Lecture 34: Calculus and Economics

In this lecture we look at applications of calculus to **economics**. This is an opportunity to review extrema problems.

Marginal and total cost

Recall that the **marginal cost** was defined as the derivative of the **total cost**. Both, the marginal cost and total cost are functions of the quantity of goods produced.

1 Assume the total cost function is $C(x) = 10x - 0.01x^2$. Find the marginal cost and the place where the total cost is minimal. **Solution**. Differentiate $C' = 10 + 0.02x$ Now find x which makes this vanish. We have $x = 50$.

2 You sell spring water. The marginal cost to produce it is given by $f(x) = 10000 - x^4$. For which x is the total cost maximal?

3 The following example is adapted from the book "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", where the following strawberry story appears: (verbatim citation in italics):

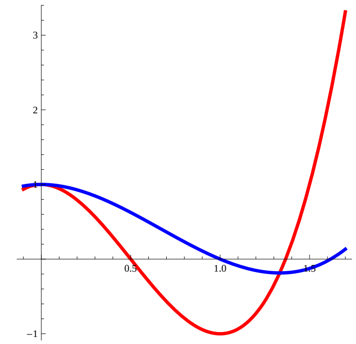
Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

Again, if $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

The function $g(x) = F(x)/x$ is called the **average cost**.

A point where $f = g$ is called a **break even point**.

4 If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

Strawberry theorem: We have $g'(x) = 0$ if an only if $f = g$.

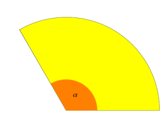
Proof.

$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g) \; .$$

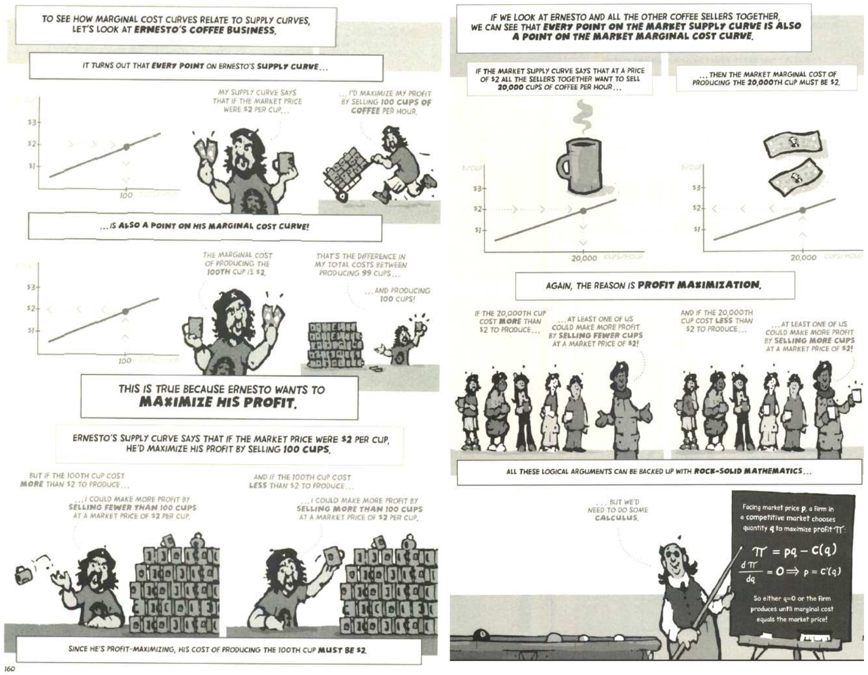
More extremization problems

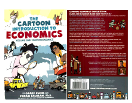
1 Find the rhomboid with side length 1 which has maximal area. Use an angle α to extremize.

2 Find the sector of radius$r = 1$ and angle α which has minimal circumference $r2r + r\alpha$ if the area $r^2\alpha/2 = 1$ is fixed.



3 Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

1 Verify the Strawberry theorem in the case when the marginal cost is $f(x) = \cos(x)$. This means that you have to compute the total cost $F(x)$ and the average cost $g(x)$ and see whether things pan out in that case.

2 The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume $Q(L) = 5000L^3 - 3L^5$. Find L which gives the maximal production.

This can be typical: For smaller groups, production usually increases when adding more workforce. After some point, bottle necks occur, not all resources can be used at the same time, management and bureaucracy is added, each person has less impact and feels less responsible, meetings slow down production etc. In this range, adding more people will decrease the productivity.

3 **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the point, where the total revenue has a local maximum.

4 Find a line $y = mx$ through the points $(3,4), (6,3), (2,5)$. which minimize the function

$$f(m) = (3m-4)^2 + (6m-3)^2 + (2m-5)^2 \ .$$

5 A function f on $[0,1]$ which has the property that its range is contained in $[0,1]$ and which has the property that $|f'(x)| < 1$ for all $x \in (0,1)$ is called a "contraction". Brower's fixed point theorem tells that a contraction has a fixed point.
a) Verify that $f(x) = 1 - x^2/2$ is a contraction.
b) Find the fixed point.

Here is a citation from the book "Mathematical Economics" by Michael Carter : *Fixed point theorems are powerful tools for the economic theorist. They are used to demonstrate the existence of a solution to an economic model, which establishes the consistency of the model and highlights the requirements minimal requirements to ensure a solution. The classic applications of fixed point theorems in economics involve the existence of market equilibria in an economy and the existence of Nash equilibria in strategic games. They are also applied in dynamic models, a fundamental tool in macroeconomic analysis.*