Lecture 33: Calculus and Music

A music piece is a function

Calculus plays a role in music because every music piece just is a **function**. If you have a loudspeaker with a membrane at position $f(t)$ at time t, then you can listen to the music. The pressure variations in the air are sound waves which reach your ear, where your eardrum oscillates with the function $f(t-T)+g(t)$ where $g(t)$ is background noise and T is a time delay for the sound reach your ear. Plotting and playing works the same way. In Mathematica, we can play a function with

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

This function contains all the information about the music piece. A music ".WAV" file contains sampled values of the function. A sample rate of 44'100 per second is usual. Since our ear does not hear frequencies larger than 20'000 KHz, a samplying rate of 44 K is good enough by a **theorem of Nyquist-Shannon**. In .MP3 files essential values are encoded in a compressed way. To get from the sample values $f(n)$ the function back, the sinc function is used. The **Whittaker-Shannon interpolation formula**

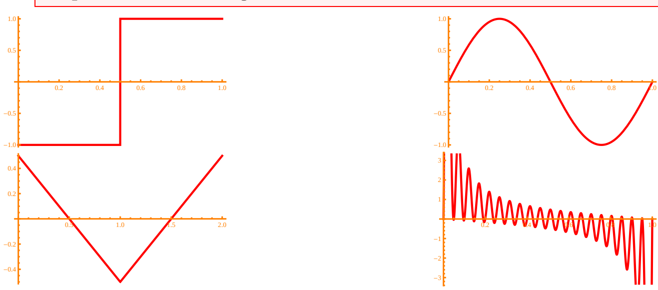
$$f(t)=\sum_n f(n)\mathrm{sinc}(t+n)$$

is especially good. We take this lecture as an opportunity to review some facts about functions. We especially see that log, exp and trigonometric functions play an important role in music.

The wave form and hull

A periodic signal is the building block of sound. Assume $g(x)$ is a 2π periodic function, we can generate a sound of 440 Hertz when playing the function $f(x)=g(440\cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A tone with 440 Hertz.

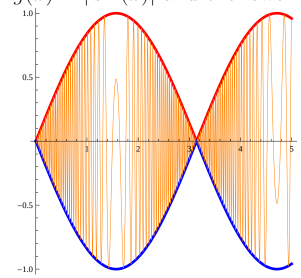
A periodic function g is called a **wave form**.



The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like "attack, vibrato, coloration, noise, echo, reverbation" and other characteristics.

The upper **hull function** is defined as the interpolation of successive local maxima of f. The lower **hull function** is the interpolation of the local minima.

For the function $f(x)=\sin(100x)$ for example, the upper hull function is $g(x)=1$ and the lower hull function is $g(x)=-1$. For $f(x)=\sin(x)\sin(100x)$ the upper hull function is approximately $g(x)=|\sin(x)|$ and the lower hull function is approximately $g(x)=-|\sin(x)|$.



We can not hear the actual function because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. Simplest examples are change of dynamics in music like **creshendi** or **diminuendi** or a vibrato. We can generate a beautiful hull by playing two frequencies which are close. You hear **interference**.

The scale

Western music uses a discrete set of frequencies. This scale is based on the exponential function. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

The **Midi numbering** of musical notes is

$$s=69+12\cdot\log_2(f/440)$$

1 What is the frequency of the Midi tone 100? **Solution**. We have to solve the above equation for f and get the **piano scale function**

$$f(s)=440\cdot 2^{(s-69)/12}\,.$$

Evaluated at 100 we get 2637.02 Hz.

The piano scale function

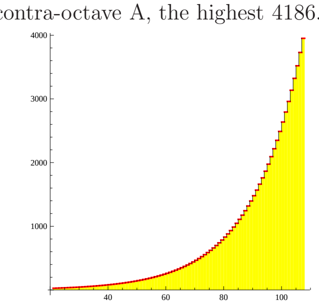
$$f(s)=440\cdot 2^{(s-69)/12}\,.$$

is an exponential function $f(s)=be^{as}$ which satisfies $f(s+12)=2f(s)$.

2 Find the discrete derivative $Df(x)=f(x+1)-f(x)$ of the Piano scale function. **Solution:** The function is of the form $f(x)=A2^{ax}$. We have $f(x+1)=2^a f$ and so $Df(x)=(2^a-1)f$ with $a=1/12$. Lets get reminded that such discrete relations lead to the important property $\boxed{\frac{d}{dx}\exp(ax)=a\exp(x)}$ for the exponential function.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

The classical piano covers the 88 Midi tone scale from 21 to 108. The lowest frequency is $27.5Hz$, the sub-contra-octave A, the highest $4186.01Hz$, the 5-line octave C.



Decomposition in overtones: low and high pass filter Every wave form can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can here melodies. Here is an example of a decomposition: $f(x) = \sin(x)+\sin(2x)/2+sin(3x)/3+\sin(4x)/4+\sin(5x)/5$. With infinitely many terms, one can also discribe discontinuous functions.

Filtering and tuning: pitch and autotune An other advantage of a decomposition of a function into basic building blocks is that one can leave out frequencies which are not good. Examples are **low pass** or **high pass** filters. A popular filter is **autotune** which does not filter but moves the frequencies around so that you can no more sing wrong. If 440 Herz (A) and 523.2 Herz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. This filtering is done on the wave form scale.

Mixing different functions: rip and remix If f and g are two functions which represent songs, we can look at $(f+g)/2$ which is the **average** of the two songs. In real life this is done using **tracks**. Different instruments can be recorded independently for example and then mixed together. One can for example get guitare $g(t)$, voice $v(t)$ and piano $p(t)$ and form $f(t) = ag(t) + bv(t) + c(p(t)$, where the constants a, b, c are chosen.

Differentiate functions: reverb and echo If f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h) - f(x)]/h$. Such a differentiation is easy to achieve with a real song. It turns out that for small h, like of order of $h = 1/1000$, the song does not change much. The reason is that a frequency $\sin(kx)$ or hearing the derivative $\cos(kx)$ produces the same song. However, if we allow h to be larger, then a **reverb** or **echo** effect is produced.

Other relations with math

We might not have time for this during the lecture.

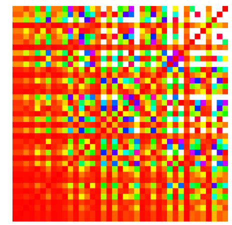
Symmetries. Symmetries play an important role in art and science. In geometry we know rotational, translational symmetries or reflection symmetries. Like in geometry, symmetries play a role both in Calculus as well as in Music. We see some examples in the presentation.

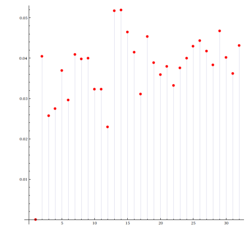
Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**, **Diophatine problems** like how good frequency rations are approximated by rationals: Why is the **chromatic scale** based on the twelfth root of 2 so good? Indian music for example uses **microtones** and a scale of 22. The 12 tone scale is good because many powers $2^{k/12}$ are close to rational numbers. I once defined the "scale fitness" function

$$M(n) = \sum_{k=1}^{n} \min_{p,q} |2^{k/n} - \frac{p}{q}| G(p,q)$$

which is a measure on how good a music scale is. It uses Euler's **gradus suavis** (="degree of pleasure") function $G(n,m)$ of a fraction n/m which is $G(n,m) = 1 + E(nm/\gcd(n,m))$, where the **Euler gradus** function $E(n) = \sum_{p|n} e(p)(p-1)$ and p runs over all prime factors p of n and $e(p)$ is the multiplicity. The picture to the left shows Euler's function $G(n,m)$, the right hand side the scale fitness function in dependence on n. You see that $n = 12$ is clearly the winner. This analysis could be refined to include scales like Stockhausens $5^{k/25}$ scale. You can listen to the Stockhausen's scale with $f(t) = \sin(2\pi t 100 \cdot 5^{[t]/25})$, where $[t]$ is

the largest integer smaller than t. Our familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 100 \cdot 2^{[t]/12})$.





3 The perfect fifth 3/2 has the gradus suavis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1 + E(12) = 1 + (2-1)(3-1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

1 **Modulation.** Draw the hull function of the following functions.

a) $f(x) = \sin(1000x) - \sin(1001x)$

b) $f(x) = \sin(x) + \cos(\tan(1000\sqrt{x}))$

c) $f(x) = \sqrt{x}\cos(10000x)$

d) $f(x) = \cos(x)\sin(e^{2x})/2$

Here is how to play a function with Mathematica. It will play for 9 seconds:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0, 10}]
```

Hint. You can play functions online with Wolfram Alpha. Here is an example:

```
play  sin(1000 x)
```

2 **Amplitude modulation (AM)**: If you listen to $f(x) = \sin(x)\sin(1000x)$ you hear an amplitude change. Draw the hull function. How many increase in amplitudes to you hear in 10 seconds?

3 **Frequency Modulation (FM)**: If we play $f(x) = x\sin(1000\sin(x))$, there are points, where the frequency is low. This is a frequency change. Draw the hull function. Try first without computer.

4 **Smoothness**: If we play the function $f(x) = \tan(\sin(3000\sin(x)))$, the sound sounds pretty nice. If we change that to $f(x) = \tan(2\sin(3000\sin(x)))$, the sound is awful. Can you see why? To answer this, you might want to plot a similar function where 3000 is replaced by 3.

5 **A mystery sound**: How would you describe the sound $f(x) = \sin(1/\sin(2\pi 3x))$? Our ear can not hear frequencies below 20 Hertz. Why can one still hear something? To answer this, plot first the function from $x = 0$ to $x = 10$.