feet chain nailed to points A,B on the wall. The center dips is 5 feet below A,B. What is the distance between A and B
Pythagoras
easy
$x^2/4 + 25 = 25$. Implies x=0.
- Addition
x,y,z are positive integers and xyz=x+y+z. Find x,y,z
arithmetic
easy
1,2,3
- A girl was ten on her last birthday, and will be twelve on her next birthday. How is this possible?
http://www.doriddles.com/Riddles/Math
calendar
easy
lap day.
- When is 1500 plus 20 and 1600 minus 40 the same thing?
http://www.doriddles.com/Riddles/Math
clock
easy
time
- How can you add eight 8's to get the number 1000?
Math riddles
888+88+8+8+8=1000
easy, needs patience
trial and error
- You have have 3 zeros and 3 ones. What is the largest number you can build.
- Finding the factors of a concrete number like the Fermat number $F_{14} = 2^(2^14) + 1$, a 4932 digit number which is known to be composite but has unknown factors. A general problem is the question whether there exists an even Gaussian prime which is not the sum of two primes with positive entries.

Some Literature

Pauls literature list is much larger and can be found in his thesis.

A. Liu
Chinese Mathematics Competitions and Olympiads
1993-2001 Book2

Heinrich Doerrie
100 Great Problems of Elmentary Mathematics, their history and solution

Harry Edwin Eiss
Dictionary of Mathematical Games, Puzzles and Amusements

Martin Gardner
Hexaflexagons

Martin Garnder
The book of Mathematical Puzzles and Diversions

Terrence Tao
Solving Mathematical Problems
Oxford University Press

Li Ta-Tsien
Problems and Solutions in Mathematics
World Scientific

Bryan Bunch
Mathematical Fallacies and Paradoxes

David Klarner
Mathematical Recreations, A collection in
Honor of Martin Garner.

Piergiorgio Odifreddi
The Mathematical Century
The 30 Greatest Problems of the Last 1000 years
More advanced problems

Clifford Pickover
Mazes for the mind

Henry Enrenest Dudeney
The Canterbury puzzles and other curious Problems

list of bibliography list of problems
http://www.artofproblemsolving.com/community/c13_contest_collections

Department of Mathematics