Dynamical Systems

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Aim: Qualitative Understanding of equations without explicit knowing of the solution.
One branch: Statistical behaviour ergodic theorie.
Here: emphasis on general differential equations not just Hamiltonian systems.

Remarks: General differential equation: $\frac{dx}{dt} = X(x)$
Space of the possible x's: state space $=: M$
It might be an euclidean space or an open set in euclidean space, a manifold or a Banachspace or an infinit dim. manifolds. Here our state space will be a manifold, but the structure of the manifolds will not be import. at the beginning so we talk about open sets in euclid. space. Think of compact euclid. spaces. We will need nonlinear functional analysis in infinit. dim. spaces. The infinit dimdit. case could be applied for partial differential equations as for example the Navie Stokes equation, but this would need a long pale of smoothing analyses and so on. If $X(\cdot)$ is Lipschitz, so local solutions exist and are unique. Ex. $\frac{dx}{dt} = x^2 \rightarrow x_0 > 0$ implies $x(t) \to \infty$ in finite time so that global solutions dont even exist in nice examples. The solution mapping $f^{t_0}(x_0)$ is defined if the d.e. has a solution $x(t)$ defined for $t \in [0, t_0)$ with $x(0) = x_0$. $f^{t_0}(x_0) = x(t_0)$. We have $f^{t_1}(f^{t_2}(x)) = f^{t_1+t_2}(x)$ whenever the left hand side is defined. Lets say that $f^t(x)$ is a semiflow. If global solutions do exist we have a semigroup $f^{t_1} \circ f^{t_2} = f^{t_1+t_2}$ such f^t is a differentiable mapping of M to itself with differentiable inverse, shortly: smooth flow if $(t, x) \to f^t(x)$ is smooth.
Then $X(x) = \frac{d}{dt} f^t(x)|_{t=0}$. Instead of f^t one could consider simply a discrete one param. group of diffeomorphism $\{f^n\}_{n \in \mathbb{Z}}$ or $\{f^n\}_{n \in \mathbb{N}}$. Clearly every thing interesting for flows has an analog for maps, the map version is technically simpler. For example f^{t_0} or surface of sections or Poincaré map constructions that reduces analysis about flow in n dimensions to maps in (n-1) dimensions. Construction: (f^t, M) Σ hypersurface in M so that Σ is transverse to $X(x)$ at each of its points. $\phi(x)$ is then defined, if $f^t(x)$ ever crosses Σ for $t > 0$ and its defined the first crossing point. By the implicit function theorem the domain of ϕ is open and ϕ is smooth where defined. For example a periodic point to ϕ corresponds to a closed orbit, a periodic solution for f^t. Important example is periodic differential equation. $\frac{dx}{dt} = X(t, x)$, $X(t+\tau, x) = X(t, x)$ with solution mappings $f^t(x)$. Here the group law fails because $f^{n\tau} = (f^\tau)^n$
The general subject of this course:

I Preliminarys:
- Spectral theory for general bounded operators on Banach spaces
- Elementary for differential calculus on Banachspaces (implicit function theorem)

II Behaviour of iterals of a map near a fixed point (invariant manifolds)
III Elementary bifurcation theory.
IV Poincaré Bendixon theory.
V Anosov systems and hyperbolic sets
VI Homomorphism of the circle classical topology theory. Moser, Hermann small divisor problem, renormalis. group analys.
VII unimodal interval mappings Topological theory
VIII Renormalisation group analysis for period doubling accumulation and for critical circle mappings.

I Spectral theory and differential calculus in B.spaces

1- Spectral theory

Look Dunford - Schwarz vol 1. Chap II - VII

Nicest case for spectral theory: normal operators on a finite dim. Hilbertspace $\mathcal{H}$ where $\mathcal{H}$ is spanned by the eigenvectors of A

Generalized eigenvectors $(A-\lambda\mathbb{1})^j \xi = 0$ for some j with Eigenspace $\{\xi : (A-\lambda\mathbb{1})^j \xi = 0 \, \exists j\}$ with Eigenvalue λ

$\mathcal{H}$ is then the direct sum of generalized Eigenspaces

In the case of normal operators on a infinite dim Hilbert. space there may be not any eigenvector

There we have $A = \int \lambda P(d\lambda)$ where $P(.)$ is a projection valued measure. Intuition: $P(E)$ = span of eigenvectors with Eigenvalue in E where E is a Borelset in $\mathbb{C}$

For a general bounded operator on a Banach space. For any isolated subset τ of the spectrum there is an associated spectral projection

Let A be a bounded operator on $\mathcal{X}$. The resolvent set of A is $\{\lambda \in \mathbb{C} \mid (\lambda\mathbb{1} - A)$ is invertible $\}$. The spectrum is the complement of the resolv. set.

If U is an invertible operator and V is some other operator, then the series is called von Neumann Serie

$$\sum_{n=0}^{\infty} (U^{-1}V)^n U^{-1} \underset{form.}{=} (U-V)^{-1}$$

If series converge in norm $\|V\| < \|U^{-1}\|^{-1}$ then $U-V$ is invertible and the equality is right.

$\sigma(A)$ is closed and the spectrum is not empty: suppose $\sigma(A)$ is empty. Look at $(\lambda\mathbb{1}-A)^{-1}$ as a function of λ is an analytic operator valued function. For sufficiently large λ

$$(\lambda\mathbb{1} - A)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}} \quad \text{for } \|A\| < |\lambda| \quad (\text{or even } |\lambda| > \rho(A))$$

$\downarrow$ for $\lambda \to \infty$

0

For any $\phi \in \mathcal{H}^*$ and any $\xi \in \mathcal{H}$ is $\phi((\lambda\mathbb{1}-A)^{-1}\xi)$ analytic, and therefore by Liouville identically zero.

$\Rightarrow (\lambda\mathbb{1}-A)^{-1} \equiv 0$ $\lightning$

we have $\|A^{n+m}\| \le \|A^n\| \|A^m\|$

$\Rightarrow$

$$\rho(A) := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

called the spectral radius exists and

$$\rho(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$$

so $\sigma(A) \subset \{\lambda \mid |\lambda| \le \rho(A)\}$. Suppose $\sigma(A) \subseteq \{\lambda \mid |\lambda| \le R \le \rho(A)\}$

$$\frac{1}{2\pi i} \int_{|\lambda| = M} \lambda^m (\lambda\mathbb{1} - A)^{-1} \, d\lambda = A^m \quad M > \rho(A)$$

Cauchy: Integration around $|\lambda| = R$

$\|A^m\| \le B_R R^m$ für alle $m \Rightarrow \rho(A) \le R$ $\lightning$

NB $\rho(A) \le \|A\|$

Ex. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\rho(A) = 0$ $\|A\| \neq 0$

If $R > \rho(A)$ then we can replace the original norm by an equivalent norm $|||\ \ |||$ where $|||A||| \le R$

for exampl. $|||\xi||| = \sup_n \|(\frac{A}{R})^n \xi\|$

We have $\|\xi\| \le |||\xi||| \le \sup_n \|(\frac{A}{R})^n\| \|\xi\|$
$\underbrace{\qquad\qquad}_{finite}$

$$|||A\xi||| = \sup_{n\ge 0} \|\frac{A^{n+1}}{R^n}\xi\| = R\sup_{n\ge 1}\|(\frac{A}{R})^n\xi\| \le R\, |||\xi|||$$

Here a Projection P means a bounded operator such that $P^2 = P$
A Projection P is associated to a direct sum decomposition:
$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ $\quad P : (\xi_1 \xi_2) \mapsto (\xi_1, 0)$ $\quad \mathcal{X}_2 = \ker P$, $\mathcal{X}_1 = \ker(1-P) = \operatorname{ran} P$
$(1-P)\xi = 0 \Rightarrow \xi = P\xi \quad \xi \in \operatorname{ran}(P)$, $\quad \xi \in \operatorname{ran}(P) \quad \xi = P\eta \quad P\xi = PP\eta = P\eta = \xi$
$\mathcal{X}_1 \cap \mathcal{X}_2 = 0 \quad \xi = P\xi + (1-P)\xi$. ($P$ is given only by specifying $\mathcal{X}_1$ and $\mathcal{X}_2$)

$[A, P] = 0 \iff A\mathcal{X}_1 \subset \mathcal{X}_1 \quad A\mathcal{X}_2 \subset \mathcal{X}_2$

A bound op on $\mathcal{X}$ diagonalisable (and by approximation in the general case)
$\sigma(A) = S_1 \cup S_2 \quad S_1 \cap S_2 = \emptyset \quad S_1, S_2$ closed

$$\frac{1}{2\pi i}\int_\gamma (\lambda I - A)^{-1} d\lambda \qquad S_1 \text{ inside } \gamma \quad S_2 \text{ outside } \gamma$$

$$= \begin{cases} 1 & \text{on eigenvectors with eigenvalues in } S_1 \\ 0 & \text{on } \quad " \quad\quad " \quad\quad " \quad\quad " \quad S_2 \end{cases}$$

<u>Proposition 1:</u>

> A bound. operator on $\mathcal{X}$
> $S_1 \cup S_2 = \sigma(A)$ $S_1 \cap S_2 = \emptyset$ S_1, S_2 closed
> γ a circle around S_1 so that each point of S_1 had windnumber 1
> and every point in S_2 w.n. 0
> then
> $$\frac{1}{2\pi i}\int_\gamma (\lambda I - A)^{-1} d\lambda =: P$$
> is a projection, commuting with every operator commuting with A. It only depends on S_1, S_2
> $\sigma(A|_{\operatorname{Ran} P}) = S_1$
> $\sigma(A|_{\ker P}) = S_2$
> The above properties characterise P

Proof: By Cauchy integral formula the integral doesn't depend on γ
take another γ_1 "inside" γ in the sense that γ winds once around each point of γ_1 and γ_1 winds zero times around each point of γ

$$P^2 = \frac{1}{(2\pi i)^2}\int_\gamma (\lambda I - A)^{-1}\int_{\gamma_1}(\lambda_1 1 - A)^{-1} d\lambda_1\, d\lambda$$

$u^{-1} - v^{-1} = u^{-1}(v-u)v^{-1}$ $\qquad \begin{matrix} u := \lambda 1 - A \\ v := \lambda_1 1 - A \end{matrix} \Big\} \Rightarrow \begin{matrix}((\lambda 1 - A)^{-1} - (\lambda_1 1 - A)^{-1})(\lambda_1 - \lambda) \\ = (\lambda 1 - A)^{-1}(\lambda_1 1 - A)^{-1}\end{matrix}$

$$= \frac{1}{(2\pi i)^2}\int_\gamma d\lambda \int_{\gamma_1} d\lambda_1 \frac{(\lambda 1 - A)^{-1}}{\lambda_1 - \lambda} - \frac{(\lambda_1 1 - A)^{-1}}{\lambda_1 - \lambda}$$

$$= \frac{1}{(2\pi i)}\int_\gamma d\lambda\, (\lambda 1 - A)^{-1} \underbrace{\frac{1}{2\pi i}\int_{\gamma_1}\frac{d\lambda_1}{\lambda_1 - \lambda}}_{0}$$

$$+ \frac{1}{(2\pi i)}\int_\gamma d\lambda\, (\lambda_1 1 - A)^{-1} \underbrace{\frac{1}{2\pi i}\int_{\gamma}\frac{d\lambda_1}{\lambda_1 - \lambda}}_{1} = P$$

If specially $S_1 = \sigma(A)$ ~~and~~ $S_2 = \emptyset$ $\quad P = 1$ if γ is big enough.
Interchanging roles of S_1 and S_2 replaces P by $1 - P$
claim: $\sigma(A|_{P\mathcal{X}}) \subseteq S_1$
i.e. if $\lambda \in S_2$ $\quad (\lambda 1 - A)|_{P\mathcal{X}}$ is invertible

$$B := \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda - \lambda_1}(\lambda_1 1 - A)^{-1} d\lambda_1$$

and $(\lambda 1 - A)B = \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda - \lambda_1}[(\lambda_1 1 - A) + (\lambda - \lambda_1)1](\lambda_1 1 - A)^{-1} d\lambda_1$

$$= \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda-\lambda_1}\,\mathbb{1}\,d\lambda_1}_{0} + \underbrace{\frac{1}{2\pi i}\int_\gamma (\lambda_1\mathbb{1}-A)^{-1}d\lambda_1}_{P} = P$$

For any ξ in $Ran(P)$ $\quad B(\lambda\mathbb{1}-A)\xi = (\lambda\mathbb{1}-A)B\xi \Rightarrow B|_{P\mathcal{X}} = (\lambda\mathbb{1}-A)^{-1}|_{P\mathcal{X}}$

$$\underbrace{\sigma(A|_{P\mathcal{X}})}_{=S_1} \cup \underbrace{\sigma(A|_{(\mathbb{1}-P)\mathcal{X}})}_{=S_2} = \sigma(A)$$

$\longrightarrow$

Let now $\hat{P}$ with all those properties $\hat{P}$ commutes with $\mathbb{1}-P$
$(\mathbb{1}-P)\hat{P}$ is a projection $\Rightarrow ran(\mathbb{1}-P)\hat{P}$ is invariant unω A
$\hat{\mathcal{X}} := (\mathbb{1}-P)\hat{P}\mathcal{X}$

$\sigma(A|_{\hat{\mathcal{X}}}) \subset S_1$ because $\hat{\mathcal{X}} \subset \hat{P}\mathcal{X}$

$\sigma(A|_{\hat{\mathcal{X}}}) \subset S_2$ because $\hat{\mathcal{X}} \subset (\mathbb{1}-P)\mathcal{X}$

$\Rightarrow \hat{\mathcal{X}} = \{0\} \Rightarrow (\mathbb{1}-P)\hat{P} = 0 \quad \hat{P} = P\hat{P}$
$\Rightarrow P = P\hat{P} = \hat{P}P = \hat{P}$ ∎

Suppose P is a one dim projection : $ran\, P = \{\alpha\xi\}$
$ker\, P = ker\, \phi \quad \phi \in \mathcal{X}^*$
o.E.d.A. $\phi(\xi) = 1$

$P\eta = \phi(\eta)\xi$. P commutes with $A \Rightarrow A\xi = \lambda\xi$ for some λ
$\phi(A\eta) = 0$ if $\phi(\eta) = 0$

$(A^*\phi)(\eta) = 0$ if $\phi(\eta) = 0$ and $A^*\phi = \lambda\phi$

Spectral theory on real vector spaces

In this case we move from $\mathcal{X}$ the real vector space to the complexification $\mathcal{X} := \mathcal{X} \times \mathcal{X}$ with action

$$(\alpha + i\beta)(\xi, \eta) = (\alpha\xi - \beta\eta, \ \alpha\eta + \beta\xi)$$

with norm $\quad \|(\xi,\eta)\|_{\mathbb{C}} := \sup_\theta \|\cos\theta\xi - \sin\theta\eta\|$

if we have a real linear operator A on $\mathcal{X}$

$$A_{\mathbb{C}} \text{ on } \mathcal{X}_{\mathbb{C}} : (\xi,\eta) \mapsto (A\xi, A\eta)$$

The spectral theory for A means now the spectral theory ~~of~~ for $A_{\mathbb{C}}$.

Proposition 2 Let A be a bounded real linear operator on $\mathcal{X}$. Then

1) $\sigma(A)$ is invariant under complex conjugation
2) $\sigma(A) = S_1 \cup S_2 \quad S_1 \cap S_2 = \emptyset$ seperated with S_1, S_2 invariant under complex conjugation, then there is a unique projection P on $\mathcal{X}$ commuting with everything commuting with A such that

$$\sigma(A|_{P\mathcal{X}}) = S_1$$
$$\sigma(A|_{\ker P}) = S_2$$

Proof : An operator B on $\mathcal{H}_{\mathbb{C}}$ is the complexification of a real-linear operator A $\Longleftrightarrow$ B commutes with $k : (\xi, \eta) \mapsto (\xi, -\eta)$, (complex conjugation)

$$A_{\mathbb{C}} - \lambda \mathbb{1} \text{ invertible} \Rightarrow \underset{\substack{\| \\ (A_{\mathbb{C}} - \bar{\lambda}\mathbb{1})}}{k^{-1}(A_{\mathbb{C}} - \lambda \mathbb{1})k} \text{ is invert.}$$

$\Rightarrow$ $\sigma(A)$ invariant under complex conjugation

$\sigma(A) = S_1 \cup S_2$

$P_{\mathbb{C}} = k^{-1} P_{\mathbb{C}}\, k$ by the uniqueness (Prop 1)

$P_{\mathbb{C}}$ is indeed the complexification of a real projection. ■

A is called ~~hyperbolic~~ hyperbolic, if the spectrum of A does not intersect the unit circle!

Proposition 3 :

A hyperbolic then $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$ so that $\mathcal{H}_s$, and $\mathcal{H}_u$ are invariant under A and $\sigma(A/\mathcal{H}_s) \subset \{ |\lambda| < 1 \}$

$\sigma(A/\mathcal{H}_u) \subset \{ |\lambda| > 1 \}$

We can choose a norm, so that

$$\| A|_{\mathcal{H}_s} \|_{\mathcal{H}_s} < 1 \qquad \| (A|_{\mathcal{H}_u})^{-1} \| < 1$$

Note : $\mathcal{H}_s$ is called the stable subspace for A and $\mathcal{H}_u$ is called the unstable subspace for A

Exercise : ① $\mathcal{H}_s = \{ \xi : \| A^n \xi \| \text{ is bounded} \}$

$= \{ \xi : A^n \xi \to 0 \}$

How can one caracterize $\mathcal{H}_u$ without assuming A invertible? $\{ \xi \mid \exists \lambda > 1 \, \|A^n \xi\| \geq \lambda^n \}$ doesn't function

① $A^n + 1$ ist invertierbar
$(A^n + 1)^{-1} \xi \to \xi$ für $\xi \in \mathcal{H}_s$
$(A^n + 1)^{-1} \xi \xrightarrow{n \to \infty} 0$ für $\xi \in \mathcal{H}_u$

$\mathcal{H}$ complex vector space
A complex linear

② We can again recomplexify A ~~???~~ (looked at) as a real linear Operator

What is the relation between $\sigma(A)$, spectral projections for A and the corresponding objects for $A_{\mathbb{C}}$?

KA is (not yet) real linear

③ What's the realisation between the spectrum of A and the spectrum of KA.

② $\lambda \in \sigma(A)$ $\lambda = \alpha + i\beta$
$A_{\mathbb{C}}(\xi, \eta) - \lambda(\xi, \eta)$
$= (A\xi - \alpha\xi + \beta\eta, A\eta - \alpha\eta - \beta\xi)$

λ reell $\Rightarrow$ $\lambda \in \sigma(A_{\mathbb{C}})$
λ

Differential calculus on Banach spaces

Here the Banach spaces are assumed to be real Banach spaces.

X, Y Banach spaces $f: U \subseteq X \to Y$

Def f is differentiable at $\xi_0 \in U$ means

- $\xi_0 \in \operatorname{int}(U)$
- there is a linear operator $L: X \to Y$ so that

$$\| f(\xi_0 + \delta\xi) - f(\xi_0) + L\delta\xi \| = o(\|\delta\xi\|)$$

i.e for ~~all~~ any $\varepsilon > 0$ $\exists \delta > 0$ so that

$$\| f(\xi_0 + \delta\xi) - f(\xi_0) + L\delta\xi \| \le \varepsilon \|\delta\xi\|$$

when $\|\delta\xi\| \le \delta$

*

L is unique and we call it the derivative of f at ξ_0 and we call it $Df(\xi)$. L_1, L_2 are derivatives : $L := L_2 - L_1$ $\|L(\xi)\|/\|\xi\| \to 0 \Rightarrow L = 0$

we require L to be bounded then it follows that f is continuous.

There is a weaker notion of differentiability:

Def f is ~~Fréchet differentiability~~ Gateau " at ξ_0 if $\forall \eta$ the mapping $\eta \longmapsto f(\xi_0 + t\eta)$ is Gateau differentiable in t at $t = 0$ *

~~If f is f F.differentiable~~ Relation between F differentiability and G differentiability. $F \to G$

$f: \mathcal{X} \to Y$
$Df(\xi) \in \mathcal{L}(X, Y)$

and we can build higher differentials.

$(D^2 f)\xi \in \mathcal{L}(X, \mathcal{L}(X, Y))$ = space of bilinear mapping $X \times X \xrightarrow{L} Y$

$f: U \subseteq X \to Y$

f is of class C^n on U if:

$D^n f(\xi)$ exists for all $\xi \in U$, and $\xi \longmapsto D^n f(\xi)$ is continuous in the standard norm of $\{ A : X^n \to Y \}$ multilinear space

$(X_1, g_1), (X_2, g_2)$ pair of metric spaces

$f: X_1 \to X_2$

we say f is Hölder continuous of order α ($\alpha > 0$), if $\exists k$ so that

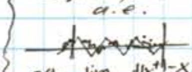
$$g_2(f(x_1), f(x_2)) \le k\,[g_1(x_1, x_2)]^\alpha$$

f is lipschitz continous, if $\exists k$ so that

$\delta_2 (f(x_1), f(x_2)) \leq k \cdot \delta_1(x_1, x_2)$ $\alpha_1 = 1$

Hölder (α) $\alpha > 1$ is not interesting: then its differentiable

in $\mathbb{R}$
Lipschitz continuous $\Longrightarrow$ differentiable a.e.



$g(x) = \lim\limits_{x^+ \to x} \frac{f(x^+) - x)}{x^+ - x} - \lim\limits_{x^- \to x} \frac{f(x^- - x)}{x^- - x}$

Zeigen $\int g(x) dx = 0$

$g(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$

$\int g(x) dx < \infty$ $g(x)$ integrable

$\int g(x) dx = \int g(x) dx - \int g(x) dx = 0$

$r > 0$ $r = n + \alpha$ $0 \leq \alpha < 1$

C^n is defined. f is of Class C^r, if

- $D^n f(\xi)$ exists for all $\xi \in U$
- $\xi \to D^n f(\xi)$ is Hölder continous for α on a neighborhood of each point of U

- Chain rule, Taylor, Existence of partial derivati-
  implicit function theorem ... go over.

Differences: Any argument which uses local compactes fails in this case.

finite: $f \in C^0 \longrightarrow f$ bounded on bounded sets.

here: $f \in C^\infty \not\longrightarrow f$ bounded on bounded sets
fails

Exercises: Construct a unbounded C^∞ Function defined on some Banach space X which vanishes identically outside the unit ball.
A continuous funct. need not be unif. contin. on any neighborhood of a given point

We have nonexistence of smooth partitions of unity.
In a given Banach space, there need not exist non-zero continuosly differentiable functions vanishing outside bounded sets.

$\underline{Ex}$: In a Hilbertspace H. Norm is C^∞ away from 0

$\underline{Ex}$: ℓ^p if $n < p$ $\|x\|_p^p$ is of class C^n
 if $n = p$ $\|x\|_p^p$ is everywhere n-times differentiable

case 1: p even $D^p(\|\cdot\|_p^p) = \text{const}$
$\Rightarrow \|\cdot\|_p^p$ C^∞

in finite dim $\frac{d^p}{dx^p} |x|^p = \begin{cases} p! & p \text{ even} \\ p! \operatorname{sign} x & p \text{ odd} \end{cases}$

case 2: p odd $D^p \|\cdot\|_p^p$ nowhere continous

p odd integer $n > p \Longrightarrow$ the only C^n function vanishing outside the unit ball $\equiv 0$

consequence of Bonic & Frampton: BAMS 71 (1965) 383-395

Theorem: (Bonic, Frampton)

$p \geq 1$ not an even integer
D connected not empty open bounded set in ℓ^p
f: continous real valued function on $\bar{D}$
C^n on D itself $n > p$

than $\sup\limits_{x \in D} |f(x)| = \sup\limits_{x \in \partial D} |f(x)|$

Exercices: -1: $A \longmapsto A^{-1}$

(1) $\mathcal{L}(X,Y) \to \mathcal{L}(Y,X)$

A^{-1} is C^1 compute the derivative (von Neumann)

(2) $C^n((-1,1)) \ni f$ (that means with $|f^n(x)|$ is bounded on $(-1,1)$)

with norm $\|f\|_{C^n} = \max\{\sup_x |f(x)|, \sup_x |f^{(n)}(x)|\}$

consider $f \longmapsto f \circ f$

which maps the unit ball into itself

show that this mapping is <u>nowhere</u> differentiable.

It is differentiable: $C^n \to C^{n-1}$

<u>Fixpoint mappings.</u>

<u>Contraction Principle:</u>

$\boxed{\begin{array}{l} (X,\rho) \text{ complete metric space} \\ f: X \to X \\ \exists \varkappa < 1 \text{ s.t. } \rho(f(x),f(y)) \le \varkappa \cdot \rho(x,y) \quad \forall x,y \in X \\ \text{Than } f \text{ has exactly one fixpoint in } X. \\ \text{For any } x_0 \in X \quad f^n(x_0) \to \bar{x} \quad n \to \infty \end{array}}$

Γ $\rho(f^n(x_0), f^{n-1}(x_0)) = \varkappa^n \rho(x_0, f(x_0))$

$\rho(x_0, \bar{x}) \le \frac{1}{1-\varkappa} \rho(x_0, f(x_0))$ $\lrcorner$

<u>Corollary</u>

$\boxed{\text{The fixpoint changes continously with the mapping:}}$

Γ f_1, f_2 both contractive $(\varkappa)$ with fixpoints $\bar{x}_1, \bar{x}_2$

$\rho(\bar{x}_1, \bar{x}_2) < \frac{1}{1-\varkappa} \rho(\bar{x}_1, f_2(\bar{x}_1)) \le \frac{1}{1-\varkappa} \sup_x \rho(f_1(x), f_2(x))$ $\lrcorner$

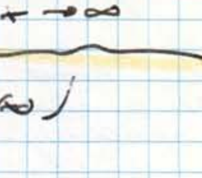
$x_1, x_2 \in X$

f differentiable on $[x_1, x_2]$ then

$\| f(x_1) - f(x_2) \| \le \|x_1 - x_2\| \cdot \sup_{x \in [x_1, x_2]} \{ \| Df(x) \| \}$

<u>Proposition:</u>

$\boxed{\begin{array}{l} f \text{ differentiable on } B(x_0, \rho) \\ \|Df(x)\| \le k < 1 \text{ on } B(x_0, \rho) \\ \text{If } \quad \|f(x_0) - x_0\| \le \rho(1-k) \\ \Rightarrow f \text{ maps this ball contractivly into itself} \end{array}}$



Γ $\|x - x_0\| \le \rho$

$\| f(x) - x_0 \| \le \| f(x) - f(x_0) \| + \| f(x_0) - x_0 \|$

$\le \underbrace{k \cdot \|x - x_0\|}_{\le \rho} + (1-k)\rho \le \rho$ $\lrcorner$

What do we do if we have no contraction?

<u>Newton's Method</u>: Reduce root-finding problem to looking for a fixed point of a contraction

$F: X \longrightarrow Y$

A <u>root</u> of F := $\bar{x} \in X$ with $F(\bar{x}) = 0$

If x_0 is an approximate root:

Near x_0 $F(x) \approx F(x_0) + DF(x_0)(x - x_0)$

If $DF(x_0)$ is invertible, we have just by algebra

$$F(x_0) + DF(x_0)(x - x_0) \text{ vanishes at}$$

$$x_1 = x_0 - [DF(x_0)]^{-1} F(x_0)$$

Look for a root of F by looking for a fixed point of the mapping

$$x \longmapsto \underbrace{x - (DF(x))^{-1} F(x_0)}_{\phi(x)}$$

by iteration.

We have to show that $D\phi$ exists

$$D\phi(x) = 1\!\!1 - D[[DF(x)]^{-1}]] \quad F(x) + \underbrace{[DF(x)]^{-1} DF(x)}_{1\!\!1}$$

$$= - D[[DF(x)]^{-1}] \quad F(x)$$

If $F(\bar{x}) = 0$, then $D\phi(\bar{x}) = 0$

$F \in C^2$

$D\phi$ contractive on a small neighborhood of $\bar{x}$

$F \in C^3$

$\Rightarrow \| \underbrace{\phi(x)}_{x_{n+1}} - \bar{x} \| \leq k \, \|x_n - \bar{x}\|^2$ "Number of digits will double"

<u>Propos</u>: F of class C^2 with bounded 2^{nd} derivatives on a ball of radius ς about x_0 and

$DF(x_0)$ invertible

$M = \| DF(x_0)^{-1} \|$

$\varepsilon = \| F(x_0) \|$

$k_1 = \sup \{ \| DF(x) \| \ / \ \|x - x_0\| \leq \varsigma \}$

$k_2 = \sup \{ \| D^2F(x) \| \ / \ \|x - x_0\| \leq \varsigma \}$

and

Dφ exists and V.N series converg $D\phi < 1$ → (1) $k_1 \varsigma M < 1$

(2) $k := k_2 \left(\frac{M}{1 - k_1 \varsigma M} \right)^2 (\varepsilon + k_1 \varsigma) < 1$

start value is right → (3) $M\varepsilon \leq \varsigma(1 - k)$

then F has exactly one root in $\{ x \ / \ \|x - x_0\| \leq \varsigma \}$

$\phi(x) = x - (DF(x))^{-1} F(x)$

$\| \phi(x_0) - x_0 \| \leq \varsigma(1 - k)$

by Neum. series and (1) $DF(x)$ is invertible on $\{ \|x - x_0\| \leq \varsigma \}$ and

$\| DF(x)^{-1} \| \leq M / (1 - k_1 \varsigma M)$

$$D\phi(x) = \underbrace{(DF(x)^{-1})^2}_{\leq M^2/(1-k_1\varsigma M)^2} \underbrace{D^2F(x)}_{\leq k_2} \underbrace{F(x)}_{\leq \varepsilon + k_1 \varsigma}$$

because $\| DF(x)^{-1} \| = \| (DF(x_0) + (x - x_0) DF(x))^{-1} \|$ $\underset{V.N}{=}$ $DF(x_0)^{-1} \sum (DF(x_0)^{-1} DF(x_0)(x - x_0))^n$

Variant from Newton's method:

Propos

F differentiable mapping of $\{x / \|x-x_0\| \le \delta\}$ into Y:

Γ invertible linear operator $X \to Y$

Assume $\exists k < 1$ s.that

(1) $\|\Gamma^{-1}(DF(x) - \Gamma)\| \le k < 1$
$\forall \|x - x_0\| \le \delta$

(2) $\|\Gamma^{-1}F(x_0)\| \le (1-k)\delta$

Then F has a unique root in $\{x / \|x-x_0\| \le \delta\}$

$\ulcorner$ $\phi(x) := x - \Gamma^{-1}F(x)$

root of F = fixed point of ϕ

$D\phi(x) = 1\!\!1 - \Gamma^{-1}DF(x)$

$= 1 - \Gamma^{-1}(DF(x) - \Gamma) - \Gamma^{-1}\Gamma$

$= -\Gamma^{-1}(DF(x) - \Gamma)$ $\lrcorner$

Theorem

Implicit funct. theorem

Let X, Y, Z be Banach spaces
F a function defined on a neighborhood U of $(0,0)$ on $X \times Z$ with values in Y

$$F : U \subseteq (X \times Z) \longrightarrow Y$$

(we want to solve $F(x,z) = Y$ for x as a function of z

Assume

(1) For each $(\xi_0, \zeta_0) \in U$
$\xi \mapsto F(\xi, \zeta_0)$ is differentiable in ξ near ξ_0

$(\xi, \zeta) \mapsto D_1F(\xi, \zeta)$ is continous

(2) $F(0,0) = 0$

(3) $D_1F(0,0)$ is invertible

Then

a) $\delta_x > 0$ is small enough
$\Rightarrow \exists! \xi$ with $\|\xi\| \le \delta_x$ so that
$F(\xi, \zeta) = 0$

b) Fix δ_x, δ_z small enough, so that $(D_1F)^{-1}(\xi, \zeta)$ is invertible for all $(\xi, \zeta) \in \Delta = \{\|\xi\| \le \delta_x, \|\zeta\| \le \delta_z\}$

Denote the ξ corresponding to ζ by $X(\zeta)$

If F is differentiable on Δ then

X is differentiable on $\{\xi \,/\, \|\xi\| \le d_z\}$ and
$$DX(\xi) = -[D_1F(X(\xi),\xi)]^{-1} D_2F(X(\xi),\xi)$$
c) If F is of class C^r $(r \ge 1)$ so is X

Proof : a) Apply the preceeding proposition for $F^{\xi}(\xi) = F(\xi,\xi)$
$\Gamma = (D_1F)(0,0)$

At $\xi = 0$ $DF^{\xi}(0) = \Gamma$

By continuity, we can make
$$\| \Gamma^{-1}(DF^{\xi}(\xi) - \Gamma \| \le \frac{1}{2}$$
For $\|\xi\| \le d_x$ $\|\xi\| \le d_z$

By making d_z smaller if necessary, we can also assume
$$\| \Gamma^{-1} F^{\xi}(0) \| \le \frac{1}{2} d_x \quad \text{if } \|\xi\| \le d_z$$
Apply preceding proposition with $k = \frac{1}{2}$, $\rho = d_x$

b) To prove X differentiable

X is continous
$$X(\xi + d\xi) = X(\xi) + dX$$
$$F(X,\xi) = 0$$
$$F(X+dX, \xi+d\xi) = 0$$
$$(D_1F)(X,\xi)\,dX + D_2F(X,\xi)\,d\xi = o(\|dX\| + \|d\xi\|)$$
$$dX = -(D_1F)^{-1} D_2F\,d\xi + o(\dots)$$
$$o(\|dX\| + \|d\xi\|) = o(\|d\xi\|).$$

c) o.k. by induction with
$$DX(\xi) = -[D_1F(X(\xi),\xi)]^{-1} D_2F(X(\xi),\xi)$$

$\phi : x \to x - Df(x)^{-1} f(x)$

$f : X \to Y$, $\left.\begin{array}{l} Df(x) : X \to Y \\ Df(x)^{-1} : Y \to X \end{array}\right\} \quad \phi : X \to X$

$D\phi(x) = -[D\{\underbrace{\underbrace{Df(x)^{-1}}_{\in \mathcal{L}(Y,X)}\}}_{\in \mathcal{L}(X,\mathcal{L}(Y,X))}] f(x)$

$D\phi(x)(\xi) = -\{[D\{Df(x)^{-1}\}(\xi)\}(f(x))$

$I : A \to A^{-1} \qquad A \in \mathcal{L}(X,Y)$

$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + O(\|B\|^2)$

$[DI(A)]I(B) = -A^{-1}BA \in \mathcal{L}(Y,X)$

$D(\underbrace{I \circ Df}_{X \to \mathcal{L}(Y,X)}) = DI \circ D^2 f \quad : X \to \mathcal{L}(Y,X)$

$D\phi(x)(\xi) = Df^{-1}(D^2 f(\xi, Df^{-1} f))$

$\| \quad \| \le \|D^2 f\| \, \|Df^{-1}\|$

Theorem

Inverse Function theorem

> Let f be a ~~contie~~ continuously differentiable mapping defined on a neighborhood of $x_0 \in X$ with values in Y.
> Assume $Df(x_0)$ is invertible. Then f maps any sufficiently small nbourhood of x_0 homöomorphically onto a nbhd of $f(x_0)$; the inverse homöom. is continously differentiable (C^n if f is) and
>
> $$df^{-1}(f(x)) = Df(x)^{-1}$$

Γ Apply impliz. func. thm. to $(x,y) \mapsto f(x) - y$ near $(x_0, f(x_0))$ and solve for x as a function of y $\lrcorner$

II Local behaviour of differentiable mappings near a fixed point

Assume f at least C^1 : $f : E \to E$ Banachspace

$f(x_0) = x_0$

We do not assume $Df(x_0)$ is invertible.

Near x_0 : $f(x) \approx x_0 + Df(x_0)(x - x_0)$

We can have one nice situation: f is exactly $Df(x_0)$ expressed in non linear coordinates, i.e. there exists

$\phi : U(0) \to U(x_0)$ invertible

$\phi(0) = x_0$

and $f(x) = \phi \circ Df(x_0) \circ \phi^{-1}$ near x_0

We say then ϕ is a linearisation of f. If ϕ is a homöomorphism we speak of topological linearisation

- Topological linearisation usually exist
- Smooth linearisation: Among C^∞ maps C^∞ linearisations exist for most f's

Invariant manifolds

If f admits a smooth linearisation $f(x) = \phi \circ Df(x) \circ \phi^{-1}$, $f \in C^1$ and if E_1 is an invariant subspace for $Df(x_0)$, then ϕE_1 is a locally invariant manifold for f

Linear stability analysis

Def x_0 is an attracting fixed point for f, if there is a nbhd U of x_0 s.t. $f^n U \to \{x_0\}$ in the sense that $f^n U \subseteq V$ for every V nbhd of x_0 and suff. large n.

We can always assume $fU \subseteq U$: take $\bigcap_{n=0}^{\infty} f^n U$

x_0 is a assymptotically stable fixed point
For Hamiltonian systems we can't have a. stable Fixed points.

Proposition: $f \in C^1$ $f(x_0) = x_0$

(1) $\rho(Df(x_0)) < 1 \Rightarrow x_0$ is attraction
(2) If $Df(x_0)$ is compact and $\rho(Df(x_0)) > 1$ then x_0 is not attracting

Γ (1) choose a norm so that $\|Df(x_0)\| < 1$
choose k $\|Df(x_0)\| < k < 1$
choose δ $\|Df(x)\| \leq k$ for $\|x - x_0\| \leq \delta$

$U := \{x \mid \|x - x_0\| \leq \delta$

$f^n(U) \subset \{x \mid \|x - x_0\| \leq k^n \delta\}$

Linear stability analysis for flows

$$\frac{dx}{dt} = X(x)$$

A fixed point is a stationary solution $x_0 : X(x_0) = 0$
attracting, if $\exists U$ s.t. $f^t U \subset V$ for all V all suff. large t
$\Longrightarrow$ $(f^1)^n U \subseteq V$ for suff. large n

$\{f^t : 0 \leq t \leq 1\}$ is equicontinuous

$\Longleftarrow$ x_0 is an attracting fixed pt. for f^1

First Variational Equation:

$V(t) := (D_x f^t)$ ~~...~~

$$\left.\begin{aligned} \frac{dV}{dt} &= D_x X(f^t(x))\, V(t) \\ V(0) &= 1\!\!1 \end{aligned}\right\} \text{determine } V(t)$$

$f^t : E \to E$
$D_x f^t : \in \mathcal{L}(E, E)$ $(D_x f^t) x \in E$
$\frac{dV}{dt} = \lim_{h \to 0} \frac{(D_x f^{t+h})(x) - D_x f^t(x)}{h}$
$= \lim D_x \lim_{h \to 0} \frac{f^{t+h}(x) - f^t(x)}{h}$
$= D_x X(f^t(x))$.

$f^t(x) = x_0$, $V(t) = e^{t D_x X(x_0)}$

x_0 is attractive, if the spectrum of $DX(x_0)$ is in the open left half plane

Def x_0 is hyperbolic fixed point, if $D\cancel{f}(x_0)$ is a hyperbolic operator
$E = E_s \oplus E_u$

There exists one invariant manifold tangent to E_s named stable manifold

Notation conventions:

- Put fixed point at the origin
- $E = E_- + E_+ \ni (x, y)$
- Λ_- restriction $Df(v)$ to E_-
  Λ_+ restriction $Df(v)$ to E_+
- $\varrho_- := \varrho(\Lambda_-)$
  $\varrho_+ := 1/\varrho(\Lambda_+^{-1})$
- choose norm s.t. $\|\Lambda_-\|, \|\Lambda_+^{-1}\| < 1$
  $\|\Lambda_-\| < 1$ if $\varrho_- < 1$ possible

Def A gap in the modulus of the spectrum of $Df(x_0)$
$(\varrho_-, \varrho_+) \subset (0, \infty)$:
$\sigma(Df(x_0)) \supset \{\lambda \mid \varrho_- < |\lambda| < \varrho_+\}$
$\sigma(Df(x_0)) \not\supset \{\lambda \mid |\lambda| = \varrho_\pm\}$

$Df(x_0)$ hyperb. $\Leftrightarrow$ 1 is in the gap

$\varrho_- < 1 < \varrho_+$: stable manifold
$\varrho_- < \varrho_+ \leq 1$ strong stable manifold
$\varrho_- = 1$ center stable manifold

$E = E' \oplus E''$

$\|(x, y)\| = \max \|x\|, \|y\|)$

$f(x,y) = (\Lambda_- x + f_-(x,y), \Lambda_+ y + f_+(x,y))$ $f^\mp$ are C^1 and vanish together with first derivatives at $(0,0)$

Proposition

Existence of stable and strong stable manifold

Assume $\varrho_- < 1$. For $\varepsilon > 0$ sufficiently small, there exist exactly one

$$\omega_- : \{x \in E_- \mid \|x\| \leq \varepsilon\} \to E_+$$

so that

1) The graph of ω_- is mapped into itself by f
2) $\omega_-(0) = 0$
3) $\|\omega_-(x_1) - \omega_-(x_0)\| \leq \|x_1 - x_0\|$

Step 1 By taking ε small, we make the first derivatives of $f_\mp$ as small as desired on ball of radius ε

Replace $f(x,y)$ by $\frac{1}{\varepsilon} f(\varepsilon \cdot x, \varepsilon \cdot y)$ w.l.o.g.: the on unit ball as small as I like

Reexpress invariance of the graph of $\omega : x \to \omega(x)$, $E_- \to E_+$

not assuming Df invertible but Λ_+ is invertible

$$f(x, \omega(x)) = (\Lambda_- x + f_-(x, \omega(x)), \Lambda_+ \omega(x) + f_+(x, \omega(x)))$$

$$\Lambda_+ \omega(x) + f_+(x, \omega(x)) = \omega(\Lambda_- x + f_-(x, \omega(x)))$$

$$\omega(x) = \Lambda_+^{-1} \{\omega(\Lambda_- x + f_-(x, \omega(x))) - f_+(x, \omega(x))\}$$
$$=: (F\omega)(x)$$

Seek now for a fixed point of F

$$(F\omega)(x) = \Lambda_+^{-1}\{\omega(\Lambda_- x + f_-) - f_+\}$$

$\psi_1, \psi_2 \in C[0,1]^3$
$T\psi = f(\psi(x))$
$\psi_1 \sim \psi_2 \Rightarrow T\psi_1 \simeq T\psi_2$
$\|T\psi_1 - T\psi_2\| = 2$ unless $\psi_1 = \psi_2$
$\to$ Reason why different compos. operator is not differentiable

Idea: consider this operator acting on a space of Lipschitz continuous functions, with (essentially) sup-norm.

$$(X_1, \varrho_1) \xrightarrow{f} (X_2, \varrho_2)$$

$\mathrm{Lip}(f) =$ smallest k so that $\varrho_2(f(x_1), f(x_2)) \leq k\, \varrho_1(x_1, x_2)$

"Lipschitz norm"

Proposition $\boxed{\begin{array}{l}(X_1,\rho_1)\xrightarrow{f}(X_2,\rho_2)\xrightarrow{g}(X_3,\rho_3)\\ Lip(g\circ f)\le Lip(f)\cdot Lip(g)\end{array}}$

Γ $\rho_3(g\circ f(x_1), g\circ f(x_2)) \le Lip(g)\, \rho_2(f(x_1), f(x_2)) \le Lip(g)\cdot Lip(f)\, \rho_1(x_1,x_2)$ $\lrcorner$

Proposition

E_+, E_- Banach spaces $\Lambda_+ : E_+ \to E_+$ inverting
$\Lambda_- : E_- \to E_-$; $\|\Lambda_-\| < 1$

$\|\Lambda_+^{-1}\| \, \|\Lambda_-\| < 1 \implies f_\pm$ is Lipschitz cont., vanish at the orig.
and if $Lip\, f_\pm$ small enough
there is exactly one
ω_- from ~~$B(0,1)$~~ unitball in $E_- \to$ unitball in E_+
$\omega_-(0)=0$
$Lip(\omega_-)\le 1$
f maps the graph of ω_- into itself

Γ $X := \{\omega$: unitball of E_- into E_+ with $\omega(0)=0$
$Lip(\omega)\le 1\}$

Metric : $\rho(\omega_1,\omega_2) = \sup_x \left\{ \frac{\|\omega_1(x)-\omega_2(x)\|}{\|x\|} \right\} \ (\le 2)$

is weaker than lip norm; but X is complete w.r.t ρ

Smallness conditions :

$Lip(f_\pm) \le \theta$ (θ will be determined later)

1. If $\omega \in X$, $F\omega$ is defined on the unit ball

$lip\{x \longmapsto \Lambda_- x + f_-(x,\omega(x))\} \le \|\Lambda_-\| + \theta$

(1) $\boxed{\|\Lambda_-\| + \theta \le 1}$ 1. condition on θ

$\|x\|\le 1 \Rightarrow \|\Lambda_- x + f_-(x,\omega(x))\| \le 1$

$\Rightarrow Lip(F\omega) \le 1$

2. $(F\omega)(0) = 0$

$Lip(F\omega) \le \|\Lambda_+^{-1}\| \{ (\|\Lambda_-\| + \theta) + \theta \}$

(2) $\boxed{\|\Lambda_+^{-1}\| (\|\Lambda_-\| + 2\theta) \le 1}$ 2. smallness condition for θ

$x_-^{(1)} := \Lambda_- x + f_-(x, \omega_1(x))$
$x_-^{(2)} := \quad \cdot \quad \cdot \quad \cdot \quad \omega_2(x)$

3. contraction : $\omega_1, \omega_2 \in X$

show : $(F\omega_1)(x) - (F\omega_2)(x) = \Lambda_+^{-1}\{ \omega_1(x_-^{(1)}) - \omega_2(x_-^{(2)})$
~~$+\omega_2(x_-^{(1)}) - \omega_2(x_-^{(2)})$~~
$+ (f_+^{(1)} - f_+^{(2)}) \}$

$\|F\omega_1(x) - F(\omega_2)(x)\| \le$

$\le \|\Lambda_+^{-1}\| \{ \rho(\omega_1,\omega_2)(\|\Lambda_-\| + \theta) + 2\theta\rho(\omega_1,\omega_2) \} \|x\|$

$\|f_-(x,\omega_1(x)) - f_-(x,\omega_2(x))\|$
$\le \theta\|\omega_1(x) - \omega_2(x)\| \le \theta\cdot\rho(\omega_1,\omega_2)\|x\|$

$$\Longrightarrow \quad \frac{\| \mathcal{F}\omega_1(x) - \mathcal{F}\omega_2(x) \|}{\|x\|} \le \|\Lambda_+^{-1}\| (\|\Lambda_-\| + 3\theta) \, \rho(\omega_1, \omega_2)$$

$$\rho(\mathcal{F}(\omega_1(x)), \mathcal{F}\omega_2) \le \quad " \quad "$$

(3) $\boxed{\|\Lambda_+^{-1}\| (\|\Lambda_-\| + 3\theta) < 1}$ $\lrcorner$

Further trick more general $\|\Lambda_-\| \nless 1$ (finite dim-case)

If $\|\Lambda_-\| \nless 1$ than, multiply non-linear terms by a cut off function to make it than globally small in Lipschitz norm

Take $\psi(x,y)$ a Lipschitz continuous function $\equiv 1$ on a nbhd of (0 $\equiv 0$ o. for $|(x,y)| \ge 1$

Compose norm with appropriate smooth f on $\mathbb{R}$

$$\tilde{f}(x,y) = (\Lambda_- x + \psi(x,y) f_-(x,y) + \psi(x,y) f_+(x,y))$$

and $\tilde{f} = f$ on a small enough nbh. of $(0,0)$

Apply arguments above working on all of E_- rather than unit ball. show that $\mathcal{F}$ admits a unique globally defined ω whose graph is mapped into itself by $\tilde{f}$.

The graph of ω is locally invariant for f

Def X is X is locally invariant for f at z_0, if there is a nbh. $U(z_0)$ of z_0 so that $\forall z \in U \cap X \Rightarrow f(z) \in X$

The general results are much less good than those for $\|\Lambda_-\| < 1$:

1) In the infinit dimens. situation, smooth cut off functions need not exist
(you can't hope to prove that the inv. manifold is smooth if they don't)
Can arrive : non linear terms small in Lipschitz norm on $\{(x,y) : \|y\| \le \|x\|\}$
you need only a cutoff function in E_-. If E_- is finite dim then the proof goes through.

2) If you can make a C^1 cut-off, than ω is C^1 (even if $\|\Lambda_-\| \nless 1$)
If $\|\Lambda_-\| < 1$ and $f \in C^r$ than ω is C^r. If $\|\Lambda_-\| \ge 1$, you also need $\|\Lambda_-\|^r \|\Lambda_+^{-1}\| < 1$
r good norms.

3) Uniqueness fails in the case $\rho_- \ge 1$ situation you change ω by changing the cut off even in a neigh.h. of $(0,0)$.

Smoothness questions

New language

Rewrite the functional equation as

$$\boxed{\begin{aligned} &\omega(x) = F(x, \omega(x), \omega(v(x, \omega(x))) \\ &F(x, y, y') = \Lambda_+^{-1} (y' - f_+(x,y)) \\ &v(x, \omega(x)) = \Lambda_- x + f_-(x, \omega(x)) \end{aligned}}$$

Def. $\boxed{\begin{aligned} Lip_i F &= \text{Lip norm of } F \text{ w.r.t. } i^{th} \text{ variable} \\ &= \text{sup norm of the } i^{th} \text{ partial derivative}\end{aligned}}$

$$\boxed{\begin{aligned} Lip_3(F) &\lessapprox \|\Lambda_+^{-1}\| = d_+^{-1} \\ Lip_1(v) &\lessapprox \|\Lambda_-\| = d_- \end{aligned}}$$ everything else is of $O(Lip\, f_1)$

$v(0,0)=0$
$F(0,0,0)=0$

Study functional equal. $\boxed{\begin{aligned} \omega(x) &= F(x, \omega(x), \omega(v(x,\omega(x)))) \\ &= (\mathcal{F}\omega)(x)\end{aligned}}$

$\omega(0)=0 \Rightarrow \mathcal{F}\omega(0)=0$

$$Lip(\mathcal{F}\omega) \leq Lip_1(F) + Lip_2(F)\, Lip(\omega) + Lip_3(F) \cdot Lip\, v$$

assuming $Lip\, \omega < 1$

$$\begin{aligned} &Lip\,(x \mapsto v(x,\omega)) \\ &\leq Lip_1(v) + Lip_2(v) \cdot Lip(\omega) \\ &\leq Lip_1(v) + Lip_2(v) \\ &=: Lip(v) \end{aligned}$$

$Lip\, \mathcal{F}\omega$

$\Rightarrow$ we need

$$\boxed{Lip_1(F) + Lip_2(F) + Lip_3(F)\, Lip\, v \leq 1}$$

To have contractivity

$$\boxed{Lip\, v \equiv Lip_1(v) + Lip_2(v)}$$

$$\begin{aligned} (\mathcal{F}\omega_1)(x) - (\mathcal{F}\omega_2)(x) = \; & F(x, \omega_1, \omega_1(v(x,\omega_1(x)))) - F(x, \omega_2, \omega_1(v_1)) \\ & + F(x, \omega_2, \omega_1(v_1)) - F(x, \omega_2, \omega_2(v_1)) \quad \text{obvious shortcuts} \\ & + F(x, \omega_2, \omega_2(v_1)) - F(x, \omega_2, \omega_2(v_2)) \\ \leq \; & Lip_2(F)\, \|\omega_1 - \omega_2\|\, \|x\| + Lip_3(F)\, \|\omega_1 - \omega_2\|\, \|v_1\| \\ & + Lip_3(F) \cdot Lip(\omega_2)\, \underbrace{\|v_1 - v_2\|}_{\leq^{*} Lip_2(v) \cdot \|\omega_1 - \omega_2\|\, \|x\|} \end{aligned}$$

$*$ $\boxed{\begin{aligned} &\|v_1\| \leq Lip(v) \cdot \|x\| \\ &\| \\ &v(x, \omega_1(x)) \end{aligned}}$

$$\leq \underbrace{\{Lip_2(F) + Lip_3(F) \cdot Lip(v) + Lip_3(F)\, Lip_2(v)\}}_{L} \|\omega_1 - \omega_2\|\, \|x\|$$

$\mathcal{F}$ contractive provided $L < 1$

$$\boxed{Lip_2(F) + Lip_3(F)\, Lip(v) + Lip_3(F)\, Lip_2(v) < 1} \qquad \boxed{D_1 F(0,0,0) = 0}$$

Differentiability

1. If ω is differentiable we can differentiate the functional equation

$$\begin{aligned} D\omega &= D_1 F + D_2 F\, D\omega + D_3 F\, D\omega(v) [D_1 v + D_2 v\, D\omega] \\ \sigma &= D_1 F + D_2 F \sigma + D_3 F\, \sigma(v)\, [D_1 v + D_2 v\, \sigma] \\ &= K\sigma \end{aligned}$$

Show that K is contractive on an appropriate fn. space

2. Show using the fnl equations that the unique fixed point does have the defining property of $D\omega$

Assume $F \in C^2$ claim: ω is C^2

$D\omega = D_1 F + D_2 F \, \overset{\sigma}{D\omega} + D_3 F \, [D_1 v + D_2 v \, D\omega] \, \sigma(v(x))$

$\times \boxed{\sigma = \bar{F}(x, \sigma, \sigma(\bar{v}))}$ $\quad \bar{v}(x) = v(x, \omega(x))$ (not depending on ω)

$\bar{F}(x, \sigma, \sigma') = D_1 F + D_2 F \sigma^* + D_3 F \, [D_1 v + D_2 v \, \sigma] \, \sigma'$ (2)

Differences with the situation before (1)

In (2) v depends only on x
in (1) we have a bound $\| D\omega \| < 1$
and we have no such bound for $D\sigma$

Look at the functional equation by differentiating $\times$

$\sigma^{(2)} = D_1 \bar{F} + D_2 \bar{F} \sigma^{(2)} + D_3 \bar{F} \, \sigma^{(2)}(\bar{v}) \, D\bar{v}$

This is a linear equation for $\sigma^{(2)}$ (in sup norm)

r.h.s. is a contraction in the sup norm, provided

~~$\| Lip_3 \bar{F} \|$~~ :

$Lip_2 \bar{F} + Lip_3 \bar{F} \, Lip(\bar{v}) < 1$

$Lip_2 \bar{F} \le Lip_2 F + Lip_3 (F) \, Lip_2 (v)$
$Lip_3 \bar{F} \le Lip_3(F) \cdot Lip(v)$
$Lip(\bar{v}) \le Lip \, v$

$\boxed{Lip_2 (F) + Lip_3 F \, Lip(v)^2 + Lip_3(F) \, Lip_2(v) < 1}$ (**)

To proof the differentiability we remark first:

The estimates used $\| \sigma \| \le 1$ aren't used.
Every term involving norm of σ also had a factor of $Lip_2(v)$
which is zero in the present case.
Estimates needed here was the same as that needed for contractivity.

Similarly: If $D^2 F$ is α Holder cont. and if
~~$Lip_2 F + Lip$ (**)~~

$\boxed{Lip_2(F) + Lip_3 F \, Lip(v)^{2+\alpha} + Lip_3 F \, Lip_2 v < 1}$

than $D^2\omega$ is α H.c.

Proposition: Lanford

$\boxed{\begin{array}{l} \text{Assume } F, v \text{ of class } C^r \quad 1 \le r \le \infty \text{ and} \\ Lip_1(F) + Lip_2(F) + Lip_3(F) \, Lip \, v \le 1 \\ Lip_2(F) + Lip_3(F) \, Lip(v) + Lip_3(F) \, Lip_2(v) < 1 \\ Lip_2(F) + Lip_3(F) (Lip(v))^r + Lip_3(F) \, Lip_2(v) < 1 \\ \quad \text{(for } r = \infty \text{ assume } Lip(v) < 1\text{)} \\ \text{than the functional equation has a} \\ \text{unique } C^r \text{ solution} \end{array}}$

Idea: Show that for an appropriately large B, K preserves the condition that the Hölder norm $\leq B$

Restrict the fn space on which K acts to the space of fn's satisfying Höldercond. with constants B; Show K maps the restricted space into itself, the restricted space is still complete in original norm. K has a fixed point in the restricted space.

and the unique fixed point is in the restr. space.

Name: $Hol_\alpha(g) := \inf\{k \mid \|g(z_1)-g(z_2)\| \leq k\cdot\|z_1-z_2\|^\alpha\}$

given σ which is α Höld. cont. Want to estimate $Hol_\alpha(K\sigma)$.

$$
\begin{aligned}
(K\sigma)(x_1)-(K\sigma)(x_2) &= (D_1F)_1 - (D_1F)_2 \\
&\quad + \overset{*}{\left((D_2F)_1-(D_2F)_2\right)}\sigma_1 + D_2F_2(\sigma_1-\sigma_2) \\
&\quad + (D_3F_1 - D_3\overset{*}{F_2})\,\sigma(v_1)\,[\ldots] \\
&\quad + D_3F_2[\sigma(v_1)-\sigma(v_2)]\,[\ldots] \\
&\quad + D_3F_2\,\sigma(v_2)\Big[(D_1v_1 - \overset{*}{D_1v_2}) + D_2v_1 - \overset{*}{D_2v_2})\sigma_1 \\
&\qquad + D_2v_2(\sigma_1-\sigma_2)\Big]
\end{aligned}
$$

for the $*$ Terms, can get an estimate of the form $A\,\|x_1-x_2\|^\alpha$ where A doesn't depend on σ...

$$
\|(K\sigma)x_1 - (K\sigma)(x_2)\| \leq A\,\|x_1-x_2\|^\alpha + Hol_\alpha(\sigma)\,\|x_1-x_2\|^\alpha \underbrace{\{Lip_2F + Lip_3F(Lip\,v)^\alpha + Lip\,(v) + Lip_3F\cdot Lip_2(v)\}}_{\{\#\}}
$$

$$Hol_\alpha(K\sigma) \leq A + \{\#\}\,Hol_\alpha(\sigma)$$

if $\{\#\} < 1 \Rightarrow \{\sigma : Hol_\alpha(\sigma) \leq \frac{A}{1-\{\#\}}\}$ is preserved

$$\boxed{Lip_2(F) + Lip_3(F)\,(Lip\,v)^{1+\alpha} + Lip_3(F)\,Lip_2(v) < 1}$$

If $Lip\,v > 1$ this condition is more restrictive and the lack of smoothness begins to bite.

In the case $F \in C^r$ replace $1+\alpha$ by $1+r$

Assume $F \in C^2$ claim: ω is C^2

$D\omega = D_1 F + D_2 F \, D\omega + D_3 F \left[D_1 v + D_2 v \, D\omega \right] \sigma(v(x))$

$\ast \boxed{\sigma = \bar{F}(x, \sigma, \sigma(\bar{v}))}$ $\bar{v}(x) = v(x, \omega(x))$ (not depending on ω)

$\bar{F}(x, \sigma, \sigma') = D_1 F + D_2 F \sigma^{*} + D_3 F \left[D_1 v + D_2 v \, \sigma \right] \sigma'$ (2)

Differences with the situation before (1)

in (2) v depends only on x
in (1) we have a bound $\| D\omega \| < 1$
and we have no such bound for $D\sigma$

Look at the functional equation by differentiating $\ast$

$\sigma^{(2)} = D_1 \bar{F} + D_2 \bar{F} \sigma^{(2)} + D_3 \bar{F} \, \sigma^{(2)}(\bar{v}) \, D\bar{v}$

This is a linear equation for $\sigma^{(2)}$ (in sup norm)

r.h.s. is a contraction in the sup norm, provided

~~$\| Lip_2 \bar{F} \|$~~ :

$Lip_2 \bar{F} + Lip_3 \bar{F} \, Lip(\bar{v}) < 1$

$Lip_2 \bar{F} \le Lip_2 F + Lip_3 (F) \, Lip_2 (v)$
$Lip_3 \bar{F} \le Lip_3 (F) \cdot Lip (v)$
$Lip (\bar{v}) \le Lip \, v$

$\boxed{Lip_2 (F) + Lip_3 F \, Lip(v)^2 + Lip_3 (F) \, Lip_2 (v) < 1}$ $(\ast\ast)$

To proof the differentiability we remark first:

The estimates used $\| \sigma \| \le 1$ can't used.
Every term involving norm of σ also had a factor of $Lip_2(v)$
which is zero in the present case.
Estimates needed here was the same as that needed for contractivity.

Similarly: If $D^2 F$ is α Holder cont. and if
~~$Lip_2 F + Lip$ $(\ast\ast)$~~

$\boxed{Lip_2(F) + Lip_3 F \, Lip(v)^{2+\alpha} + Lip_3 F \, Lip_2 v < 1}$

than $D^2 \omega$ is α H.c.

<u>Proposition:</u>
<u>Lanford</u>

$\boxed{\begin{array}{l} \text{Assume } F, v \text{ of class } C^r \quad 1 \le r \le \infty \text{ and} \\ Lip_1 (F) + Lip_2 (F) + Lip_3 (F) \, Lip \, v \le 1 \\ Lip_2 (F) + Lip_3 (F) \, Lip(v) + Lip_3 (F) \, Lip_2 (v) < 1 \\ Lip_2 (F) + Lip_3 (F) (Lip(v))^r + Lip_3 (F) \, Lip_2 (v) < 1 \\ \quad \text{(for } r = \infty \text{ assume } Lip(v) \le 1\text{)} \\ \text{than the functional equation has a} \\ \text{unique } C^r \text{ solution} \end{array}}$

other approaches to proving smoothness.

- Proof $\mathcal{F}$ is contractive in C^1 norm but have to assume f is C^2.
- Choose a clever metric equivalent to C^1 norm: it's enough to assume $f \in C^{1+\delta}$
- In finite dim. case

$$\omega_0,\ \mathcal{F}^n\omega_0 \longrightarrow \omega$$

get a uniform bound on derivatives of $\mathcal{F}^n\omega_0$
Arzela, ascoli thm, → subsequence
Need Df is uniformly continuous on nbhd of a fixed pt.

Intrinsic structure of this invariant manifolds

$$f : (x,y) \to (\Lambda_- x + f_-(x,y), \Lambda_+ x + f_+(x,y))$$

$$\cancel{\#}\ y = \omega(x)$$

$$\mathrm{Lip}(x \longmapsto \Lambda_- x + f_-(x,\omega(x))) \le \|\Lambda_- + \mathrm{Lip}(f_-)\|$$

suppose $\|\Lambda_- + \mathrm{Lip}(f_-)\|$ is less than 1

If we take an point $(x, y=\omega(x))$ on the manifold. and form

$$f^n(x,\omega(x)) = (x_n, \omega(x_n))$$

$$\|x_n\| \le (\|\Lambda_-\| + \mathrm{Lip} f_-)\|^n \|x\|_-$$

$$\|\omega(x_n)\| \le \|x_n\|$$

$$\|x_n, \omega(x_n)\| \le \|\Lambda_-\| + \mathrm{Lip}(f_-)\|^n \|x, \omega(x)\|$$

$$f(x,y) = (\Lambda_- x + f_-(x,y), \Lambda_+ y + f_+(x,y))$$

$$\|\Lambda_-\| > \|\Lambda_+^{-1}\|^{-1}$$

$\|\Lambda_-\| + \mathrm{Lip}(f_-) < 1$ supposed

$z = (x, \omega(x))$ a point on the manifold

$$\|f^n(z)\| \le [\|\Lambda_-\| + \mathrm{Lip}(f_-)]^n \|z\|$$

Near a hyperbolic fixed point z_0 of f are given two points z_1, z_2 such that $z_1 - z_2$ looks roughly in the expanding direction. Than we have

$\exists k > 1$ indipend. of z_1, z_2, so that

$$\|f(z_1) - f(z_2)\| \ge k\|z_1 - z_2\|$$

and $f(z_1) - f(z_2)$ lookes roughly in the expanding direction and as long as the orbits $f^n(z_1)$ and $f^n(z_2)$ remain near the fixed point,

$$\|f^n(z_1) - f^n(z_2)\| \ge k^n \|z_1 - z_2\|$$

so at least one these orbits must escape from the vicinity of the fixed point.

Theorem

Let f be C^r $1 \le r \le \infty$
z_0 fixed pt for f

(ρ_-, ρ_+) a gap in the modulus for the spectrum of $Df(z_0)$

$\mathcal{E} = \mathcal{E}_- \oplus \mathcal{E}_+$ the corresponding splitting of the state space

If $\rho_- < 1$ r finite assumed
If $\rho_- > 1$ $\rho_-^{\,r} < \rho_+$

1) There is a C^r manifold W_{loc}^- passing through z_0, tangent there to $\mathcal{E}_-$, locally invariant for f at z_0

2) If $\rho_- < 1$, then $W^{(-)}_{loc}$ can be taken to be mapped into itself by f. It is locally unique in the case that any locally invariant C^1 manifold passing through z_0 and tangent there to $\mathcal{E}_-$ must coincide near z_0.
If $z \in W_{loc}^{(-)}$ and z is near enough to z_0, then $f^n(z) \to z_0$ and in fact

$$\limsup_{n\to\infty} \frac{1}{n} \log \| f^n(z) - z_0 \| \le \log(\rho_-)$$

If U is any sufficiently small nbhd of z_0 and if $\rho_- \le r \le \rho_+$, then $W_{loc}^{(-)}$ coincides near z_0 with the set of z, so that

$$f^n(z) \in U$$
$$f^n(z) \to z_0$$
$$\limsup_{n\to\infty} \frac{1}{n} \log \| f^n(z) - z_0 \| < \log(r)$$

3) If $\rho_+ > 1$, $W_{loc}^{(-)}$ any invariant manifold U, small enough nbhd of z_0, then any z is $f^n(z) \in U$ for all n (and must therefore be in $W^{(-)}_{loc}$.

4) $\rho_- < 1 < \rho_+$ (hyperbolic case)
then for any sufficiently small nbhd U of z_0
$W_{loc}^{(-)}$ coincides near z_0 with $\{ z \mid f^n(z) \in U \ \forall n \}$

$\mathcal{E} = \mathcal{E}_- \oplus \mathcal{E}_+$

Invariant manifolds tangent to $\mathcal{E}_+$ exists, if
- $Df(z_0)$ is compact of finite dim
- f ~~not~~ smoothly invertible
- $\mathcal{E}_+$ finite dimensional

(b) If $z \in$ Unit ball $- \Gamma(\omega)$, then either some $f^n(z)$ is outside the unit ball or

$$\liminf_n \frac{1}{n} \log \|f^n(z)\| \geq \log(v_+)$$

$(\rho_- < v_- < v_+ < \rho_+)$

If $v_+ > 1$, then some $f^n(z)$ is outside the unit ball

c) If $z \notin \Gamma(\omega)$ but $f^n(z)$ nevertheless converge to 0, then

(can't happen, if $\rho_+ > 1$)

$$\liminf_n \frac{1}{n} \log \|f^n(z)\| \geq \log(\rho_+)$$

Proof: a) Can get an estimate like this with ρ_- replaced by v_-

$$\|f^n(z)\| \leq (v_-)^n \|z\|$$

$f^n(z)$ is eventually in any small ball about $(0,0)$

We can make $\mathrm{Lip}(f.)$ small by making the ball small

$$\limsup_{n \to \infty} \frac{1}{n} \log \|f^n(z)\| \leq \log \|A_+\|$$

independent of norm — minimize over norms

b) $z = (x, y)$ $z' = (x, \omega(x))$
(separation between z and z' is exactly vertical

As long as $f^n(z)$ and $f^n(z')$ stay in the unit ball, the separation between $f^n(z)$ and $f^n(z')$ is prov. vertical and

$$\|f^n(z) - f^n(z')\| \geq v_+{}^n \|z - z'\|$$

$f^n(z')$ does stay in the unit ball

Either some $f^n(z)$ is outside unit ball, or

$$\liminf_{n \to \infty} \frac{1}{n} \log \|f^n(z) - f^n(z')\|$$

$$\geq \liminf_{n \to \infty} \frac{1}{n} \log \|f^n(z')\| \geq \log v_+$$

b) $\Rightarrow$ c) just like a). $\lrcorner$

Notations: - X locally invariant for f at z_0, if $\exists$ nbh U of z_0 so that $z \in U \cap X \Rightarrow f(z) \in X$

- X_1, X_2 coincide near z_0, if $\exists$ nbhd U of z_0 $X_1 \cap U = X_2 \cap U$

Theorem

Let f be C^r $1 \le r \le \infty$
z_0 fixed pt for f

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If $z \in W^{(-)}_{loc}$ and z is near enough to z_0, then $f^n(z) \to z_0$ and in fact

$$\limsup_{n\to\infty} \log \frac{1}{n} \| f^n(z) - z_0 \| \le \log(\rho_-)$$

If U is any sufficiently small nbhd of z_0 and if $\rho_- \le r \le \rho_+$, then $W^{(-)}_{loc}$ coincides near z_0 with the set of z, so that

$$f^n(z) \in U$$
$$f^n(z) \to z_0$$
$$\limsup_{n\to\infty} \frac{1}{n} \log \| f^n(z) - z_0 \| < \log(r)$$

3) If $\rho_+ > 1$, $W^{(-)}_{loc}$ any invariant manifold U, small among nbhd of z_0, then any z is $f^n(z) \in U$ for all n (and must therefore be in $W^{(-)}_{loc}$.

4) $\rho_- < 1 < \rho_+$ (hyperbolic case)
then for any sufficiently small nbhd U of z_0 $W^{(-)}_{loc}$ coincides near z_0 with $\{ z \mid f^n(z) \in U \ \forall n \}$

$\mathcal{E} = \mathcal{E}_- \oplus \mathcal{E}_+$

Invariant manifolds tangent to $\mathcal{E}_+$ exists, if
- $Df(z_0)$ is compact of finite dim
- f ~~not~~ smoothly invertible
- $\mathcal{E}_+$ finite dimensional

Example: $(x,y) \overset{f}{\longmapsto} (\lambda x, \lambda^2 y + x^2)$

$Df(0) = \begin{pmatrix} \lambda & \\ & \lambda^2 \end{pmatrix}$

$g_- = \lambda \quad g_+ = \lambda^2$

There exists invariant manifold tangent to $(1,0)$ $g_-^{\,r} < g_+$ for $r < 2$

Claim: there is no C^2-invariant manifold

that means, no ~~C^2 invariant manifold~~ ~~a~~ manifold invariant as a graph of a C^2 fkt.

$(x, \omega(x)) \overset{f}{\longmapsto} (\lambda x, \lambda^2 \omega(x) + x^2)$

$\boxed{\omega(\lambda x) = \lambda^2 \omega(x) + x^2}$ functional equation

$\omega''(\lambda x)\lambda^2 = \lambda^2 \omega''(x) + 2$

$x = 0 \quad \lambda^2 \omega''(0) = \lambda^2 \omega''(0) + 2 \quad \lightning$

Remark: in the case $\lambda \underset{\neq}{\cong} \lambda^2$ there exists an inv. manifold C^2

in the same way for arbitrary n

$(x,y) \mapsto (\lambda x, \lambda^n y + x^n)$

$\omega^{(n)}(\lambda x)\lambda^n = \lambda^n \omega^{(n)}(x) + n!$

$\lightning$ in $x = 0$

The condition $g_-^{\,r} < g_+$ is an analytically invariant condition. 'Real' conditions: absence of certain algebraic dependence amoung eigenvalues.

Invariant manifolds tangent to E_+ (without assuming f invertible): Need to chang definition of incidence

consider case where $f_\pm \equiv 0$

Diskussion of the linear case

Λ_- has a non trivial kernel.

E_- generalizes stab manifold satisfies $f^{-1}E_- = E_-$

characterize $\ast$ by $f^{-1}E_- \supset E_- \Leftrightarrow fE_- \subset E_-$

It needs not be true, that $fE_- \supset E_-$ if f is not invertible

E_+ satisfies $fE_+ \supset E_+$.

characterize $\ast$ by $f^{-1} E_+ \supset E_+$

need not satisfie $f^{-1}E_+ \subset E_+$ because Nullspace of f in E_- is obtained in $f^{-1}E_+$

Diskussion of the unstable manifold

Translate $fW \supset W$ in the functional equation.

$\omega : E_+ \longrightarrow E_-$ defined on the unit ball

$f(x,y) = (\Lambda x + f_-(x,y), \Lambda_+ y + f_+(x,y))$

$z = (\omega(y), y)$ on the graph when is $z = f(\bar{z})$, $\bar{z} = (\omega\bar{y}, \bar{y})$?

(a) $\Lambda_+ \bar{y} + f_+(\omega(\bar{y}), \bar{y}) = y$

(b) $\Lambda_- \omega(\bar{y}) + f_-(\omega(\bar{y}), \bar{y}) = \omega(y)$

Treat these as follows

Show that (a) regarded as an equ. for $\bar{y}$ has a unique solution

$$\bar{y} = \Lambda_+^{-1} y - \Lambda_+^{-1} f_+(w(\bar{y}), \bar{y}) \qquad \text{Contraction etc.}$$

Call the solution $v(y, w)$

$$z \in \text{graph}(w) \implies z = f(\bar{z})$$

$$\Updownarrow$$

$$w(y) = \Lambda_- w(v(y,w)) + f_-(w(v(y,w)), v(y,w))$$

$f(\text{graph}\, w) \supseteq \text{graph}$

$\iff$ w satisfies $\boxed{w(y) = F(w(v)), v)}$

Do the same for as in the stable manif. case.

The conditions for C^r invariant manifold : (Stable manif. $\varrho_-^r < \varrho_+$ Bites if $\varrho_- \geq 1$) good case $\varrho_- < 1$

here $\varrho_- < \varrho_+^r$ bites for $\varrho_+ < 1$ good case $\varrho_+ > 1$

(If we have $\varrho_+ > 1$ and $\varrho_- < 1$ then hyperb. case)

unstable maifold	stable manifold
$\varrho_- < \varrho_+^r$	$\varrho_-^r < \varrho_+$
Bites if $\varrho_+ < 1$	Bites if $\varrho > 1$
good case $\varrho_+ > 1$	good case $\varrho_- < 1$
if $r = \infty$ need $\varrho_+ > 1$	if $r = \infty$, need $\varrho_- < 1$

What becomes with the contractive and expansiv wedges arguments in this situations?

Given z, a backward orbit (not exist not unique) for z means $z_{-1}, z_{-2}, \dots z_{-n}$ so that $f(z_{-1}) = z$, $f(z_{-j}) = z_{-j-1}$ $j \geq 2$

May not exist, and when existing need not be unique.

- If $z \in \text{graph}(w)^r$, and $\varrho_+ > 1$ then z has an infinite backward orbit in the graph(w). It's unique

$$\limsup_{n\to\infty} \frac{1}{n} \log \| z_{-n} \| \leq -\log(\varrho_+)$$

- If $z \notin \text{graph}(w)$, then z admits <u>no</u> infinite backward orbit in the unit ball with

$$\liminf_{n\to\infty} \frac{1}{n} \log \| z_{-n} \| \leq -\log(r_-)$$

- If $r_- < 1$, z admits <u>no</u> infinite backward orbit in the unit ball.

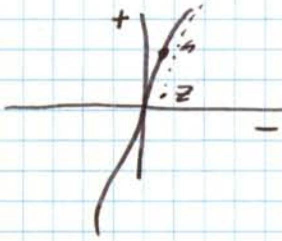
· Assume $r_- < 1$ (not $r_+ > 1$); let z be in the unit ball, assume also, that $f(z), f^2(z), \cdots f^n(z)$ are all in unit ball, then

$$\mathrm{dist}(f^n(z), \mathrm{graph}(\omega)) \le 2 r_-^n$$

If $f^n(z)$ remains in the unit ball for all n, then

$$f^n(z) \to \text{graph of } \omega$$

"unstable manifold is a ~~contra~~ attractor"



z very near $z_0 \Rightarrow f^n(z)$ stays to a long time ~~red~~ in the unit ball.
not on the stable manifold. $f^n(z)$ eventually gets out

Remarks: Invariant manifolds for flows.

$\dot z = X(z)$ with solution flow f^t
0 fixed point of f^t : $X(0) = 0$.

The proper analog of a gap in the modulus of the spectrum is a band σ_-, σ_+

so that the spectrum $DX(0)$ doesn't intersect

$\{\sigma_- < \mathrm{Re}(\lambda) < \sigma_+\}$

but does intersect both vertical lines $\mathrm{Re}(\lambda) = \sigma_\pm$

$E = E_- \oplus E_+$ splitting, where E_- corresponds to the spectrum in $\{\mathrm{Re}\{\lambda\} \le \sigma_-\}$, because

$Df^t(0) = e^{t DX(0)}$

$(\rho_- = e^{t\sigma_-}, \rho_+ = e^{t\sigma_+})$ is a gap for f^t

f^t, t fixed, has a locally invariant manifolds tangent to E_- and E_+. (not necessarily unique)

Question: Do there exist such ~~man~~ locally invariant for all f^t simultaneously.

First case: $\sigma_- < 0$ $e^{t\sigma_-} < 1$: good case
We know that stable manifold for f^1 is locally unique
Because the f^t commute, $f^t \omega_1$ is another inv. manif for f^1 invariant manifold tangent to E_-.
So, $f^t \omega_1 = \omega_1$ near fixed point. true for all t.

In general case: Constructed ω is got by cutting off the nonlinear terms in f outside a small nbhd of 0.
The cut off f has a unique globally invariant manifold

trick: Instead of cutting off the non linear terms in f^t's cut of the nonlinear terms in $X(x)$.

$X(x,y) = (A_- x + X_-(x,y), A_+ y + X_+(x,y))$
$\tilde X(x,y) = (A_- x + \psi(x,y) X_-, A_+ y + \psi(x,y) X_+(x,y))$

$\tilde f^t$: flow generated by $\tilde X$
$\tilde f^t$ has uniformly small non linear terms

$\tilde f^t$ has a unique invariant set having the form of the graph of a fn with Lip norm ≤ 1. $\tilde W$

Can I conclude, that $\tilde f^t \tilde W = \tilde W$?

$\tilde f^t = f^t$ near enough to $(0,0)$

By the group property it's enough to prove this for small t. For small t, $\tilde f^t$ is uniformly C^1 near the identity.

$Lip(\tilde f^t \tilde W) \lesssim Lip(\tilde W)$

Going back to the proof of exist. of $\tilde W$: show that, by putting more smallness conditions in non linear terms, cann guarantee $Lip(\tilde W) \le 1/2$.

Uniqueness still holds in class with $Lip \le 1$
uniqueness $\longrightarrow$ $f^t \tilde W = \tilde W$.

Good case: Can ~~arrange~~ make W = graph w (w defined on ball of radius ε in a well chosen norm)

$fW \subset W$

For flows, should make W so that $f^t W \subset W$ for all $t \ge 0$.
Can do this by choosing a norm, so that

$$\| e^{tA_-} \| \le e^{ts} \quad s<0 \quad \forall t \ge 0$$

In the bad case, the argument given doesn't work very well for partial differential evolution equs:

f^t is <u>not</u> uniformly near id for small positive t

Maybe: f^t uniformly near $e^{-tDX(0)}$

or : p. 48 Marsden Mc Cracken.

What happens if f is real analytic?

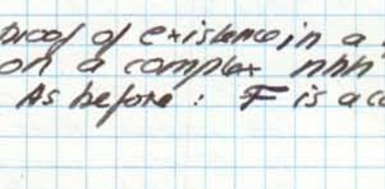
Good case: $\rho_- < 1$

Make the fundamental proof of existence in a space of complex analytic w's on a complex nbh'd of 0 and the same norm as before. As before: $\mathcal{F}$ is a contraction.

In the bad case $\rho_- > 1$

cutting off spoils analyticity.

Center manifolds

$\sigma(Df(z_0))$ looks like 

$\mathcal{E}_s \oplus \mathcal{E}_c \oplus \mathcal{E}_u$
inside on outside

An invariant manifold tangent to E_c is called the centre manifold.

By the unstable manif. theory, $\exists$ invar. manifold tangent to $E_c \oplus E_u$: center unstable manifold W_{cu}

Restrict f to W^{cu}.

$\sigma(Df \restriction W^{cu})$ = part of the spectrum outside and on the unit circle.

Apply stable manifold theorem to get an invariant manifold tangent to E_c.

Reminder : Need smooth cut-off fn's
Preserve arbitrary finite degree of differentiability.

Counterexamples about center manifolds

$$\left.\begin{aligned} \frac{dx}{dt} &= -x \\ \frac{dy}{dt} &= y^2 \end{aligned}\right\} \quad \frac{dz}{dt} = X(z)$$

with solutions $f^t(x,y) = \left(e^{-t}x, \frac{y}{1-ty}\right)$

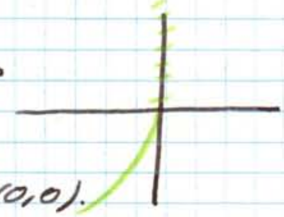
$t < \frac{1}{y}$ $y>0$
$t > \frac{1}{y}$ $y<0$

$(0,0)$ is a fixed point $\quad DX(0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

E_s : x axis
E_c : y axis

Center manifold = invariant manifold tangent to y axis.
y axis itself is a centre manifold

Take any solution curve with $y<0$. Let $t \to \infty$



$f^t(x,y) \to (0,0)$

$f^t(x,y)$ has an infinite order contact with y-axis at $(0,0)$.

$\{f^t(x,y)\}$ $\ldots < t < \ldots \} \cup \{0,0\} \cup \{$positive y axis$\}$ is another C^∞ center manifold.

$\exists!$ analytic center manifold

$$\frac{dy_1}{dt} = -y_2$$
$$\frac{dy_2}{dt} = 0$$
$$\frac{dx}{dt} = -x + h(y_1)$$

h : analytic function, vanishing to 2^{nd} order at 0, not entire.

$$DX(0) = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

y_1, y_2 plane is a center subspace
x axis is a stable subspace

A center manifold is a graph of w - $\{x = w(y_1, y_2)\}$ $\quad w(0,0)=0$, $Dw(0,0)=0$

Invariance : $\frac{dx}{dt} = \frac{d}{dt} w(y_1, y_2)$ when $x = w(y_1, y_2)$

$$-x + h(y_1) = D_1 w \cdot (-y_2)$$
$$-w(y_1,y_2) + \underbrace{y_2 D_1 w(y_1,y_2)}_{\text{preserves degree}} + h(y_1) = 0$$

partial differential equation

$$\omega(y_1, y_2) = \sum_{k=2}^{\infty} \sum_{j=0}^{k} \omega_j^{(k)} y_1^j y_2^{k-j}$$

$$h(y) = \sum_{k=2}^{\infty} h_k y^k$$

Invariance condition term by term:

$$-\sum_{j=0}^{k-1} j\,\omega_{j+1}^{(k)} y_1^{j+1} y_2^{k-j} + \sum_{j=0}^{k} \omega_j^{(k)} y_1^j y_2^{k-j} = h_k y_1^k$$

$$\Longrightarrow \quad \omega_k^{(k)} = h_k$$

$$\omega_{j+1}^{(k)}(j+1) - \omega_j^{(k)} \qquad j = 0, 1, 2 \ldots$$

$$\omega_j^{(k)} = h_k \frac{k!}{j!}$$

$$\omega(0, y_2) = \sum_{k=2}^{\infty} h_k\, k!\, y_2^k$$ has zero radius of convergence unless h is entire

$\rightarrow$ have no analytic centre manifold

but there are a lot of C^∞ center manifolds

Let $\tilde{h}$ be a bounded C^∞ fn agreeing with h near 0

$$u(t) = \omega(y_1 - t y_2, y_1)$$

$$\frac{du}{dt} = -y_2 D_1 \omega = -u(t) + \tilde{h}(y_1 - t y_2)$$

$$\frac{d}{dt} e^t u(t) = e^t \tilde{h}(y_1 - t y_2)$$

$$u(0) = \int_{-\infty}^{0} e^t \tilde{h}(y_1 - t y_2) = \omega(y_1, y_2)$$

Check: does in fact solve the equation and is a C^∞ solution of the p.d.e, vanishing to 2nd order at 0.

Two ω's made in this way differ by $O(e^{-\mathrm{const}/|y_2|})$

Claimed p. 44. Marsden - Mc Cracken

there exist C^∞ mappings, which admit no C^∞ center manifold

Derivatives of invariant manifolds:

Remark: $(D^n \omega)(0)$ can be computed by algebra.

$$\omega(x) = \Lambda_+^{-1} \{ \omega(\Lambda_- x + \ldots) \}$$

$$D\omega(x) = \ldots$$

$$D^2\omega(x) = \ldots\ldots$$

$$D^2\omega(0) = \Lambda_+^{-1} \{ D^2\omega(0) \Lambda_-^2 - D_1^2 f_+(0) \}$$

is a linear inhomogeneous eqn. for $D^2\omega(0)$

$$\|\Lambda_+^{-1}\| \|\Lambda_-\| < 1$$

$D^n w = \Lambda_+^{-1}\{D^n w(0)\Lambda_-^{\ n} + \psi_n\}$ — ψ_n contains derivative of f_1 at 0 up to order n and contains derivat. of w at 0 up to order $n-1$

$\|\Lambda_+^{-1}\|\|\Lambda_-\|^n < 1 \Rightarrow$ equation can be solved.

If f_1 vanish to order n at 0, so does w. (Not obvious)

Global stable and unstable manifolds

Assume z_0 be a hyperbolic fixed point for f.
The stable set $= \{z : f^n(z) \xrightarrow[n\to\infty]{} z_0\} = W^s$

If f is invertible smoothly, then W^s is a "manifold".

W^s_{loc} local stable manifold. $W^s_{loc} \subset W^s$

W^s_{loc} contains all orbits which stay near z_0 for all time and converge to z_0.

$z \in W^s$: $f^n(z)$ near z_0 for all sufficiently large n.

$f^{n_0}(z) \in W^s_{loc}$ for some n_0. $z \in (f^{n_0})^{-1} W^s_{loc}$

$W^s = \bigcup_{n=0}^{\infty} (f^n)^{-1} W^s_{loc}$

f invertible, implies $(f^n)^{-1} W^s_{loc} = \underbrace{(f^{-1})^n W^s_{loc}}_{\text{imbedded disk}}$

$(f^{-1})^n W^s_{loc}$ can bend back on themselves, accumulate on themselves but they can't across themselves.

W^s is a connected, injectively immersed submanifold

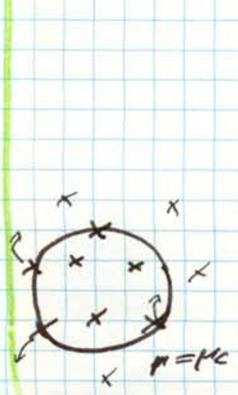
S. J. van Strien Math. Z. 166 143–145 (1979)
J. Carr: Applications of the Centre Manifold thm.
Springer Applied Math. Series

Example of an analytic mapping with no C^∞ center manifold

Two ideas: Ruelle-Takens trick:

$f_\mu(0) = 0$ μ Parameter
$Df_\mu(0)$ has some spectrum on unit circle for $\mu = \mu_c$

(x, y) splitting of coordinates
(x: centre part at $\mu = \mu_c$; y: rest)



For μ near μ_c there is an invariant manifold tangent to the spectral subspace coming from following the part of the unit circle at $\mu = \mu_c$

Manifold depends on μ. Does it behave smoothly on μ

Look at the mapping

$(z,\mu) \longmapsto (f_\mu(z),\mu)$

$(0,\mu)$ is fixed for all μ.
Look at $(0,\mu_c)$

$DF(0,\mu_c)$ has $m+1$ dimensional centre.
Applie center manifold thm to F:

$\{y=\omega(x,\mu)\}$

For each μ near enough to μ_c $\{(x,y): y=\omega(x,\mu)\}$ invariant manifold for f_μ (varying smoothly with μ)

Suppose ω is C^∞:
Then for each μ sufficiently near to μ_c, f_μ admits a C^∞ invariant of the form $y=\omega(x)$ passing through 0.

2. Argument: $f(x,y) \longrightarrow (\lambda x + h.o(x^2), \lambda^n y + O(y^2))$ tends not to have a C^n inv. manifold (→ Example above)

$f_\mu(x,y) = (\mu x + O(x^2), 2y + O(y^2))$

For $\mu = 2^{\frac{1}{n}}$, this tends not to have a C^n invariant manifold.

To make a counterexample, we have to ~~choose~~ show, how to choose higher order terms. So, $f_{2^{1/n}}$ has no C^n center manifold ~~of~~ for any n.

$f_\mu(x,y) = (\mu x, 2y + h(x))$ h analytic, not a polynomial, vanishes to 2nd order at 0

invariant manifold: $\omega(\mu x) = 2\omega(x) + h(x)$

$\mu^n \omega^{(n)}(0) = 2\omega^{(n)}(0) + h^{(n)}(0)$ by differentiating

$\mu = 2^{1/n}$ impossible unless $h^{(n)}(0) = 0$

If $h^{(n)}(0) \neq 0$ for ∞ many n's. then there are ∞ many n's so that $f_{2^{1/n}}$ has no C^∞ invar. manifold
i.e. F can't have C^∞ inv. manifold

Two examples

1. Hyperbolic linear toral automorphisms Cat Maps

$F = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$\tilde{f}(x,y) = (2x+y, x+y)$ $\tilde{f}: \mathbb{Z}^2 \to \mathbb{Z}^2$

$\Rightarrow \tilde{f}$ induces a mapping $f: \mathbb{T}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$\det F = +1 \Rightarrow F^{-1}$ is also a integer matrix. $F^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

f is invertible and the image of $(0,0)$ under f is fixed (Nothing else

$Df(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Eigenvalues of F: $\lambda_\pm = \frac{3\pm\sqrt{5}}{2}$ not on the unit circle.
$\Rightarrow (0,0)$ is a hyperbolic fixed point ~~of~~ for f.

$\phi_\pm$ eigenvectors for $\lambda_\pm$ real.

local stable manifold $= \{t\phi_- : |t| < \varepsilon\}$ under quot. mapping
global stable manifold $=$ image under π of the line
unstable $\{t\phi_\pm : -\infty < t < \infty\}$ with irrational slope going dense in the torus

Remark: $\pi(x,y)$ periodic $\Longleftrightarrow$ (x,y) are rational
$\Rightarrow$ Period. orbits are dense

z_0 period. pt of period p
$f^p(z_0) = z_0$
$Df^p(z_0) = F^p$ same eigenvectors
Stable and unstable manifolds are also dense winding lines.)

Pick n let Δ_n = image of $\mathbb{Z}^2/n$ under quotient mapping
$f\Delta_n \subset \Delta_n$. f injective $\Rightarrow$ $f\Delta_n = \Delta_n$
$\Rightarrow$ every point of Δ_n is periodic.

$\pi(x,y)$ periodic of period p. $F^p\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}m\\n\end{pmatrix}$
$(F^p - 1)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}m\\n\end{pmatrix}$

Hénon mapping

$(x,y) \to (1 - ax^2 + by, x)$ $\quad$ a,b konstant parameters

$\downarrow$ rescale y by b

$(x,y) \to (1 - ax^2 + y, bx)$ $\qquad \begin{pmatrix}-2ax & b\\ 1 & 0\end{pmatrix}$ $\qquad b = -1 \Rightarrow$ konservativ

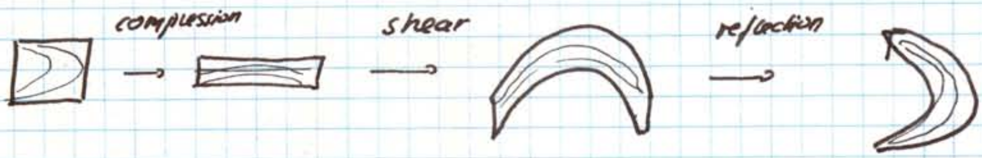
Hénon looked for it for $a = 1.4$ $b = 0.3$
Compute orbits. They fill a complicated selfsimilar set with Cantor like structure.

$a =$ $\quad (-1.33, 0.42)$ $\quad (1.32, 0.1333)$

$(-1.06, -0.5)$ $\quad (1.245, -0.14$

is mapped onto itself.

Analisation: $(x,y) \mapsto (x, by) \mapsto (x, 1 - ax^2 + by) \mapsto (1 - ax^2 + by, x)$



Remarks: 1. Mapping is invertible for $b \neq 0$
The inverse is

$$f^{-1}: (x,y) \to \left(y, \frac{1}{b}\left[x - (1 - ay^2)\right]\right)$$

2. It contracts area by $|b|$
Jacobean of $= -b$ $\qquad \{b = \pm 1$ area preserving$\}$

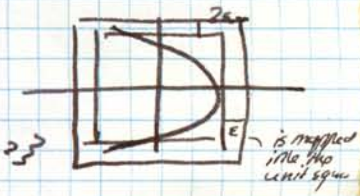
b positiv $\Rightarrow f$ is orientation reversing
z_0 periodic with odd period
$\det(Df^p(z_0)) = (-b)^p < 0$ $\Rightarrow$ $Df^p(z_0)$ can't be a rotation followed by a contraction

To what extend is the picture a theorem?
(Ignoring recent rumors)
It is conceivable, that for every pair (a,b) almost every orbit converges to the attracting cycle!
Hénon's picture might be a numerical artifact, the true orbits converging to some long attracting periodic cycle.
Carleson

Almost certainly, the set of pairs (a,b) for which f has an attracting cycle is dense.

It's likely, that the set of pairs, for which f has infinitely many attracting periodic orbits is dense.

Look at small b



Start with $b \neq 0$: $(x,y) \mapsto (by, 1-ax^2)$

Everything is projected to the parabola $\{x = 1 - ay^2\}$

By continuity, if b is small enough, f maps the rectangle

$$\Delta = \{(x,y) : -(1+\varepsilon) \le x \le (1+\varepsilon), -(1+2\varepsilon \le y \le 1+2\varepsilon)\}$$

into its interior.

$f\Delta \subseteq \Delta$ $\quad$ Area $(f\Delta) = |b|$ Area (Δ)

$\Delta \supset f\Delta \supset f^2\Delta \dots \supset \underset{\text{area}\,|b|^n|\Delta|}{f^n\Delta} \supset \dots$ $\qquad f^n\Delta \to X \quad f(X) = X$

$\mu_{Leb}(X) = 0$

Is X an attractor? (all points on full are recurrent)

What, if f has an attracting periodic orbit in Δ?
Every periodic orbit for f in Δ is in X.

If z_0 is an attracting periodic orbit of period p, U small neighborhood of z_0,

$f^{np}U \searrow \{z_0\}$ $\quad U \setminus \{z_0\}$ is non recurrent, transient

X is connected! $\Rightarrow$ $U \setminus \{z_0\}$ must contain some points of X

Distinguish between: - attracting set : X compact $fX = X$
there exists a open $U \supset X$ with
$f^nU \to X$ in the sense that for every open $V \supset X$, $f^nU \subset V$ for large enough n.

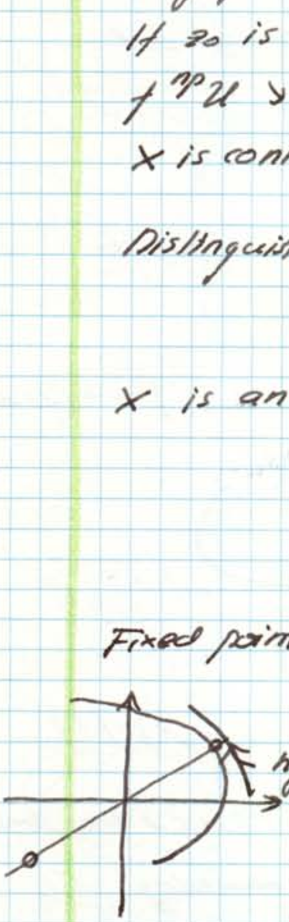
X is an attracting set.

- an attractor is an attracting set with additional properties:

(a) Every point recurrent
(b) "indecomposable"

Fixed points: (x_0, y_0) $\Rightarrow$ $x_0 = y_0$

$x_0 = 1 - ax_0^2 + bx_0$ $\quad$ Fixed point condition



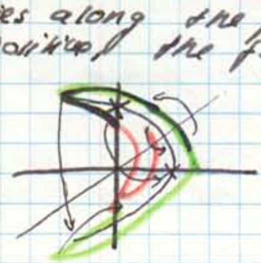
$a > 0$: one positive one negative root

with 1^d unstable manifold for $a \ge \frac{3}{4}(1-b)^2$

What does the unstable manifold look like.

$b = 0$: Stable manifold = verticle line through fixed point
The unstable manifold lies along the parabola.
By making b small, positive, the fixed point moves up

What happens globally?

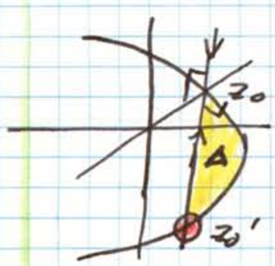


$\lambda_s < 0$

what goes on for $b=0$?

$$\bigcup_{n=0}^{\infty} f^n W^u_{loc} = \{(1-ay^2, y) \mid 1-a \le y \le 1\}$$

is not a manifold (singularities at the ends) but a closed arc.



For small positive b W^s and W^u cross at some place z_0' roughly below z_0.

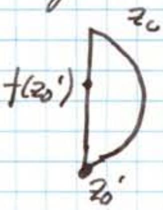
z_0' is a homoclinic point, even a transversal homoclinic point.

This implies the existence of infinitely many periodic orbits.

What happens with Δ? Δ is bounded by a piece of stable and unstable manifold.

Claim: Every orbit starting in Δ converges to the unstable manifold.

Γ $f^n\Delta$ is topolog. a disk with a part of the boundary made up by a piece of stable and unstable manifold.



The piece of unstable manif. has exponentially small length. The area of $f^n\Delta$ is exponentially small. That means: Given ε, if n is big enough, every point of $f^n(\Delta)$ is within ε of boundary of $f^n\Delta$ if n is big enough.

Given $\varepsilon > 0$. $\exists n_0$ s.t. that $n > n_0$ every point of $f^n(\Delta)$ is within distance 2ε of the unstable manifold. $\perp$

If there would be a Henonattractor, it would be the closure of the unstable manifold.

Computing stable and unstable manifolds for fixed points of the Henon mapping.

Taylorseries ...
More clever: (Franceschini & Russo)
Look for the desired manifold as a parametrized curve

$$x = x(t)$$
$$y = y(t)$$

$$f(x(t), y(t)) = (x(\lambda t), y(\lambda t)) \quad !$$

$x(0), y(0)$ is a fixed point

$$(1 - a x(t)^2 + b y(t),\ x(t)) = (x(\lambda t), y(\lambda t))$$

$$\Rightarrow \boxed{x(t) = y(\lambda t)}$$
$$\boxed{x(\lambda t) = 1 - a x(t)^2 + b\, x\!\left(\tfrac{t}{\lambda}\right)} \ \#$$

For unstable manifold
$y(t) = \sum y_n t^n$, $x(t) = \sum y_n \frac{t^n}{\lambda^n}$
~~y_n~~ b

Suppose $x(t) = \sum_{n=0}^{\infty} x_n t^n \Rightarrow y(t) = \sum_{n=0}^{\infty} x_n \frac{t^n}{\lambda^n}$

$$\sum_{n=0}^{\infty} \left\{ x_n(\lambda^n - b\lambda^{-n}) + a \sum_{j=0}^{n} x_j x_{n-j} \right\} t^n = 1 \quad \Bigg| \quad \sum_n \left(y_n(-b\lambda^{-n} + \lambda^{+n}) + a \sum_{j=0}^{n} y_j y_{n-j} \right) t^n = 1$$

$n=0$: $x_0(1-b) + a x_0^2 = 1$ ~~$x_0=0$~~ Fixed pt condition

$n=1$: $\left(\lambda - \frac{b}{\lambda}\right) x_1 + 2a x_0 x_1 = 0$

$$\left(\lambda - \frac{b}{\lambda} + 2a x_0\right) x_1 = 0 \xRightarrow{\text{for interest}} \left(\lambda - \frac{b}{\lambda} + 2a x_0\right) = 0 \quad \text{Eigenvalue condition}$$

x_1 undetermind. Fix normalisation by putting $x_1 = 1$

$$x_n\lambda^n - b\lambda^{-n} + ax_0x_n\cdot 2 = \underbrace{-a\sum_{k=1}^{n-1} x_j\cdot x_{n-j}}_{\text{know}}$$

$$x_n\{\lambda^n - b\lambda^{-n} + 2ax_0\} =$$

rekursiv condition.

Compute as many x_n as you like numerically.
In fact to compute $x(t)$ for large t, compute $x\left(\frac{t}{\lambda^n}\right)$ reasonable n and iterate mapping n times.

$$\boxed{\begin{aligned} x_n &= \frac{-a\sum_{k=1}^{n-1} x_j x_{n-j}}{(\lambda^n - b\lambda^{-n}) + 2ax_0} \\ x_0 &= \frac{(b-1) \mp \sqrt{(1-b)^2 + 4a}}{2a^2}; \\ \lambda &= \frac{-2ax_0 \pm \sqrt{4a^2x_0^2 + 4b}}{2} \end{aligned}}$$

Linearisations

$f(0) = 0$

is there a φ so that $\varphi(0) = 0$
$\varphi \circ f \circ \varphi^{-1} = \Lambda$ linear ?

- Topological case is trivial
- Differentiable case is interesting
  - Attracting or repelling fixed pts
  - Non trivially hyperbolic

. No point in considering anything other than hyperbolic fixed pts.

.

Start with one-dimensional case:

Remarks
- f defined. strictly monotone on a nbhd of 0. $f(0) = 0$
  $\varphi \circ f \circ \varphi^{-1}(x) = \lambda x$ for small enough x
  $\varphi f(x) = \lambda\, \varphi(x)$ is a linear equation

  can assume φ increasing

- If $\varphi'(0)$ exists and is different from 0, you see, that
  $\lambda = f'(0)$: $\varphi'(0)\, f'(0)\, \varphi^{-1\prime}(0) = \lambda$

  In general not : $f(x) = \lambda x$
  $\varphi(x) = \operatorname{sgn}(x)\, |x|^\alpha \quad \alpha > 0$
  $\varphi \circ f \circ \varphi^{-1}(x) = \operatorname{sgn}(\lambda)\, |\lambda|^\alpha x$

  In the topolog. situation λ isn't determind :
  but $\operatorname{sgn}(\lambda)$ is determined.

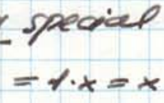
$\overrightarrow{\text{Hart}}$
- $\lambda = 1$ is very special
  If $\varphi \circ f \circ \varphi^{-1}(x) = 1\cdot x = x \iff f(x) = x$ for small x

  $\lambda = -1 \implies f^2(x) = x$ for small x

  $\lambda = 0$ is also special

  If f admits a topological linearisation with $\lambda \neq \pm 1, 0$,
  then 0 is either an attracting fixed pt or a repelling one.

  Example : $x \longmapsto x + x^4$ has no topological linearisation.

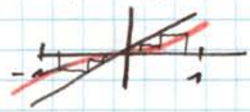


Proposition f admits a topolog. linearisation, if and only if one of the following holds

(a) $f(x) = x$ near 0 $(\lambda = 1)$
(b) f is decreasing and $f^2(x) = x$ near 0 $(\lambda = -1)$
(c) 0 is an attracting fixed point for f — can take $\lambda = \frac{1}{2}$ for f in[…], $\lambda = -\frac{1}{2}$ for f de[…]
(d) 0 is an attracting fixed point for f^{-1} — can take $\lambda = 2$ for f increas[ing], $\lambda = -2$ f decreasing

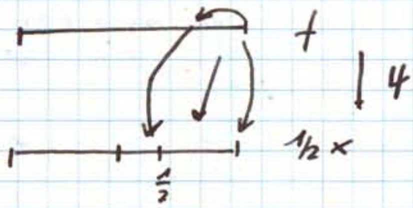
Γ consider (c) f increasing w.l.o.g. (d),(b) clear (c),(d) equiv.

Can suppose f defined, increasing on $[-1,1]$
$f(x) > x \quad -1 \le x < 0$;
$f(x) < x \quad 0 < x \le 1$



$\Longrightarrow x \in [-1,1] \quad f^n(x) \to 0$
$x \in [0,1]$ $f^n(x)$ is a decreasing sequence ~~in~~ converges to something
The limit must be a fixed pt. — it must be 0

$\boxed{\psi \circ f(x) = \frac{1}{2} \psi(x)}$ (*)



Choose any continous strictly increasing ψ mapping
$[f(1), 1] \longrightarrow [\frac{1}{2}, 1]$

Claim: can extend ψ uniquely to $[0,1]$ making (*) hold

1. Step R.h.s. defined for $f(1) \le x \le 1$
Read off from R.h.s. values of ψ on $[f^2(1), f(1)]$
Get so an extension of ψ, which is still strictly increasing continous.

$\psi(f(1)) = \frac{1}{2} \qquad [f^2(1), 1]$

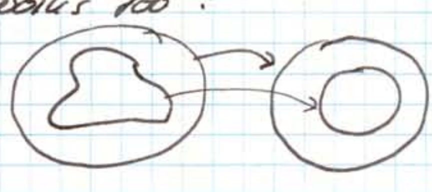
... Repeat n times, and get ψ extended to $[f^n(1), 1]$.
$\underset{0}{\overset{n\to\infty}{\downarrow}}$

$\psi(f^n(1)) = \frac{1}{2^n} \to 0$

... ψ is defined on $(0,1]$. Define $\psi(0) = 0$.
and have a continous ψ on the hole intervall $\lrcorner$

Topol. linearis. are not unique.
On the other hand differ. linea. are unique. Problem. Make ψ differentiable on 0.
Thats a bad way. It never works, to look at the continous solutions and ^then^ pick out a differentiable one.

Remark: In the multidimensional attracting case, the above method works too:



$\psi \circ f = \lambda \circ \psi$ on the unit sphere.
have to construct ψ more seriously.

The smooth situation (1 dim)

Rem: Suppose $\exists \psi$ s.t. $\psi'(0) \neq 0$ specially $\psi'(0) = 1$ w.l.o.g.
Assume 0 is attracting for f.
Claim: $\psi(x) = \lim_{n\to\infty} \frac{f^n(x)}{\lambda^n}$ for all small x $\psi \circ f = \lambda \psi$

Proof: Iterate linear. equation (*) $\psi \circ f^n(x) = \lambda^n \psi(x)$
$\psi(x) = \frac{\psi(f^n(x))}{\lambda^n} \xrightarrow{n\to\infty} \psi(x)$
for small enough x

$\frac{\psi(f^n(x))}{f^n(x)} \xrightarrow[n\to\infty]{} 1 \; ; \; \frac{f^n(x)}{\psi(f^n(x))} \quad \psi(x) = \frac{f^n(x)}{\lambda^n}$

Rem: Take $\psi_n(x) := \frac{f^n(x)}{\lambda^n} \qquad \psi_n f(x) = \lambda \psi_{n+1}(x)$
If the limit exist $\psi_n(x) \to \psi(x)$ then $\psi(x)$ is strictly increasing

$$\frac{d}{dx} \psi_n(x) = \frac{1}{\lambda^n} \prod_{j=0}^{n-1} f'(f^j(x)) = \prod_{j=0}^{n-1} \left[\frac{f'(f^j(x))}{\lambda}\right]$$

Define $\frac{f'(z)}{\lambda} = 1 + \eta(z)$ "To kand the hadl horse"

If f is C^1, $\eta(z) \xrightarrow[z\to 0]{} 0$

$$\frac{d}{dx} \psi_n(x) = \prod_{j=0}^{n-1} \left[1 + \eta f^j(x)\right]$$

f is C^1 is not enough because you can make the infinit product ~~to~~ diverge to ∞.

(we have honverg $\Leftrightarrow$ $\sum_{j=0} \eta(f^j(x))$ honverges

$f^j(x) \longrightarrow 0$ exponentially

If f' is Holder continous $f \in C^{1+\alpha}$ $\alpha > 0$ then the infinite product converges nicely. $\eta(f^j(x)) \leq C \cdot x^{(1+\alpha)j}$

$\psi_n \to \psi$ C^1 & strictly increasing

and $f \in C^{1+\alpha} \Rightarrow \psi$ is $C^{1+\alpha}$

$\boxed{f \in C^r \quad 1 < r \leq \infty \quad \text{then } \psi \in C^r}$

$\boxed{f \in C^\omega \quad \text{so is } \psi}$ (look at the infin. product too)

In the case $|f'(0)| > 1$, look at f^{-1}.

The analytical case is a classical theorem from Schröder 1871 with taylorserious.

There is no need for f to be real.

All this holds only for $|\lambda| \neq 1$.

Siegel has looked at the complex analytic case:

$|\lambda| = 1$ $\lambda = e^{2\pi i \theta}$ θ: "very irrational".

Then there exists also complex analytic linearisation.

Formal linerarisations

Given $f \in C^n$ smooth $f(0) = 0$. We always assume the mappings to be invertible and always we ~~and~~ deal with the finite dimensional case. $Df(0) = \Lambda$

$\exists$ of n'the order linear.

$\boxed{\text{Does there exist } \phi \text{ so that } \phi \circ f \circ \phi^{-1} = \Lambda z + o(z^n) \text{ with } \phi \text{ smooth. } \phi(z) = z + \dots}$

This is an algebraic question. But it contains the heart of the question

$$\phi \circ f(z) = \Lambda \phi z + \overset{o(\phi(z^n))}{o(z^n)} \quad \text{linear}$$

is solvable for general f, iff and only if some algebraic relations among eigenvalues of Λ do not hold.

$$\Lambda^{-1} D^j [\phi \circ f] = \Lambda^{-1}[(D^j \phi) \circ f](Df)^j + \text{lower order terms}$$

$\quad + \Lambda^{-1}[(D\phi) \circ f] \, D^j f$

$f_j := (D^j f)(0)$

$\phi_j = (D^j \phi)(0)$

$$(D^j [\Lambda^{-1} \phi \circ f])(0) = \Lambda^{-1} \phi_j(0) \Lambda^j + \Lambda^{-1} f_j + \omega_j$$

lower ord.

equation to be solved:

$$\phi_j = \Lambda^{-1}\phi_j(\Lambda\cdot^j) + \Lambda^{-1} f_j + \omega_j$$

ϕ_j is a symmetric j-linear mapping $\underbrace{E \times E \dots E}_{j \text{ times}} \to E$

$$\boxed{\begin{aligned} &(1\!\!1 - K)\phi_j = a_j \\ &(K\phi_j)(z_1 \cdots z_j) = \Lambda^{-1}\phi_j(\Lambda z_1 \dots \Lambda z_j) \end{aligned}}$$

$\mathcal{L}_j$ space of all symm. j linear mappings $E \times E \dots E \to E$

What ~~are the~~ is the spectrum of K

If Λ is a diagonal matrix $\lambda_1 \dots \lambda_m$, then the spectrum of K is exactly the set of products

$$\lambda_{i_0}^{-1}\lambda_{i_1}\cdots\lambda_{i_j}$$

That works also for Λ diagonable. In the general case argue as follows: Diagonable Λ's are dense. and $\Lambda \mapsto K$ is continuous and the spectrum changes continously with the operator.

Linearisations exist iff there are no relations

$$\lambda_0 = \lambda_1 \cdots \lambda_j \qquad \lambda_i \text{ eigenvalues of } \Lambda$$
Resonance $\qquad 2 \le j \le n$

Proposition

> f C^n $\quad f(0) = 0 \quad Df(0) = \Lambda$
> If no relation $\lambda_0 = \lambda_1 \dots \lambda_j \quad 2 \le j \le n$ hold among the eigenvalues of Λ, then there is an invertible C^n $\varphi(z) = z + \text{h.o.t.}$ such that $\varphi \circ f \circ \varphi^{-1}(z) = \Lambda z + o(z^n)$
>
> The derivatives of ϕ at 0 up to order n are unique.
>
> If some relations of the above kind do hold there is a set of non-trivial algebraic conditions, that the $(D^j f)(0)$ must satisfie, in order that such a ϕ exist.

Remarks : 1) If Λ has an eigenvalue λ with $|\lambda| = 1$ than $\bar{\lambda}$ is also an eigenvalue.

$$\lambda = |\lambda|^2 \cdot \lambda = \lambda^2 \cdot \bar{\lambda}$$

so you can't eliminate 3^{rd} order terms in general.

So we consider in the following only hyperbolic fixed points

2) Trivially hyperbolic case: $\{$All eigenvalues inside or all eig. outside.

is qualitiely different from the non trivially hyperbolic case.

$|\lambda_i| < 1 \quad \forall j \quad (\lambda_{min}), \ (\lambda_{max})$

$|\lambda_1 \cdots \lambda_j| \le |\lambda_{max}|^j$

If j is big enough: $|\lambda_{max}|^j < |\lambda_{min}|$

$\Rightarrow$ there are no resonances of order j.

The set of Λ's with $\rho(\Lambda) < 1$ such that there is no resonance at all is open, dense

In the non trivially hyperbolic case, the set of Λ's for which there is some resonance is dense

In particular, the set of C^∞ f's, for which there is no fine C^∞ linearisations, is dense.

Next thing: If n^{th} order linearisations for sufficiently large n exist, then C^1 linearisations exist.

Exact linearisations in the contractive case

$f(z) = \Lambda\{z + g(z)\}$ $\quad g(z)$ vanishes to high order at 0

$\Psi(z) = z + \psi(z)$ $\quad \psi(z) = O(z^2)$ at least.

$\Psi \circ f = \Lambda \Psi$

find: $\Lambda z + \Lambda g + \psi(f(z)) = \Lambda z + \Lambda \psi(z)$

$$\boxed{\psi(z) - \Lambda^{-1}\psi f(z) = g(z)}$$

$$\psi - k\psi = g$$

$$\sum_{n=0}^{\infty} k^n g \quad \text{is a formal solution}$$

If this series converges well enough, then the sum satisfies
$$\psi - k\psi = g$$

$$k^n g = \Lambda^{-n} g(\underbrace{f^n(z)}_{\to 0})$$

grows $\quad$ vanishes to high order at 0

Choose a norm so that $\|\Lambda^{-1}\| < 1$.
then $k < 1$; ε so that
$$\|Df(z)\| \le k \quad \text{when } \|z\| \le \varepsilon = 1 \text{ (by rescaling)}$$

$g(z) = O(\|z\|^N)$

$\|f^n(z)\| \le k^n \|z\| \qquad \|z\| \le 1$

So $\quad \|g(f^n(z))\| \le \text{const}\ k^{Nn} \|z\|^N$

and $\quad \|\Lambda^{-n} g f^n(z)\| \le \text{const} (\|\Lambda^{-1}\| k^N)^n \|z\|^N$

If I take N big enough, then so that $k^N \|\Lambda^{-1}\| < 1$, so the series converges.

$$D k^n g(z) = \underbrace{\Lambda^{-n} \underbrace{Dg}_{\text{vanishes to order } N-1} (f^n(z))}_{\le \text{const}\ k^{(N-1)n} \|z\|^{N-1}} \underbrace{\underbrace{Df(f^{n-1}(z)) \ldots Df(z)}_{n \text{ factors}}}_{\le k^n}$$

$$\boxed{D k^n g(z) \le \text{const}\ \|\Lambda^{-1}\|^n k^{Nn} \|z\|^{N-1}}$$

If $k^N \|\Lambda^{-1}\| < 1$, then $\sum_n k^n g$ converges in C^1.

Take an N ~~[illegible]~~ so that $k^N \|\Lambda^{-1}\| < 1$

Suppose g is C^n; $n \geq N$

What comes out is, that ψ is also C^n

$n = \infty$ is also allowed.

Remark: Can make k as close as I like to $|\lambda_{max}|$, and $\|\Lambda^{-1}\|$ as close as I like to $|\lambda_{min}|$.

$$|\lambda_{max}|^N \frac{1}{|\lambda_{min}|} < 1 \qquad N \text{ in this way, makes things work.}$$

Higher differentiability:

Construct carefully a C^n norm with respect to which k is contractive.

For $j = 0, 1, \ldots, n$

$$\boxed{\|\eta\|_j = \begin{cases} \sup_z \|D^j \eta(z)\| \, \|z\|^{N-j} & j < N \\ \sup_z \|D^j \eta(z)\| & j \geq N \end{cases}}$$

If $\eta \in C^n$ $n \geq N$, and if η vanishes to N^{th} order at 0; then

$\|\eta\|_j$ norms are finite

and the space $\{\eta \in C^n \ n \geq N, \eta \text{ vanishes at } N^{th} \text{ order at } 0\}$

is complete in $\|\eta\| = \sup_j \|\eta\|_j$

Look at $D^j k\eta(z) = \Lambda^{-1} \underset{\text{evaluated at } f(z)}{D^j \eta} \, Df(z)^j$ + lower order terms

$$\|\eta\| = \sup_j \frac{\|\Lambda^{-1} D^j \eta(f(z)) \, (Df(z))^j\|}{\|z\|^{N-j}} \|\eta\|_j \leq \|\Lambda^{-1}\| k^{N-j} k^j \|\eta\|_j$$

$$\|k\eta(z)\|_j \leq k^N \|\Lambda^{-1}\| \, \|\eta\|_j + B_j \sup_{k<j} \|\eta\|_k$$

$$\|k\eta\|_j \leq a_j \|\eta\|_j + B_j \sup_{k<j} \|\eta\|_k \qquad a_j < 1$$

Choose a sequence C_j nondecreasing $C_1 = 1$ such that

$$B_j \frac{C_{j-1}}{C_j} \leq \frac{1 - a_j}{2}$$

Define $\|\eta\| = \sup_{j \leq n} \{\|\eta\|_j \, C_j\}$ a new norm.

$$\|k\eta\| \leq \sup_j \left(\frac{1+a_j}{2}\right) \|\eta\| \quad \text{and}$$

k contractive in this norm.

What's about uniqueness?

$\Phi(z) = z + \psi(z)$ is a linearisation if and only if
$\psi(z) - \Lambda^{-1}\psi(f(z)) = g(z)$

this equation has only one solution vanishing to Nth order at 0.

This solution ψ is unique in C^N provided only that $\psi(z) = O(z^2)$

The general hyperbolic fixed point

Setting up: $E = E_s \oplus E_u$
$f(x,y) = (\Lambda_s x + f_s(x,y), \Lambda_u y + f_u(x,y))$

Proceed by supposing f_s, f_u vanish to high order

$W^s = \{y=0\}$
$W^u = \{x=0\}$ } by a smooth change of variables:

(x,y) $y = \omega(x)$
$(x',y') = (x, y-\omega(x)) = F(x,y)$
$DF(0) = \mathbb{1}$
$W^s = \{y'=0\}$ $W^u = \{x' = \omega_u(y')\}$
$y'' = y'$, $x'' = x' - \omega_u'(y')$

By this change of param. I have killed the low order terms of f too.

Can assume, that the non-linear terms vanish in a small nbh. of 0 and that f is globally near Λ.
Multiply non linear terms by a cut off fn which is 1 near 0.
Doing this doesn't spoil the above change of variables:

Little argument: $\{y=0\}$ is an invariant manifold $\Leftrightarrow$ $f_u(x,0) = 0 \quad \forall x$
$\{x=0\}$ invariant manifold $\Leftrightarrow$ $f_s(0,y) = 0$

---

correction of the estimates.

$$\| Kh \|_j \le a_j \|h\|_j + B_j \sup_{k<j} \|h\|_k \qquad a_j < 1$$

$$\|h\| = \sup_j \{ c_j \|h\|_j \} \qquad c_j \text{ decreasing} \qquad \frac{c_j}{c_{j-1}} B_j \le \frac{1-a_j}{2}$$

$$c_j \| Kh \|_j \le a_j \, c_j \|h\|_j + B_j \sup \{ \frac{c_j}{c_{j-1}} c_k \|h\|_k$$

$$\le \left(a_j + B_j \frac{c_j}{c_{j-1}} \right) \|h\| \le \frac{1+a_j}{2} \|h\| \qquad \forall j$$

$$\|Kh\| \le \sup_j \left(\frac{1+a_j}{2} \right) \|h\|$$

Look for a linearis. $\Phi(z) = z + \psi(z)$
$\psi(z) = (\psi_s(x,y), \psi_u(x,y))$

$\psi\circ f = \Lambda\psi$

gives $\Lambda_s x + f_s(x,y) + \psi_s(f(x,y)) = \Lambda_s x + \Lambda_s\psi_s(x,y)$
$\Lambda_u y + f_u(x,y) + \psi_u(f(x,y)) = \Lambda_u y + \Lambda_u\psi_u(x,y)$ } are independent

solve only one and have the other too by assuming invertible f.

$$\boxed{\underbrace{\psi_s - \Lambda_s^{-1}\psi_s\circ f}_{K\psi} = \underbrace{\Lambda_s^{-1}f_s}_{a}}$$

with formal solution

$$\boxed{\psi_s(z) = \sum_{n=0}^{\infty}\Lambda_s^{-n}a f^n(z)} \; (*)$$

has only a finite number of nonzero terms if z is not on the stable manifold

By the expanding and contracting wedges argument, all orbits converge to the unstable manifold

$\exists k<1 \quad f(x,y) = (x',y') \quad \|x'\|\le k\|x\|$

$f^n(x,y) = (x_n,y_n) \quad \|x_n\|\le k^n\|x\|$

Must have a vanishes near the unstable manifold.

Working hypothesis: $a(x,y)\le \text{const}\,\|x\|^N$ uniformly in y

If N is big enough, so that $k^N\|\Lambda_s^{-1}\|<1$, then the convergence of $(*)$ is locally uniformly.

$$D\Lambda_s^{-n}a(f^n(z)) = \underbrace{\Lambda_s^{-n}}_{\|\Lambda_s^{-1}\|^n}\underbrace{Da(f^n(z))}_{k^{(N-1)n}}\underbrace{Df(f^{n-1}(z))\cdots Df(z)}_{M^n}$$

$M = \sup\|Df(z)\| \approx g(1)$

N big enough $\quad k^{N-1}\|\Lambda_s^{-1}\|M<1 \quad \Rightarrow$ series converge in C^1 locally. $\Rightarrow \psi_s\in C^1$

Want to make $(D_1^j f_s)(0,y)\equiv 0$ for $0\le j\le N$

Coordinate change killing all low degree terms in x in f_s.

$\Phi(x,y) = (\phi(x,y),y) \qquad \phi(0,y)=0$

$\Phi\circ f\circ\Phi^{-1}(x,y) = (\Lambda_s x + O(x^{N+1}), \ldots)$

Check that this is equivalent to

$$\boxed{\begin{aligned}(D_1^j\phi\circ f)(0,y) &= (\Lambda_s D_1^j\phi)(0,y) \quad j=2,\ldots,N\\ &= \Lambda_s \;\; '' \qquad j=1\end{aligned}}$$

Solve for $D_1^j\phi(0,y)$

$D_1\phi\circ f(0,y) = D_1\phi(f(0,y))[\Lambda_s + D_1 f_s(0,y)] = \Lambda_s(D_1\phi)(0,y)$
$+ \underbrace{D_2\phi(\underbrace{f(0,y)}_{(0,y')})D_1 f_u(0,y)}_{0}$

$\left(\begin{aligned}D_1\phi(0,y) &=: \phi_1(y)\\ \phi(x,y) &= \sum_{j=1}\phi_j(y)x^j\\ \alpha_1(y) &:= D_1 f_s(0,y)\end{aligned}\right)$

($F(y) = \Lambda_u y + f_u(0,y)$ is invertible and its inverse is contractive)

$$\Rightarrow \phi_1(F(y))[\Lambda_s + \alpha_1(y)] = \Lambda_s\phi_1(y)$$

$$\boxed{\phi_1(y) = \Lambda_s\phi_1(F^{-1}y)[\Lambda_s + \alpha_1(F^{-1}y)]^{-1}} \; (**)$$

Iterate $(**)$ n times: $\Lambda_s^{+n}[\Lambda_s + \alpha_1(F^{-n}y)]^{-1}[\Lambda_s + \alpha_1(F^{-n+1}y)]^{-1}\cdots$
$\cdots[\Lambda_s + \alpha_1(F^{-1}y)]$

$= [\mathbb{1} + \Lambda_s^{n-1}\alpha_1(F^{-n}y)\Lambda_s^{-n}][\ldots][\mathbb{1} + \alpha_1(F^{-1}y)\Lambda^{-1}]$

converges nicely, if α, is small enough.
that means f_s vanishes at high enough order.

higher order terms: $\phi_j(y) - \Lambda_s \phi_j (F^{-1}y)\Lambda_s^{-j} = \alpha_j(F^{-1}y)\Lambda_s^{-j}$

Theorem

Sternberg Linearisation theorem

> Let $0<\alpha<\beta<1$ and let n be a positive integer. Then there exists $N(\alpha,\beta,n)$ such that: If z_0 is a fixed point for the mapping f $(\mathbb{C}^N)$ and if the spectrum of $Df(z_0)$ is contained in $\{\alpha<|\lambda|<\beta\} \cup \{\beta^{-1}<|\lambda|<\alpha^{-1}\}$ and if
> $\sigma(Df(z_0)) \cap [\sigma(Df(z_0))]^j = \phi$ for $2 \le j \le N$
> Then f admits a linearisation of class C^n at z_0

with some reorganisation:

- Put fixed point at origin
- non linear terms small by rescaling
- straighten out stable and unstable manifolds

$\psi(x,y) = (x + \psi_s(x,y), y + \psi_u(x,y))$

solve only for ψ_s: $\boxed{\psi_s - \Lambda_s^{-1}\psi_s \circ f = \Lambda_s^{-1} f_s}$

with formal solution $\psi_s = \sum_{n\ge 0} \Lambda_s^{-(n+1)} f_s \circ f^n$ for

which $f^n(z) \to \varepsilon_M$ with uniformity.

If $f_s(x,y) = O(x^N)$ uniformly in y, the formal solution converges and is in C^1

Make a coordin. chang, to have $f_s(x,y) = O(x^N)$ uniformly in y.
Really need only this for small y (Multiply non linear term by a cut off function)

$\phi(x,y) = (\phi(x,y), y)$ coord change
$\underbrace{\phi \circ f \circ \phi^{-1}}_{=:\hat f}(x,y) = \Lambda_s x + O(x^{N+1})$

$\hat f(x,y) = (\Lambda_s x + \hat f_s(x,y), \Lambda_u y + \hat f_u(x,y))$

$D_1^j \hat f_s(0,y) = 0$ for $1 \le j < N$

$\Updownarrow$

$\boxed{(D_1^j(\phi \circ f))(0,y) = \Lambda_s (D_1^j\phi)(x,y) \quad 1 \le j < N}$

solve this successively for $j = 1, 2, \ldots N-1$

$(D_1^j(\phi \circ f))(x,y) = (D_1^j\phi) f(x,y) \, (\Lambda_s + D_1 f_s)^j(x,y) + \text{l.o.t.}$

$$\left.\begin{aligned} F^{-1}(y) &= \Lambda_u y + f_u(0,y) \\ f(0,y) &= (\ldots, \Lambda_u y + f_u(0,y)) \\ \Lambda_s(y) &= \Lambda_s + D_1 f_s(0,y) \\ \phi_j(y) &:= D_1^j \phi(0,y) \end{aligned}\right\} \text{Notation}$$

$\phi_j(F^{-1}(y))(\Lambda_s(y))^{+j} - \Lambda_s \phi_j(y) = \alpha_j(y)$

After replacing y by $F(y)$ and multipl. the right by $\Lambda_s(y)^{-1}$

$$* \quad \boxed{\phi_j(y) - \Lambda_s \phi_j(F(y)) \Lambda_s(y)^{-j} = b_j(y)}$$

F has the origin as an attracting fixed point.

Technical lemma

Let $\mathcal{E}, \mathcal{F}$ be finitely dim. normed spaces
F a continuously diff. mapping of $\mathcal{E}_1$ ← open unit ball in $\mathcal{E}$
to itself. $F(0)=0$.
Let L be a mapping $\mathcal{E}_1 \to \mathcal{L}(\mathcal{F},\mathcal{F})$
Assume

$$\kappa := \sup_{\|x\|<1} \|DF(x)\| < 1$$

$$M := \sup_{\|x\|<1} \|L(x)\|$$

Let n be an integer # such that $\kappa^n M < 1$
and let n_0 $0 \le n_0 < n$ be another integer
Assume f, L are of class C^n (smoothness)

$1 \notin \sigma(L(0))\, \sigma(DF(0))^j$ for $n_0 \le j < N$ (non resonance)

Let $a: \mathcal{E}_1 \to \mathcal{F}$ of class C^n

$(D^j a)(0) = 0$ for $j < n_0$

then there is a unique $\psi: \mathcal{E}_1 \to \mathcal{F}$, such that

$$\psi(x) - L(x)\, \psi \circ F(x) = a(x)$$

in C^n with $(D^j\psi)(0)=0$ for $j<n_0$

Proof: Look the proof for linearisation for the contracting case.

Proof of $*$

Put $\mathcal{E} = \mathcal{E}_u$ $\mathcal{F} = \{ \mathcal{E}_s \times \dots \mathcal{E}_s \to \mathcal{E}_s \}$

$\mathcal{L}(y)\, \phi_j(x_1 \dots x_j) = \Lambda_s \phi_j(\Lambda_s(y^{-1})x_1, \dots \Lambda_s(y^{-1})x_j)$

$L(0)\, \Phi_j(x_1 \dots x_j) = \Lambda_s \phi_j(\Lambda_s^{-1}x_1, \dots \Lambda_s^{-1}x_j)$

$\sigma(L(0)) = \sigma(\Lambda_s)\sigma(\Lambda_s)^{-j}$

$1 \in \sigma(\Lambda_s)\sigma(\Lambda_s^{-j})\sigma(\Lambda_u)^{-k}$
special case of general non resonance condition
$1 \in \sigma(\Lambda^{-1})\,\sigma(\Lambda)^m$
Assuming enough smoothness and non resonance
for j can recursively proof ~~unic~~ existence of ϕ_i

(A little care for $j=1$: Impose the condition $\phi_1(0) = \mathbb{1}$)

$*$ $\psi_1 \circ f \circ \psi_1^{-1} = \Lambda$
$\psi_2 \circ f \circ \psi_2^{-1} = \Lambda$
$f = \psi_2^{-1} \Lambda \psi_2$
$f = \psi_1^{-1} \Lambda \psi_1$
$\psi_1 \psi_2^{-1} \Lambda \psi_2 \psi_1^{-1} = \Lambda$
$\psi_1 \psi_2^{-1}$ commutes with Λ

Non uniqueness

ψ is unique $\Leftrightarrow$ the only ϕ with $\phi(0)=0$
$D\phi(0) = \mathbb{1}$ which commutes with Λ is $\mathbb{1}$

want to show:

$*$

If Λ is non trivially hyperbolic even if there are no resonances at all, there exists C^∞ ϕ's commuting with Λ on a neighborhood of 0, but such that $\phi(x) - x$ does n't vanish on any nbhd of 0.

$\phi(x) = x + \phi(x)$
$\Lambda\, \phi(x) = \Lambda x + \Lambda \phi(x)$
$\phi\, \Lambda x = \Lambda x + \phi(\Lambda x)$

Must have: $\phi(x) = \Lambda^{-1}\phi(\Lambda x)$ on a nbhd of C^∞
choose $a(x)$ which vanishes on a nbhd of 0

solve $\boxed{\phi(x) - \Lambda^{-1}\phi(\Lambda x) = a(x)}$

$$\phi(x) = \sum_{n \geq 0} \Lambda^{-n} a(\Lambda^n x)$$

Take a: C^∞ with comp. support vanishes together with all its derivatives on E_n. Sum converges in $\mathcal{E}^\infty$
Sum vanishes to ∞ order at 0.

special case: Λ is diagonal matrix with positive eigenvalues. Take $a(x)$ to have all components non negative (non cancellations in the sum);

$a(x) > 0$ for $\{1 \leq \|x\| \leq \|\Lambda\|\}$
except on E_u

If x is ~~infinite~~ inside the unit ball not on E_s
$\Lambda^n x \to \infty$ for some n $\Lambda^n x$ must be in $\{1 \leq \|x\| \leq \|\Lambda\|\}$
$\to \phi(x) > 0$

The C^∞ case

Theorem

> Let z_0 be a fixed point for the C^∞ mapping f such that
>
> $$\sigma(Df(z_0)) \cap \sigma(Df(z_0))^j = \phi \qquad j \geq 2$$
>
> Then f admits a $\mathcal{E}^\infty$ linearisation at z_0

Need to make $f_s(x,y)$ vanish to ∞ order in x for each y near 0.

we know how to find a formal change of coordinates which does that:

$$\phi(x,y) = \Big(\underbrace{\sum_{j \geq 1} \phi_j(y)(x)}_{\text{converges?}}, y\Big)$$

Don't need that $\sum \phi_j(y)(x)$ converges.
we only need.

$\phi(x,y)$ with the x-derivatives at $x = 0$ converges.

General principle: every formal power series in the taylor series for a C^∞ fn

Proposition:

> Let C_α be any family of real numbers indexed by the multi indexes. Then there exists a C^∞ real valued fn ψ on $\mathbb{R}^n$
>
> $$(D^\alpha \psi)(0) = \alpha!\, C_\alpha \qquad \forall \alpha$$

Γ $\psi(x) = \sum_\alpha C_\alpha x^\alpha \xi(\eta_\alpha x)$ $\qquad \xi \in C^\infty$ $\xi = 1$ on a nbhd of 0 with compact support

and $n_\alpha \to \infty$ fast enough
For any non zero λ there are only finitely many non vanishing terms if $n_\alpha \to \infty$ $|\alpha| \to \infty$

Sum is C^∞ everywhere except at 0.
Some care is needed to get C^∞ at 0.

Show: for any β $\sum_{|\alpha|>|\beta|} c_\alpha D_\beta (x^\alpha \xi(n_\alpha x))$ converges uniformly.

$$D_\beta (x^\alpha \xi(n_\alpha x)) = \frac{1}{n_\alpha^{|\alpha|}} D^\beta((n_\alpha x)^\alpha \xi(n_\alpha x))$$
$$= n_\alpha^{|\beta|-|\alpha|} D^\beta_y y^\alpha \xi(y) \Big|_{y = n_\alpha x}$$

Take $M_\alpha := \sup_{|\beta| < |\alpha|} \sup_y | D^\beta y^\alpha \xi(y) |$

$N_j :=$ # of multiindices of total degree $|\alpha| = j$

Choose n_α, so that $\frac{|c_\alpha| M_\alpha}{n_\alpha} \le \frac{1}{2^{|\alpha|} N_{|\alpha|}}$

$|c_\alpha D^\beta (x^\alpha \xi(n_\alpha x))|$. $|\alpha| > |\beta|$

$\le \frac{|c_\alpha| M_\alpha}{n_\alpha^{|\alpha|-|\beta|}} \le \frac{|c_\alpha| M_\alpha}{n_\alpha} \le \frac{1}{2^{|\alpha|} N_{|\alpha|}}$

$\sum_{|\alpha|>|\beta|} \frac{1}{2^{|\alpha|} N_{|\alpha|}} < \infty$ ∎

Remarks: - Question: Does the existence of high order resonances get in the way of the existence of linearisations with low (non zero) degree of differentiability?

- $\lambda_{i_0} = \lambda_{i_1} \cdots \lambda_{i_n}$ Clear: Non exist. of C^n linearis.
Not clear whether there exist C^{n-1} linear.

Exercise:
1. $f \in C^2$ $Df(0)$ contraction $\Rightarrow f$ has C^1 linearis.
2. $(x, y, z) \xrightarrow{f} (\alpha x, \beta y, \alpha\beta(z + xy))$ $\alpha < 1$, $\beta > 1$, $\alpha\beta < 1$
has no C^1 linearisation
$f^n(x, y, z) = (\alpha^n x, \beta^n y, (\alpha\beta)^n (z + n x y))$
$\Lambda^n(x y z) = (\alpha^n x, \beta^n y, (\alpha\beta)^n z)$

Fixed points for mappings depending on a parameter

$z \mapsto f_\mu(z)$ jointly smooth in μ, z

$\mu_0 < \mu < \mu_1$
$\mu_0 < \bar\mu < \mu_1$

Want to use the implicit function theorem to conclude the existence of μ dependent fixed points for f_μ μ near $\bar\mu$. Can do this, provided 1 is not in the spectrum of $Df_{\bar\mu}(\bar z)$

Need: $D_z f_\mu(z) - z \Big|_{\substack{z = \bar z \\ \mu = \bar\mu}}$ is invertible

Zorn: $\exists$ maximal interval $(\mu_-, \mu_+) \ni \bar\mu$ $\mu \mapsto x_\mu$ continuous $f_\mu(x_\mu) = x_\mu$

What happens at μ_+?

$\mu = \mu_1$ trivial
$x_\mu \to \infty$ as $\mu \to \mu_1$

If $\exists \mu_n \to \mu_1$ so that $x_{\mu_n} \to x_1$
then $f_{\mu_1}(x_1) = x_1$ $\quad Df_{\mu_1}(x_1)$ has spectrum containing 1

Γ If not true, $\exists x_1(\mu)$ continuous μ dep. fixed pt. defined for μ near μ_1
by uniqueness of Impl. Fct. thm. $x_{\mu_n} = x_1(\mu_n)$ for large enough μ_n

therefore $x_\mu = x_1(\mu)$ for all μ near enough to μ_1 $\lrcorner$

Periodic orbits ... Poincaré map ... in the same way ~~that~~ with exeptions:
- Orbit may become infinite
- Orbit may shrink to a point
- Period go to ∞

Look carefully to what happens near f_μ $\quad f_0(0)=0$, $Df_0(0)$ has 1 in the spectrum

Analysis has two pieces:
- Simpelst possible ~~cor~~ mappings 1 dim mapping. (Deep part)
- Essentially same thing happens in hi[gher] dimensions

$$\boxed{\begin{array}{l} f_\mu : \mathbb{R} \to \mathbb{R} \\ f_\mu(0)=0 \quad f_\mu'(0)=1 \end{array}}$$

unlikely both.

write $f(\mu, x) := f_\mu(x)$

Assume: $\boxed{(D_\mu f)(0,0) \neq 0}$ $\qquad$ Non degeneracy condition

Idea: Solve fixed pt eqn. for μ as a fn of x

$$f(\mu, x) - x = 0 \qquad D_\mu\big(f(\mu,x) - x\big)\Big|_{\mu=x=0} \neq 0$$

$\exists \mu(x)$ s.t. x is a fixed point for $f_{\mu(x)}$ $\qquad$ (smooth if f smooth)

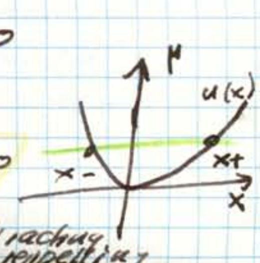
$$\frac{d\mu}{dx} = \frac{\overbrace{(D_x f)(\mu, x)}^{f_\mu'(x)} - 1}{(D_\mu f)(\mu, x)} \leftarrow \text{Vanishes at } x=0 \text{ at } \mu=x=0$$

$$\frac{d^2\mu}{dx^2} = \frac{D_x^2 f + (D_\mu D_x f \frac{d\mu}{dx}) D_\mu f - (D_x f - 1) D_\mu^2 f \frac{d\mu}{dx}}{(D_\mu f)^2} = \frac{D_x^2 f}{D_\mu f} \text{ at } \mu=x=0$$

Second nondegen. condit: $\boxed{D_x^2(f)(0,0) \neq 0}$
$$\Longrightarrow \left(\frac{d^2\mu}{dx^2}\right)\Big|_{\mu=x=0} \neq 0$$

Assume minimum $\quad$ (else $\mu \rightsquigarrow -\mu$)

For small negative μ, no fixed point at all near 0
For small $\mu > 0$, two fixed point at $x_\pm = \pm c\sqrt{\mu}$



one attracting the other repelling

$$D_x f(\mu, x_+) = D_x f(\mu, 0) + \int^{x_+} (D_x^2 f)(\mu, x)\, dx$$

For small μ $D_x^2 f(\mu, x) \approx D_x^2 f(0,0)$. and $x_+ \approx c\sqrt{\mu}$
which means

$$D_x f(\mu, x_\pm) = \underbrace{\pm \underset{\substack{\neq 0 \\ > 0 \text{ O.B.d.A.}}}{c}\left(\sqrt{\mu} + O(\sqrt{\mu})\right)}_{\text{from integral}} + \underbrace{1 + O(\mu)}_{\text{from first term}} = 1 \pm c'\sqrt{\mu} + O(\sqrt{\mu})$$

$\Longrightarrow$ x_+ is attaching
x_- is repelling

For $f(\mu, x) = x - \mu + x^2$ can do calculation, explicitly

This bifurcation is called saddle node bifurcation

Higher dimensions:

use centre manifold thm. for the reduction.
Way of Ruelle Takens.

$f_\mu : \mathcal{E} \longrightarrow \mathcal{E}$ $\quad f_\mu(x) = f(\mu, x)$ Jointly smooth C^r $1 \leq r < \infty$

$f_0(0) = 0$, ~~$1 \in \sigma(Df_0(0))$~~

Assume $\sigma(Df_0(0)) \cap$ unit circle $\neq \emptyset$ and the part of the spectrum on the unit circle is seperate from the rest.

$\mathcal{E} = \mathcal{E}_S \oplus \mathcal{E}_C \oplus \mathcal{E}_\mu$ spectral decomposition.

Assume: There is a C^r fn on $\mathcal{E}_C$ non identically zero but vanishing outside the unit ball.

Apply center Manifold theorem to $(\mu, x) \longmapsto (\mu, f_\mu(x))$ which has fixedpoint $(0,0)$: Center subspace $= \mathcal{E}_C \oplus \mathbb{R}$.

invariant manifold is a graph of C^r

$$\underbrace{(\mu, x)}_{\in \mathbb{R} \times \mathcal{E}_C} \longrightarrow \underbrace{\omega(\mu, x)}_{\mathcal{E}_S \oplus \mathcal{E}_u}$$

Need from above

> There exist an $\varepsilon > 0$, U_c a nbhd of 0 in $\mathcal{E}_C$, U_{su} a nbhd of 0 in $\mathcal{E}_S \oplus \mathcal{E}_u$. $\exists\, \omega : (-\varepsilon, \varepsilon) \times U_c \longrightarrow U_{su}$ C^r mapping
>
> so that $\exists\, \mathcal{W}_\mu$ (= graph of ω) is invariant in the sense that if $x \in \mathcal{W}_\mu$ so that $f_\mu(x) \in U = U_c \times U_{su} \Longrightarrow f_\mu(x) \in \mathcal{W}_\mu$
>
> ~~$\mathcal{W}_\mu$ is invariant in the sense that if $x \in \mathcal{W}_\mu$~~
>
> Then a) If $z \in U$ and if $f_\mu^n(z) \in U$ $\forall n \geq 0$, then $f_\mu^n(z) \to \mathcal{W}_\mu$
>
> If $z \in U$ and $f_\mu^{-n}(z) \in U$ $\forall n \geq 0$ then $f_\mu^{-n}(z) \longrightarrow \mathcal{W}_\mu$
>
> b) Any z whose full orbit stays in U must be in $\mathcal{W}_\mu$ especially fixed points and periodic points must be in $\mathcal{W}_\mu$
>
> c) If $z_0 \in \mathcal{W}_\mu$ is a fixed point, then $\sigma(Df_\mu(z_0))$

and $\sigma(Df_\mu(z_0))\big|_{\text{ambient space}} = \sigma(Df_\mu(z_0))\big|_{\text{submanif.}} \cup \{$ spectrum of an operator in $E_s \oplus E_u$ norm near $\Lambda_s \oplus \Lambda_u \}$

If $E_u = (0)$ and if z_0 is attracting in W_μ then it is attracting in the ambient space.

The same is true for periodic points.

[Spectrum in ambient space. Choose coordin. (x, y) so that $W_\mu = \{(x, 0)\}$. Suppose

$$Df(z_0) = \begin{pmatrix} (Df)_{xx} & (Df)_{xy} \\ (Df)_{yx} & (Df)_{yy} \end{pmatrix}$$

(0 by invariance of x axis) ($\approx \Lambda_s \oplus \Lambda_u$)]

$f_\mu(x)$ jointly smooth. $f_0(0) = 0$

$(Df_0)(0)$ has 1 in its spectrum as an isolated point. Spectral subspace has 1 dimension. No other points of the unit circle are in the spectrum. (In complex case nondegen. condition, else more hyp.)

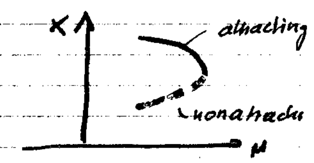
Center manifold theorem reduces analysis to the 1 dim case. Impose 1-d. non degen. condit. (complicated in ambient space).

Then as $\mu \to 0$ for appropriate side two fixed points collide and disappear. If $E_u = \{0\}$ one is attracting. The other hyperbolic with 1 dim. unstable manifold.

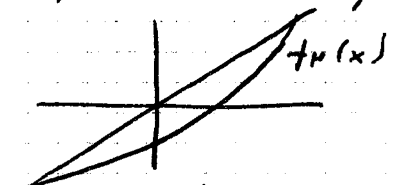
Other nondegen. conditions: $\frac{d\mu}{dx} \neq 0$

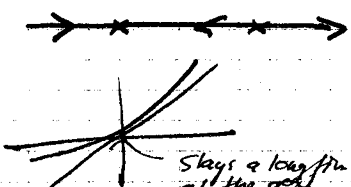
If $D_\mu f_\mu(0,0) = 0$
$f_0(0) = 0$

Perturbation argument: take $f_\mu(x) + \varepsilon_1\mu + \varepsilon_2 x^2$



Macroscopic dynamic near a bifurcation: $f_\mu(x) = x + x^2 - \mu$

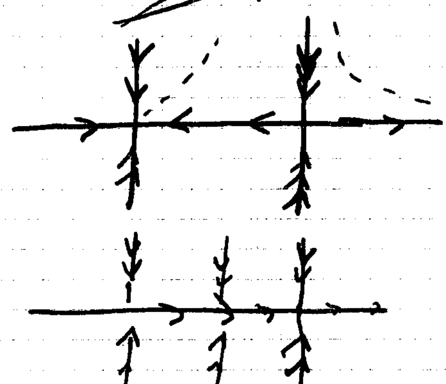




high dimens

before bifurcation

after bifurcation



Orbits stay ~~stop~~ trapped for long periods of time near where the fixed points were.

One of the kinds of intermittency discussed by Pomeau-Manneville

More is true.
There is a more fundamental justification for nondegeneracy conditions.

> can prove, that there exist an open dense set of one param. families $\mu \mapsto f_\mu$ satisfying non degeneracy conditions everywhere

Reference: Sotomayor "Generic bifurcations in dynamical systems" Dynam. Systems confer. proceedings. Peixoto (1973)

Another Bifurcations

f_μ for $\mu = \bar\mu$ $f_{\bar\mu}(\bar x) = \bar x$;
$\sigma(Df_{\bar\mu}(\bar x))$ inside the unit circle.

$\bar x_\mu$ can followed as long as it remaines attracting.
There are three non-degenerate ways to lose attractivity.

- an eignvalue goes to $+1$ $\to$ saddle node
- an eignvalue goes to -1
- a complex conjugate pair goes to $e^{\mp i\theta}$

$\boxed{f_\mu \text{ with } f_0(0)=0 \quad f_0'(0) = -1}$

the fixed point doesn't disappear in μ
can assume that the fixed point stays at 0. That means w.l.o.g. $f_\mu(0)=0$ for μ small.

Look at $f_\mu \circ f_\mu$: $D(f_\mu \circ f_\mu) = Df_\mu(f_\mu(x))\, Df_\mu(x)$
$= (-1)^2 = 1$ at $\mu = x = 0$

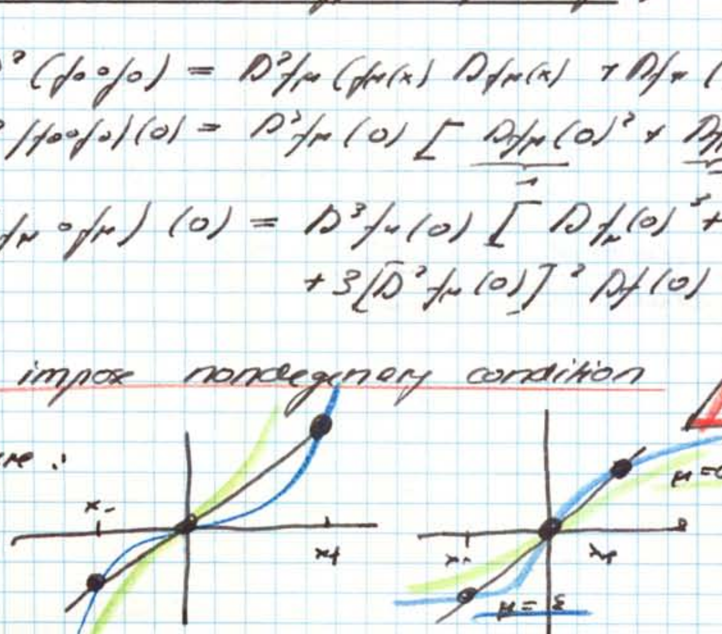
but we are not in the saddle node case.
Heart of elem. bifurc theory: $\boxed{D^2(f_0 \circ f_0)(0) = 0}$!
(We are not in a typical family.).

$\lceil D^2(f_0 \circ f_0) = D^2 f_\mu(f_\mu(x))\, Df_\mu(x) + Df_\mu(f_\mu(x))\, D^2 f_\mu(x)$
$D^2(f_0\circ f_0)(0) = D^2 f_\mu(0)\,[\underbrace{Df_\mu(0)^2}_{1} + \underbrace{Df_\mu(0)}_{-1}] = 0 \rfloor$

$D^3(f_\mu \circ f_\mu)(0) = D^3 f_\mu(0)\,[Df_\mu(0)^3 + Df(0)]$
$+ 3[D^2 f_\mu(0)]^2\, Df(0)$

Can impose nondegenery condition $\boxed{D^3(f_\mu \circ f_\mu)(0) \neq 0}$

Picture:



$f_\mu \circ f_\mu$ has 3 fixed points, 0, and two others $x_\pm$

sign $f(x) = -$ sign(x) $\Rightarrow$ f can't admit non zero fixed point near 0

$\Rightarrow$ The new fixed points for $f\circ f$ are a cycle of period 2 for f.

$(f\circ f)'(x_\pm) < 1$ tells you that ~~$x_\pm$ are~~ the cycle is attracting.

In the first case get a cycle of period 2, which is repelling.

<u>Oversimplified model</u>: $f_\mu \circ f_\mu(x) = (1+\varepsilon)x - bx^3$

$$b \equiv -\frac{(f_\mu\circ f_\mu)'''(0)}{3!}$$

$$(1+\varepsilon)x - b^3x = x$$

$$\varepsilon x - bx^3 = 0 \qquad x \neq 0$$

$$\varepsilon - bx^2 = 0$$

Two cases: $b>0$ sol. for $\varepsilon > 0$

$$x_\pm = \pm\sqrt{\frac{\varepsilon}{b}}$$

$b<0$ solutions are $x_\pm = \pm\sqrt{\frac{-\varepsilon}{b}}$

$$(f\circ f)'(x_\pm) = 1+\varepsilon - 3bx^2$$

$$= 1 - 2bx_\pm^2 \quad \text{at } x_\pm$$

$$= 1 - 2\varepsilon \quad \text{at } x_\pm$$

cycle is attracting if $b>0$
repelling if $b<0$

Two situations: $b>0$, fixed point attracting: no periodic cycle
fixed point repelling: there is a periodic cycle of period 2 at $x_\pm \simeq O(\sqrt{\varepsilon})$
the attracting cycle is emitted by the fixed point at the instant when it ceases to be attracting.

f
f^{-1}

Normal period doubling or flip bifurcation

$b<0$ when fixed point is repelling: no cycle.
cycle exists if fixed point is attracting
cycle is repelling and is absorbed by fixed point at the instant when it ceases to be attracting

inverted period doubling or flip bifurcation

$f_\mu : \mathbb{R} \to \mathbb{R}$ $\quad C^n$ in (μ, x) $\quad n \geq 4$

$f_\mu(0) = 0 \quad f_0'(0) = -1$

$(f_0 \circ f_0)''(0) = 0$ comes out

$-\frac{(f_0 \circ f_0)'''(0)}{3!} =: b \neq 0$

Look for $\boxed{f_\mu \circ f_\mu(x) = [f_\mu'(0)]^2 x + \frac{(f_\mu \circ f_\mu)''(0)}{2} x^2 - \beta(\mu, x) x^3}$ $\ast$

$\beta(\mu, x) = \frac{-1}{2!} \int_0^1 (1-t)^2 (f_\mu \circ f_\mu)'''(tx)\, dt \quad \in C^{n-3}$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} x^k + \frac{1}{n!} \int_0^1 t^n f^{(n+1)}(tx)\, dt$$

Assume : $\boxed{D_\mu f_\mu'(0) < 0}$ at $\mu = 0$

Reparametrisation : $f_\mu'(0) = -\sqrt{1+\mu}$

(arrange μ so that this is the case)

$\frac{(f_\mu \circ f_\mu)''(0)}{2!} = \frac{f_\mu''(0)}{2!} [f_\mu'(0)^2 + f_\mu'(0)]' = \mu \alpha(\mu) \in C^{n-2}$

$\ast \Rightarrow (1+\mu)x + \mu\alpha(\mu)x^2 - \beta(\mu, x)x^3 = x$

$\boxed{\mu + \mu\alpha(\mu)x - \beta(\mu, x)x^2 = 0}$

$b = \beta(0,0)$

Assume $b > 0$

$\sigma := \sqrt{\mu}$

$\mu(1 + \alpha(\mu)x) = \beta(\mu, x)x^2$

$\pm\sqrt{\mu}\sqrt{1+\alpha(\mu)x} = \sqrt{\beta(\mu, x)}\, x$

$\sigma\sqrt{1+\alpha(\mu)x} = \sqrt{\beta(\sigma^2, x)} \cdot x$

$\boxed{x\sqrt{\beta(\sigma^2, x)} - \sigma\sqrt{1+\alpha(\mu)x} = 0}$ Equation to be solved for x, everything is C^1 in σ

$\sigma = 0$: solution is $x = 0$

$D_x(\ \dots\ )_{\sigma=0} = \sqrt{\beta(0,0)} \neq 0$

Implizit function theorem : $\exists\, x(\sigma) \in C^{n-3}$ solving the equation

μ small $> 0 \implies x(\pm\sqrt{\mu})$ is a fixed point for $f_\mu \circ f_\mu$ and form a cycle of period 2 for f_μ

For $\mu < 0$ we have no periodic pts near 0 other than 0 itself

For $\mu > 0$ cycle is attracting.

For $\mu < 0$ there exists a nbhd V of 0 in $\mathbb{R}$, such that if $\mu \leq 0$ is sufficiently small and $x \in V$, then $f_\mu^n(x) \to 0$.

("Domain of attraction doesn't go to zero")

This is a consequence of

$$\frac{|f_\mu(f_\mu(x))|}{|x|} = \frac{(1+\mu)x + \mu\alpha(\mu)x - \beta(\mu, x)x^2)}{x}$$

(x, μ) small enough, $\mu \leq 0$, $x \neq 0$ $\implies \mu(1+\alpha(\mu)x) - \beta(\mu, x)x^2 < 0$

(just take μ, x small enough that $\alpha(\mu)x > -1$

want to compute $\quad D_x f_\mu \circ f_\mu (x_\pm) = 1+\mu+2\mu\alpha(\mu)x - 3\beta(\mu,x)x^2 - D_x\beta$

$= 1-2\mu - \mu\alpha(\mu)x_\pm - D_x\beta(\mu,x)x_\pm^3$

$x_\pm \simeq \pm\,\text{const}\sqrt{\mu}$

$= 1-2\mu + O(\mu^{3/2}) < 1$

$\mu > 0$

$\longrightarrow$ period-cycle is attracting ∎

Some Details:

- Can even have $f \in C^3$: Claim: $f_\mu \in C^n$, then $n \geq 3$

$x\sqrt{\beta(\mu,x)} \in C^{n-2}$

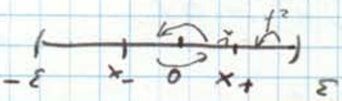
Remark: Suffices to prove that the first partials extend continuously through $x=0$ and this is the same as showing $\partial_x\beta(\mu,x)$ is C^{n-2}

$$x\cdot\beta(\mu,x) = \frac{1}{2}\int_0^1 (1-t)^2 x\, g'''(tx)\,dt \qquad g = f_\mu \circ f_\mu$$

$$= -\frac{1}{2}g''(0) + \int_0^1 (1-t)\,g''(tx)\,dt$$

$$= \int_0^1 (1-t)\left[g''(tx) - g''(0)\right]dt \quad \in C^{n-2}$$

Consider $\mu > 0$: $\quad x_\pm(\mu) = x(\pm\sqrt{\mu})$ $\qquad x_-(\mu) < 0 < x_+(\mu)$



- Claim: ε small enough: For all $x \in (-\varepsilon,\varepsilon)$ except $x=0$, $f^n(x) \longrightarrow \{x_\pm\}$ $\quad n \to \infty$

For $x_- < x < x_+$ $\quad \varepsilon$ suff. small $\quad f^{-n}(x) \to 0$
For $-\varepsilon < x < x_-$
or $\varepsilon < x < x_+$ $\qquad f^{-n}(x)$ eventually gets out.

Claim: $\mu + \mu\alpha(\mu)x - \beta(\mu,x)x^2 < 0$ for $0 < x < x_+ \Rightarrow |f_\mu \circ f_\mu(x)| > |x|$

> 0 for $x_+ < x < \varepsilon \Rightarrow |f_\mu \circ f_\mu(x)| < |x|$

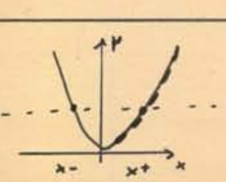
- $b < 0$ $\quad$ substit. $\sigma = \pm\sqrt{-\mu}$ $\quad \sigma^2 = \mu$ $\quad$ and as before

$D_x f_\mu \circ f_\mu(x_\pm) = 1 - 2\mu + O(\mu^{3/2})$

$\mu < 0$ $\quad$ cycle repelling

Bifurkationen

$f_\mu : \mathbb{R} \to \mathbb{R}$ $\quad C^2$
$f_\mu(0) = 0 \quad f_\mu'(0) = 1$
$D_\mu f(0,0) \neq 0 \quad (\Rightarrow \frac{d\mu}{dx} = 0)$
$D_x^2 f(0,0) \neq 0 \quad (\Rightarrow \frac{d^2\mu}{dx^2} > 0)$

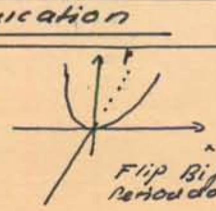


$\exists$ Intervalle $(\mu_1, 0)$, $(0, \mu_2)$ $\quad \varepsilon > 0$
$\mu \in (\mu_1, 0)$ $\quad f_\mu$ hat kein Fixpunkt in $(-\varepsilon, \varepsilon)$
$\mu \in (0, \mu_2)$ $\quad f_\mu$ hat zwei Fixpunkte x_+, x_- einer attraktierend, der andere repellierend

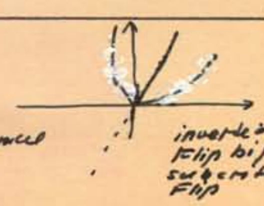
Saddle-Node- or Fold Bifurcation

$f_\mu : \mathbb{R} \to \mathbb{R}$ $\quad C^3$
$f_\mu(0) = 0 \quad f_\mu'(0) = -1 \quad (\Rightarrow D(f_\mu \circ f_\mu) = 0)$
$\qquad D^2(f_\mu \circ f_\mu) = 0$
$D_x^3(f_\mu \circ f_\mu) \neq 0 \quad D_\mu D_x f_\mu(0) < 0$
$\quad \overset{\shortparallel}{-b}$

$b > 0$



Flip Bifurcation, normal period doubling



inverted Flip bifurcat. subcritical Flip

$\exists$ Intervalle $(\mu_1, 0), (0, \mu_2)$ $\quad \varepsilon > 0$
$\mu \in (\mu_1, 0)$ $\quad f_\mu$ hat einen stabilen Fixp. in $(-\varepsilon, \varepsilon)$
$\mu \in (0, \mu_2)$ $\quad f_\mu$ hat einen stabilen Orbit und einen unstabilen Fixpunkt.

In many dimensions: $f_\mu(0)=0$. At $\mu=0$ Df_0 has -1 in its spectrum, which is a simple eigenvalue, rest of the spectrum strictly inside unitcircle

Eigenvalue $\lambda_\mu = -1$ at $\mu=0$
Assume $\frac{d}{d\mu}\lambda_\mu > 0$ at $\mu=0$

Then there is an analytic function $b=b(f_0)$ of Df_0, D^2f_0, D^3f_0 such that if $b>0$

we have for $b>0$ normal bifurcation
$\mu>0$: attracting 2 cycle $x_\pm(\mu)$ and $x_\pm(\mu)$ smooth function of $\sqrt{\mu}=\sigma$ with non vanishing derivat. at 0:

$$x_\pm(\mu) = y_\pm\sqrt{\mu} \qquad y_\pm \neq 0 \text{ vectors}$$

Every orbit starting near zero converges to this cycle $x^\pm$ excepting the fixed point itself.
For $\mu<0$ every orbit starting near zero converges to 0.

If $b<0$ we have a inverted bifurcation
$\mu<0$ cycle of period two is hyperbolic (as fixed point for f^2) with 1-dim unstable manifold
For small positive μ every orbit starting near zero is pushed away from 0
(excepting orbit living on the codim 1 unstable manifold)

- More general: the condition "strictly inside unitcircle" can be replaced by "There is no spectrum other than -1 on the unit circle." If $b\neq 0$ as μ passes through 0, ~~either~~ a periodic circle of period two is either emitted or absorbed by the fixed point.

- look at the normal bifurcation $\mu>0$. The fixed point at 0 has a 1 dim unstable manifold.

$x_\pm$ points of the attract. circle

because in 1 dim the unstable manifold is (x_-, x_+)

More global geometric view

First the 1 dim version:

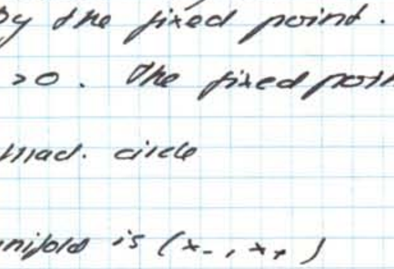
Look at the Banachspace $\{C^3([-1,1]) \quad , f(0)=0\} = X$

$\Sigma := \{f\in X \quad f'(0)=-1 \quad (f\circ f)'''(0)<0\}$ is an open subset of a hyperplane, a smooth codim 1 submanifold

bifurcation surface for the normal period doubl. bifurcation

f_μ is a curve in X

A normal period doubling bifurcation is a curve crossing Σ.



For any $f_0\in\Sigma$ there are:

- a nbhd U of f_0 in X
- a nbhd V of 0 in $\mathbb{R}$

such that if $f\in U$, $f'(0)\geq -1$ then every orbit for f starting in V converges to 0.

and if $f\in U$ $f'(0)<-1$ f admits an attracting cycle of period 2 $x_\pm(f)$. For any $x\in V$ $x\neq 0$ $f^n(x)\to\{x_\pm f\}$

the size of cycle is $O(\underbrace{\sqrt{-f'(0)-1}}_{\lambda(f)})$

$\frac{x_\pm(f)}{\sqrt{\lambda(f)}}$ is a continuous non vanishing function on $\{f \in U \ \ \lambda(f) \geq 0\}$

what's about the smoothness of the dependence of $x_\pm$ on $\sqrt{\mu}$?

$f \in X$ is determined by $\underset{\mathbb{R}}{f'(0)}$, $\underset{C^{n-1}}{f''(x)}$. (coordinates in X)

$\lambda := -f'(0)-1$

λ and $f''(x)$ are coord. in X for f.

$x_\pm = X(\sqrt{\lambda}, f'')$ $\quad$ X is C^{n-2} $\quad$ (to check)

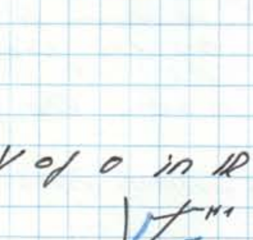
what happens if we drop the nondegeneracy condition $(f_0 \circ f_0)'''(0) \neq 0$.

general argument: $\boxed{\text{Passage of } f_\mu'(0) \text{ through } -1 \text{ produces period 2 cycles. (independent of the nondegen. condition)}}$

$f_\mu(x)$, $f_\mu'(x)$ vary continuously with μ and x.

$f_\mu(0) = 0$;
$f_{\mu_0}'(0) > -1$
$f_{\mu_1}'(0) < -1$

there is a nbhd V of 0 in $\mathbb{R}$, such that $\frac{f_{\mu_0} \circ f_{\mu_0}(x)}{x} < 1$ for $x \in V$ $x \neq 0$

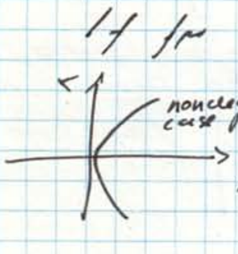


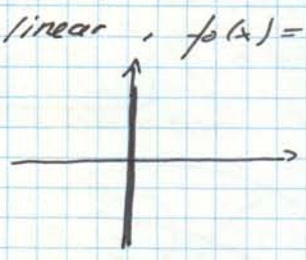
and $\frac{f_{\mu_1} \circ f_{\mu_1}(x)}{x} > 1$ for $x \in V$ $x \neq 0$

For any nonzero $x \in V$ $\exists$ some μ s.that x is a fixed point for $f_\mu \circ f_\mu$ and therefore a periodic pt of period 2 for f_μ itself.

Conclusion: at least a ~~continuous~~ uncountable family of period 2 cycles associated with the transition, organized nicely if $(f_0 \circ f_0)'''(0) \neq 0$

If f_μ is linear, $f_0(x) = -x$ every pt periodic





Suppose $f_\mu(x)$ is jointly C^2

$\lambda_\mu = \sqrt{-f_\mu'(0)-1}$

$f_\mu \circ f_\mu(x) = (1+\lambda_\mu)x + \beta_2(\mu,x)x^2$

Fixed point equation:

$\lambda_\mu = \beta_2(\mu,x)x$

If $\frac{d}{d\mu}\lambda_\mu > 0$, then we can solve for μ as a function of x.

μ is a C^{n-1} fn of x

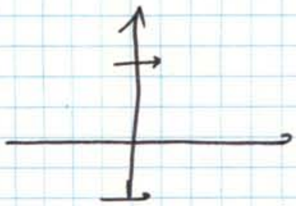
Question: $\boxed{\text{with what generality is it true that cycles with } \mu>0 \text{ are attracting and } \mu<0 \text{ are repelling?}}$

Bifurcations in differential equations

$$\frac{dx}{dt} = X_\mu(x)$$

$\bar{x}$ stationary solution $X_{\bar\mu}(\bar{x}) = 0$
can be continued in the parameter, provide $0 \notin \sigma(DX_\mu(\bar{x}))$
stationary sol'n attracting, if the spectrum of $DX_\mu(\bar{x})$ is in left half plane.

Non degenerate ways to lose attractivity:



- An eigenvalue can go to 0 (saddle node bifurcation)
- complex conjugate pair of eigenvalues passes through the imagin. axis at $e^{\pm i\phi}$ $\phi > 0$

As by the period doubling bifurcations there are two nondegenerate possibilities

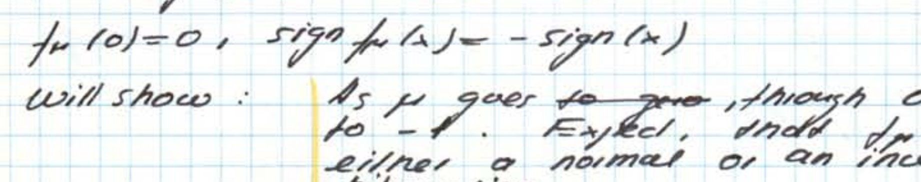
Normal Hopf bifurcation – Stationary sol'n emits an attracting periodic solution

inverted Hopf bifurcation – Stationary sol'n absorbs a non attracting periodic solution.

experiment:

Start as usual with the lowest dimens. situation: 2 dim.
Stationary solut. at 0. Eigenvalues of $DX_\mu(0) = \lambda(\mu) \pm i\phi(\mu)$
$\lambda(0) = 0$ $\phi(\mu) > 0$

Make a linear change of variables so that the linearized flow is at $\mu = 0$ is a rotation



Let $f_\mu(x_1)$ be the first return map to sol'n flow to the x_1 axis. (not only posit. x achse)

$f_\mu(0) = 0$, $\operatorname{sign} f_\mu(x) = -\operatorname{sign}(x)$

Will show: As μ goes ~~to zero~~, through 0, $f_\mu'(0)$ goes to -1. Expect, that f_μ will undergo either a normal or an inverted period doubling bifurcation.
Pt of period 2 for f_μ = intersect. of x_1 axis with periodic orbit for $\dot{x} = X_\mu(x)$
the orbit is attracting for $\dot{x} = X_\mu(x)$ iff the cycle is attracting for f_μ.

History: Ideas of Poincaré, Hopf, Andronov

Problems / Proofs:

a) Find regularity conditions on X_μ which guarantee $f_\mu(x)$ is jointly C^3.

b) Show that $f_\mu'(0)$ passes through -1 when $\lambda(\mu)$ passes through 0

c) Show that $(f_0 \circ f_0)'''(0)$ is generally non zero.

(a) Go to polar coordinates

$$x_1 = r\cos\theta \qquad \frac{dr}{dt} = R(r,\theta) \qquad R = \cos\theta\, X_1(r\cos\theta, r\sin\theta) + \sin\theta\, X_2(r\cos\theta, r\sin\theta)$$
$$x_2 = r\cdot\sin\theta \qquad \frac{d\theta}{dt} = \Theta(r,\theta) \qquad \Theta = \frac{1}{r}(\sin\theta X_1 - \cos\theta X_2)$$

If $X_\mu(x_1,x_2) \in C^n$ then $R_\mu(r,\theta) \in C^n$
$\Theta_\mu(r,\theta) \in C^{n-1}$ ← cheap way not to loose the derivat.

Meaning: • Given any solution of the r,θ equal., if I construct corresponding $x_1(t), x_2(t)$, satisfies orig. equal. whatever $r(t)$ does.

• $R_\mu(0,\theta) = 0$ because $x_1 = \lambda_1$ in origin
$\Theta_0(0,\theta) = \phi(0)$

If μ, r are small enough, the sol'n to $r,\theta)$ eqn. with $r(0)=r$ $\theta(0)=0$ is approxim $r=0, \theta = t\phi(0)$ which will have $\theta'(t) > 0$ up to at where $\theta(t) > \pi$
Can find therefor $t(\mu,r)$ = first t, such that this solution has $\phi(t) = \pi$ for $r>0, r<0, r=0$.
this $t(\mu,r) \in C^{n-1}$

$f_\mu(r) \equiv r(t(\mu,r))$

that means $f_\mu(r)$ is jointly C^{n-1}

$X_\mu(x) \in C^4$ is good enough.

(b) $f_\mu(0)=0$. Compute $f_\mu'(0)$: ~~$f_\mu(0)$~~

compute $f_\mu(r)$ to first order in r

$t_\mu(r) \simeq \dfrac{\pi}{\phi(\mu)}$ $\qquad r(t) \approx r_0 e^{+t\cdot\lambda(\mu)}$

$f_\mu(r) \approx -r e^{+\frac{\lambda(\mu)}{\phi(\mu)}\pi}$

$f_\mu'(0) = -e^{\lambda(\mu)\pi/\phi(\mu)}$ does indeed go through -1 as $\lambda(\mu)$ goes through 0.

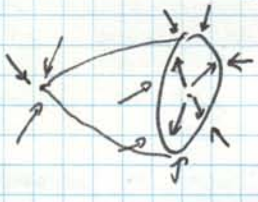
(c) $(f_0 \circ f_0)'''(0)$ is some fixed analyt. function of $DX_0(0), D^2X_0(0), D^3X_0(0)$

it's either identicly 0 or generically not 0.
Have therefor only to show that it's not identically 0.
Just need an example: $\dfrac{dx_1}{dt} = (x_1^2+x_2^2)x_1 + \phi x_2$

$$\frac{dx_2}{dt} = (x_1^2+x_2^2)x_2 + \phi x_1$$

$$\frac{d\phi}{dt} = \phi$$
$$\frac{dr}{dt} = r^3$$

Many dim situation: Most interesting: Rest of spectrum stays to the left.
Normal bifurc. → attract. period. orbit. attracting everything except a codim 2 manifold
unstable manifold of fixedpt is a little disc bounded by the periodic orbit.



inverted bifurcation. Periodic orbit hyperbolic with twodim unstable manifold

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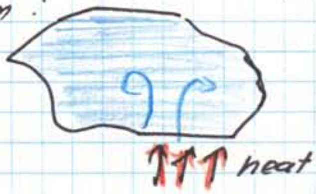
Bifurcationsurface picture: codim 1 surface in space of all C^4 differential equations with station. solutions at the origin.
Remove non degen. hypothesis: non possib.

It's easy to recognise a hopf bifurcation in a physical complex system without being able to calculate the transition explicitly.

Experim:



... oscillations ... hopf bifurcations

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f_μ $f_\mu(0)=0$. At $\mu=0$ a complex conj. pair of non real eigenvalues passes through unit circle at $e^{\pm i\phi}$ $\phi\in(0,\pi)$

Consider 2dim situation:

Before the transition:

After transition:

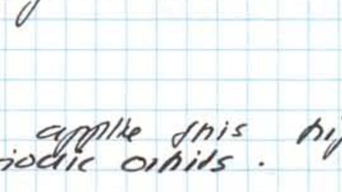
Analysis shows: 2 possible nondegenerate possibilitys. a normal and a inverted version of the same bifurcation.

In the normal situation, the mapping aquires for $\mu>0$ an invariant attracting circle

Analysis is harder because the invariant circle is not a single orbit anymore. Finding an invariant circle entails solving a funktional equation (insted of a lowdim fixed point problem)

History: Announced by Naimark (1959)
R. Sacker (1964) unpublished
und J. Mose
Ruelle & Takens (1971) rediscovered.

Can apply this bifurcation to Poincaré maps for periodic orbits.



invariant circle for the return map.
$\leadsto$ invariant twodim. torus for the flow.

The motion on the torus is complicated, but for at least some parameter values, can introduce coord. so that flow is const. velocity flow.

$$\frac{d\theta_1}{dt} = \omega_1$$
$$\frac{d\theta_2}{dt} = \omega_2$$

$\rightarrow$ orbits starting near the torus are asymptotically doubly periodic. 1 period $\rightsquigarrow$ 2 period

corresponding two the rotation number infinitly many bifurcations in

---

$f_\mu : \mathbb{R}^2 \to \mathbb{R}^2$
$f_\mu(0) = 0$

Assume eigenvalues go through unit circle at $\mu = 0$ with non vanishing speed.
reparametrize, so that the spectrum of $Df_\mu(0) = (1+\mu)e^{\pm i\phi(\mu)}$

Make a μ depend. coord. change, so that

$$Df_\mu(0) = (1+\mu)\begin{pmatrix} \cos\phi(\mu) & \sin\phi(\mu) \\ \sin\phi(\mu) & \cos\phi(\mu) \end{pmatrix} = \Lambda_\mu \quad 0 < \phi(0) < \pi$$

Linearize to finite degree f_μ.

Do this in a way, which is smooth in μ.
At $\mu = 0$ there are resonances. $\Rightarrow$ some terms can't be transformed away.

Resonances : $\lambda_0 = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ $\quad$ λ eigenvalues $n \geq 2$

$n = 2$ $\quad$ $\mu = 0$ resonances impossible if not

$$e^{i\phi(0)} = e^{-2i\phi(0)} \qquad \phi(0) = \frac{2\pi}{3}$$

rule this out.

$n = 3$ $\quad$ $e^{i\phi(0)} = e^{i\phi(0)} e^{i\phi(0)} e^{-i\phi(0)}$ $\leftarrow$ unavoidable

assume $\phi(0) \neq \frac{\pi}{2}$ $\quad$ i.e. $e^{ki\phi(0)} \neq 1$

complex notation: $\quad v = \left(\frac{1}{2}, -\frac{i}{2}\right) \in \mathbb{C}^2$

eigenvector for $Df_\mu(0)$ with eigenvalue $\lambda_\mu = (1+\mu)e^{i\phi(\mu)}$

other eigenvector : $\bar{v}$

$av + b\bar{v} \in \mathbb{R}^2 \iff b = \bar{a}$

$(x_1, x_2) \in \mathbb{R}^2 = zv + \bar{z}\bar{v}$ $\qquad z = x_1 + ix_2$
general point

a general polynomial (complex valued) homogen. of degree n

$$\sum_{j=0}^{n} a_j z^j \bar{z}^{n-j}$$

ϕ homogenious of degree n : $\mathbb{R}^2 \to \mathbb{R}^2$
is specified by uniquely by

$$\sum_{j=0}^{n} a_j z^j \bar{z}^{n-j}$$

general mapping: $\quad z \to z' = \sum_{j=0}^{n} a_j z^j \bar{z}^{n-j}$
arbitrary homog.

$Df_\mu^{-1}\, \phi\, (Df_\mu)^n$ transforms

$$\phi \longrightarrow \sum_{j=0}^{n} \lambda_\mu^{j-1} \bar{\lambda}_\mu^{n-j} a_j z^j \bar{z}^{(n-j)}$$

Terms which can be elimin. are those, for which $\lambda^{j-1}\bar{\lambda}^{n-j} \neq 1$

$n=3$: $a_2 |z|^2 z$

$$\phi(z) = \sum_{j=0}^{n} a_j z^j \bar{z}^{n-j}$$

$$\Lambda_\mu^{-1}\phi(\Lambda_\mu z) = \sum_{j=0}^{n-1} \lambda_\mu^{j-1}\bar{\lambda}_\mu^{n-j} a_j z^j \bar{z}^{n-j}$$

and can elemin. terms for which $\lambda^{j-1}\bar{\lambda}^{n-j} \neq 1$

Non resonance conditions $\cdot\ \lambda_0^3 \neq 1 \longrightarrow$ elim all 2 order terms
$\cdot\ \lambda_0^4 \neq 1 \longrightarrow$ elim 3 order terms except $j-1=3-j \quad j=2$

can arrive at $\phi_\mu \circ f_\mu(z) = \lambda_\mu \phi_\mu(z) + a_\mu |z|^2 z + O(|z|^4)$
$\phi_\mu(z) = z + O(z^2)$

Replace z by $\phi(z)$

$$\phi_\mu \circ f_\mu \circ \phi_\mu^{-1}(z) = \lambda_\mu z + a_\mu |z|^2 z + O(|z|^4)$$

Polar coordinates : $(r,\theta) \longmapsto (r',\theta')$ C^4 fund. vanishing to 4th order for $r \to 0$ uniform in θ and μ

$r' = (1+\mu) r \,|1+\tilde{a}_\mu r^2| + O(r^4) \qquad = (1+\mu)r - br^3 + O(r^4)$

$\theta' = \theta + \phi_\mu + \arg(1+\tilde{a}_\mu r^2) + O(r^3) = \theta + \phi + cr^2 + O(r^3)$

Assume $b>0$ (else if $b<0$ replace f by f^{-1}) ($b=0$ nondegen. conditi)

look for a fixed point of $r' = (1+\mu)r - br^3$ which is an invariant circle for the simplified mapping.

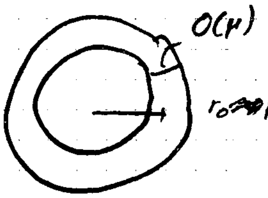
$r_0 = \sqrt{\frac{\mu}{b}}$ is a fixed point $\rightarrow$ $r' = (1+\mu)r - br^3 + O(r^4)$ has inv. circle. attracting
because $\frac{d}{dr}\left((1+\mu)r - br^3\right)\Big|_{r=r_0} = 1-2\mu < 1$

Problem:
Strength of attract. goes to 0 as μ goes to 0

claims : 1) There are constants A, R_0 such that for sufficiently small μ and $0<r<r_0-A\mu$

$0 < r < r_0 - A\mu \implies r' > r$
$r_0 + A\mu < r < R_0 \implies r' < r$

Hence the annulus $r_0 - A\mu < r \leq r_0 + A\mu$ attracts all orbits on $\{0 < r \in R_0\}$ and is mapped onto itself.



Replace r by y where $r = r_0 + A\mu y \qquad -1 \leq y \leq 1$ and write the mapping as $F = F_\mu$

Outline : Show that F maps graphs of Lipschitz fns $\{y = w(\theta)\}$ to graphs of Lipschitz fns, and define an operator $\tilde{F}$ on the space of Lipschitz cont. fund. $\underset{\text{graph}}{\Gamma}(\tilde{F}w) = F\Gamma(w)$

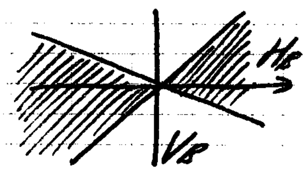
Show then, that $\tilde{F}$ is contractive in the sup norm. Here so Lipschitz inv circle which is smooth.

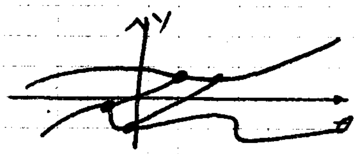
$$DF_\mu(\theta, y) = \begin{pmatrix} 1 + O(\mu^{3/2}) & O(\mu^{3/2}) \\ O(\mu) & 1 - 2\mu + O(\mu^{3/2}) \end{pmatrix}$$

Expanding, attracting wedges:

$$H_B = \{ (v_\theta, v_y) : |v_y| \le B |v_\theta| \}$$

$$V_B = \{ (v_\theta, v_y) \ |v_\theta| \le B^{-1} |v_y| \}$$





Need to argue: $z_1 - z_2 \in H_B$ then $F(z_1) - F(z_2) \in H_B$

suffices even: DF maps H_B to H_B preserving the sign of the θ component for all relevant (μ, θ, y)

$$v_\theta' = (1 + O(\mu^{3/2}))\, v_\theta + O(\mu^{3/2})\, v_y$$
$$v_y' = O(\mu)\, v_\theta + (1 - 2\mu + O(\mu^{3/2}))\, v_y \quad \dots$$

$$(O(\mu) + (1 - 2\mu + O(\mu^{3/2}))\, B)\, v_\theta \le B(1 + O(\mu^{3/2}) - B O(\mu^{3/2}))$$

$\to$ any sufficiently large B makes this inequality hold for all sufficiently small μ.

Pick some such B, say B_1, work in space of $\omega : \theta \to y$ with $|\omega(\theta)| \le 1$ ~~coo~~ where

$$\mathrm{Lip}(\omega) \le B$$

And so $\mathcal{F}$ is well defined.

Contractivity. Pick special norm equivalent to supremum norm

$$\|\omega_1 - \omega_2\| = \sup \{ |z_1 - z_2|_y \mid z_1 \in \Gamma(\omega_1),\ z_2 \in \Gamma(\omega_2),\ z_1 - z_2 \in V_{B_2} \}$$

$\wedge$ bigger than sup norm; V_{B_2} — to be chosen

$$\mathrm{Lip}(\omega_i) \le B_1 \implies \| \quad \| \le \left(\frac{B}{B_2 - B_1} \right) \| \quad \|_{\sup}$$

By taking B_2 large enough, ~~can~~ and μ small enough, can guarantee, that in V_{B_2}

$$\left(DF \begin{pmatrix} v_\theta \\ v_y \end{pmatrix} \right)_y \le (1 - \mu) |v_y|$$

DF contracts y components of elements of V_{B_2} by at least $1 - \mu$.

Can also arrange: $DF : H_{B_2} \to H_{B_2}$

$$DF^{-1} : CH_{B_2} \to CH_{B_2}$$
$$: V_{B_2} \to V_{B_2} \quad \text{by continuity}$$

$$\|\mathcal{F}\omega_1 - \mathcal{F}\omega_2\| = \sup \{ |F(z_1) - F(z_2)|_y \mid z_1 \in \Gamma(\omega_1),\ z_2 \in \Gamma(\omega_2),\ \underbrace{F(z_1) - F(z_2) \in V_{B_2}}_{\to\ z_1 - z_2 \in V_{B_2}} \}$$

$$\le \sup \{ |F(z_1) - F(z_2)|_y \mid \dots z_1 - z_2 \in V_{B_2} \}$$

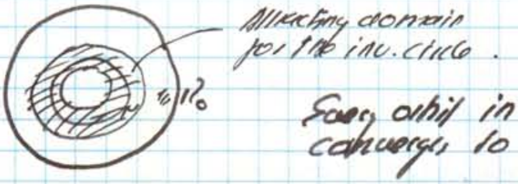
$$\left(z_1 - z_2 \in V_{B_2} \implies |F(z_1) - F(z_2)|_y \le (1 - \mu)\, (z_1 - z_2)_y \right)$$

$$\le (1 - \mu) \|\omega_1 - \omega_2\|$$

$\to \exists!$ Lipschitz cont. circle

■

Remark :



Every orbit in the annulus converges to the invariant circle.

Poincaré - Bendixson

Def ω limit sets :

Assume $\{x \mid \{f^n(x)\}_{n \le m} \text{ defined}\}$ relatively compact in D_f

$\omega(x,f)$ = set of limit points of converg. sequ. $f^n(x)$

$= \bigcap_{m \le n} \overline{f^m O^+(x,f)}$ is compakt non empty

$f^n(x) \xrightarrow[n\to\infty]{} \omega(x,f)$

for flows : $O^+(x,f) = \{f^t(x) \mid t \ge 0\}$

$\omega(x,f^t) = \bigcap_{s \to \infty} \overline{f^s O^+(x,f^t)}$ is connected

$L(f) = \overline{\bigcup_x \{\omega(x,f)\}}$

Every orbit converges to $L(f)$, which may be much smaller (Ex. Henon mapping) but complicated.

Theorem Poincaré Bendixson

Let $X(x)$ be a differential equation defined on an open set of $\mathbb{R}^2$. Let x_0 be a point such that $f^t(x_0)$ is defined for all t and relatively compakt :

If $\omega(x_0,f^t)$ contains no stationary solution, then it reduces to a single periodic orbit to which $f^t(x_0)$ converges

Terminology : A transverse arc : $\gamma \in D(X)$ such that $X(x)$ is transverse to γ at each point of $x \in \gamma$.

Lemma : $x(t)$ solut. curve , γ transverse arc .

The successive crossings of γ by $x(t)$ form a monotone sequence on γ.

Γ

Make a closed curve . $\{x(t) : t_0 \le t \le t_1 \cup \text{segment}\}$ This is a Jordan curve and seperates therefore the plane in an inside and an outside .

$x(t)$ $t = t_1 + \varepsilon$ } are of opposite side

$\Rightarrow$ $x(t)$ comes back next must be to the right of $x(t_1)$ $\square$

Cor There can't be more than one point of $\omega(x(t))$ on any transverse arc

Cor Any $y_0 \in \omega(x_0) \cap \gamma$ must be a limit of crossing point limit of crossing points has only one accumul. point

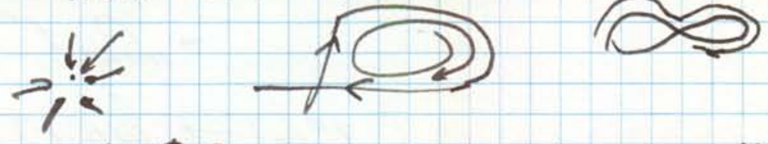
$y_0 \in \omega(x_0)$ $\omega(x_0)$ compact.
$f(y_0)$ defined and stays in $\omega(x_0)$ (because $\omega(x_0)$ is for all t compact)

$f^+(y_0)$ is relativ compact and has a ω limitset $\omega(y_0) \subseteq \omega(x_0)$

$y_1 \in \omega(y_0)$, $X(y_1) \neq 0$ $\Rightarrow$ can draw a little transverse arc γ through y_1. ~~the only~~ y_1 is the only point of γ which is also in $\omega(x_0)$. Therefore $f(y_0)$ must cross γ infinitly often. but it can only cross it in one point because $f^+(y_0) \subset \omega(x_0)$)
$\Rightarrow f^+(y_0)$ is periodic. can find t_j ? $x(t_j) \to y_0$
$\Rightarrow$ $x(t)$ asympt. period. sol. ∎

Remarks: • 'There is some orbit in $\omega(x_0)$ which is not asympt. to a stationary solution.' can replace "there is no station. solut"

also sit.



• Extensions of P.B to general two manifolds (Hartmann, $\overline{VII}$ 12.

• Constant veloc. ination flow on the torus has no circle as ω limitset. $\omega(x) = T^2$ $\forall x$

General principle: • Interesting dynamics can happen for non invert. mappings in d dim, $d \geq 1$ for invert. mappings in ? $d \geq 2$ (Hénon mapp)

• For differential equat. $d \geq 3$ (Lorenz system)

Hyperbolic sets

Idea: Study phenomena of exponential seperation of nerby orbits. (Technically strong version) One has sensit. depend. on initial conditions. That leads to appearantly stochasticy.

Exampels: Hyperbolic torus automorph: Arnolds cat $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$: $T^2 \to T^2$
$\lambda = \frac{3-\sqrt{5}}{2} < 1$
ϕ_s, ϕ_u correspond. eigenvectors

distance: minim dist. between a pair of repr. on $\mathbb{R}^2$
Look at $F^n z$, $F^n z'$ z, z' so that $\|z - z'\| \ll 1$

Expand $z - z'$ as linear combin of ϕ_s and ϕ_u
If non zero ϕ_u comp. $\|F^n z - F^n z'\|$ grows exponen like λ^{-n}

If ~~in~~ $z - z'$ is in direct. of ϕ_s $\|F^n z - F^n z'\| \to 0$ exponentialy.

$\eta > 0$ fixed
$W_\eta^s(z) = \{ z' \,|\, \|f^n(z) - f^n(z')\| \leq \eta \ \ \forall n \geq 0 \}$

$=$ a little segment of straight line in the direct of ϕ_s

Backward orbits for nearby orbits also diverge exponentially except in the direct. of ϕ_u

$W_\eta^u(z) = \{ z' \,|\, \|f^{-n}(z) - f^{-n}(z')\| \leq \eta \ \ \forall n \geq 0 \}$

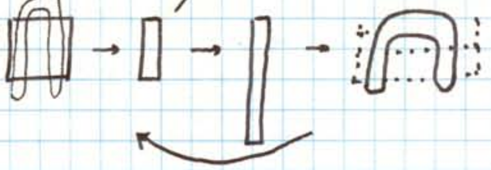
$W_\eta^s(z) \cap W_\eta^u(z) = \{z\}$
$\Rightarrow$ No orbit other than z stays near z in $n>0$ $n<0$.

Tangent spaces of W_η^u and W_η^s are complement.
Periodic orbits are dense. and can be counted by combinat. consideration.

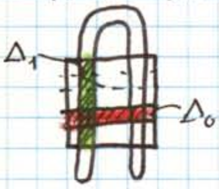
System is structurally stable.

The Smale Horseshoe

f : unit square $\to \mathbb{R}^2$



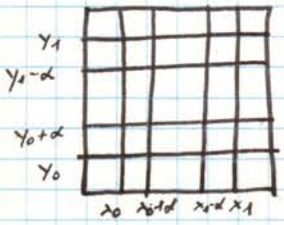
What does the set of orbits, which lie in the unit square Δ for all time look like.



Assume f is exactly linear on each Δ_0, Δ_1

$z \longmapsto M_i z + z_i$ on Δ_i $\quad i = 1, 2$

Pick $\alpha \in (0, \frac{1}{2})$ contraction factor, α^{-1} expansion factor,
Pick $0 < x_0 < x_0 + \alpha < \frac{1}{2} < x_1 - \alpha < x_1 < 1$



$\Delta_0 = \{(x,y) \mid y_0 \le y \le y_0 + \alpha\}$

$\Delta_1 = \{(x,y) \mid y_1 \le y \le y_1 + \alpha\}$

f on Δ_0 : $\quad (x,y) \longmapsto (x_0 + \alpha x, \alpha^{-1}(y - y_0))$
f on Δ_1 : $\quad (x,y) \longmapsto (x_1 - \alpha x, \alpha^{-1}(y_1 - y))$

Look at a z for which $f^n(z)$ is defined : $f(z), f^2(z) \dots f^{n-1}(z)$ is in the unit square Δ

$\Longrightarrow z \in \Delta_0 \vee \Delta_1$
$\quad z \in \Delta_{i_0}$ $\quad i_0$ unique

and $f(z) \in \Delta_{i_1} \quad \dots f^k(z) \in \Delta_{i_k}$

$\Delta^{(0,n)}_{i_0 \dots i_n} = \{z \mid f^j(z) \in \Delta_{i_j} \quad 0 \le j \le n\} \qquad i_0 \dots i_n \in \{0,1\}^n$

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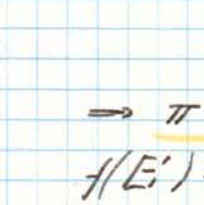

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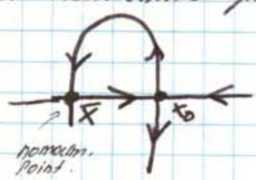

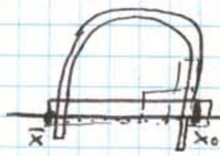

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$\to$ Angle between $E^s(x)$ and $E^u(x)$ is bounded away from zero.

By uniqueness: $\hat{h}(f(z_0)) = \hat{f}(\hat{h}(z_0))$

$\hat{h} \circ f = \hat{f} \circ \hat{h}$ on Λ

Can show: $\hat{h}$ is continuous, injective $\therefore$ A homöom. onto $\hat{\Lambda}$

$\hat{f}|_{\hat{\Lambda}} = \hat{h} \circ f|_{\Lambda} \circ \hat{h}^{-1}$

__Theorem__

Structural stability of hyperbolic sets

> Let Λ be a hyperbolic set for the C^1 diffeom f. $\hat{f}$ suff. near f in C^1 topol.
> $\exists \hat{h}$ on Λ onto a compact set $\hat{\Lambda}$ invariant for $\hat{f}$ such that $\hat{f} = \hat{h} \circ f \circ \hat{h}^{-1}$ on $\hat{\Lambda}$
> $\hat{h}$ can be made as close to $\mathbb{1}$ as we desire.

__Rem__: $\hat{h}$ not diff even in the best cases.

Argument:

Λ contains some ~~pts~~ period. pt x_0

$\hat{h}(x_0)$ is a periodic pt for $\hat{f}$

if $\hat{h}$ where differentiable at x_0

then the spectrum of $\sigma(D\hat{f}^p(\hat{x}_0))$

$= \sigma(Df^p(x_0))$ is independent by chain rule..

which can be distorted by changing f

Γ only: $\hat{h}$ is cont. and inj.

(i) • injecl. follows from expansivity

(ii) • continuity follows from uniqueness of shad orbit + compactness

(i) • Take ε small enough.

Any two distinct f orbits on Λ are somewhere separated by more than ~~than~~ 2ε.

Take two dist. pts $x_0 \neq x_0'$ in Λ

Find j so that $\| f^j(x_0) - f^j(x_0') \| > 2\varepsilon$

$\| f^j(x_0') - \hat{f}^j(\hat{h}(x_0')) \| \le \varepsilon$

$\| \hat{f}^j(\hat{h}(x_0)) - \hat{f}^j(\hat{h}(x_0')) \| > 0 \Rightarrow \hat{h}(x_0) \neq \hat{h}(x_0')$

(ii) • continuity $\left.\begin{array}{l} x_n \to \bar{x} \\ \hat{h}(x_n) \to z \end{array}\right\} \Rightarrow z = \hat{h}(\bar{x})$

$\| f^j(x_n) - \hat{f}^j(\hat{h}(x_n)) \| \le \varepsilon$

$\| f^j(\bar{x}) - \hat{f}^j(z) \| \le \varepsilon \quad \forall j$

$\rightarrow z = \hat{h}(\bar{x})$ by uniqueness of shadowing $\lrcorner$ ■

__Def__: $f: M \to M$ is called __structural stable__ (with resp. to some topologie), if for every $\hat{f}$ near f there is a homöom. $\hat{h}$ with $\hat{f} = \hat{h} \circ f \circ \hat{h}^{-1}$

__Corollary__

> Suppose f is a C^1 Anosov diffeom of M, then for any $\hat{f}$ near f in C^1 Topologie, there is a homöomorphism $\hat{h}: M \to M$ so that $\hat{f} = \hat{h} \circ f \circ \hat{h}^{-1}$

Γ Still to show: $\hat{h} \circ f$ maps M onto M.

$h(x_0) = y_0$

Shadowing: For each $y_0 \in M$ $\exists x_0$ so that $\| f^n(x_0) - \hat{f}^n(y_0) \| \le \varepsilon$ because, if $\hat{f}$ near enough f in C^0 make $\hat{f}^n(y_0)$ δ pseudoorbit for f with δ as small as desired uniformly in y_0.

$\lrcorner$

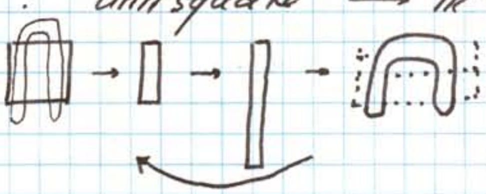
$W^s_\eta(z) \cap W^u_\eta(z) = \{z\}$
$\Rightarrow$ No orbit other than z stays near z in $n \geq 0$ $n \leq 0$.

Tangent spaces of W^s_η and W^u_η are complement.
Periodic orbits are dense. and can be counted by combinat consideration.

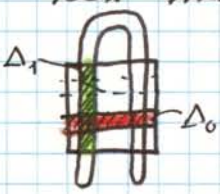
system is structurally stable.

The Smale Horseshoe

f : unit square $\longrightarrow \mathbb{R}^2$



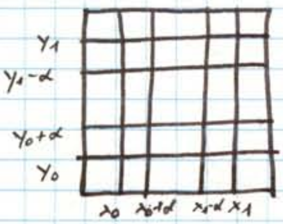
What does the set of orbits, which lie in the unit square Δ for all time look like.



Assume f is exactly linear on each Δ_0, Δ_1

$z \longmapsto M_i z + z_i$ on Δ_i $\quad i = 1,2$

Pick $\alpha \in (0, \frac{1}{2})$ contraction factor, α^{-1} expansion factor,
Pick $0 < x_0 < x_0 + \alpha < \frac{1}{2} < x_1 - \alpha < x_1 < 1$



$\Delta_0 = \{(x,y) \mid y_0 \leq y \leq y_0 + \alpha\}$

$\Delta_1 = \{(x,y) \mid y_1 \leq y \leq y_1 + \alpha\}$

f on Δ_0 : $\quad (x,y) \longmapsto (x_0 + \alpha x, \alpha^{-1}(y - y_0))$

f on Δ_1 : $\quad (x,y) \longmapsto (x_1 - \alpha x, \alpha^{-1}(y_1 - y))$

Look at a z for which $f^n(z)$ is defined : $f(z), f^2(z) \ldots f^{n-1}(z)$ is in the unit square Δ

$\Rightarrow$ $z \in \Delta_0 \cup \Delta_1$
$\quad z \in \Delta_{i_0}$ $\quad i_0$ unique

and $f(z) \in \Delta_{i_1} \ldots f^k(z) \in \Delta_{i_k}$

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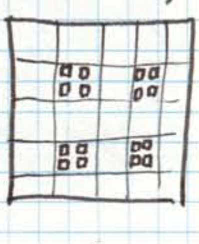

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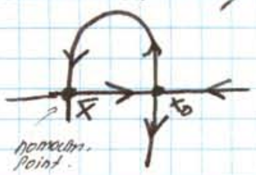

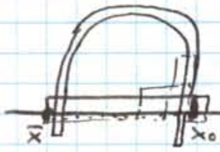

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If the whole space is a hyperb. set, one speaks of Anosov diffeomorphisms. (Ex. Torus autom.)

$\xi \in E^u(x)$
$\eta = Df^n(x)\xi \quad \in E^u(f^n(x))$
$\| Df^{-n} f^n(x)\, \eta \| \le c\lambda^n \|\eta\|$
inv. Funkt. $\quad Df^{-n}(f^n(x)) = [Df^n(x)]^{-1}$
$\| Df^n(x)\xi \| \ge c^{-1}\lambda^{-n} \|\xi\|$

$\xi \notin E^s(x)$ has a nonzero comp. in $E^u(x)$
$\|Df^n(x)\xi\|$ grows exponentially with n.

$\{(x,\xi) : x \in \Lambda\ \xi \in E^s(x)\}$ ← is closed

$\{(x,\xi) : x \in \Lambda\ \ \|Df^n(x)\xi\| \le c\lambda^n \|\xi\| \}$

$\ast$
$x_n \to x$
$\xi_n \to \xi \quad \xi_n \in E^s(x_n)$
$\Rightarrow \xi \in E^s(x)$

Proposition: $E^s(x)$ and $E^u(x)$ vary continously with x for $x \in \Lambda$

$P^s(x)$ projectio onto $E^s(x)$ along $E^u(x)$: Assertion $P^s(x)$ varies continuously.

$\Gamma \quad x_n \to x \implies P^s(x_n) \to P^s(x)$

Suffic
a) $x_n \to x \quad P^s(x_n) \to \hat{P} \implies \hat{P} = P^s(x)$
b) $\|P^s(x)\|$ uniformly bounded in x

a) $(P^s(x_n))^2 = P^s(x)$
$\hat{P}^2 = \hat{P} \quad \hat{P}$ is a proj.
$\xi = \hat{P}\xi = \lim_{n\to\infty} P^s(x_n)\xi \quad \in E^s(x_n)$
$\overset{\ast}{\Rightarrow} \xi \in E^s(x)$
$\Rightarrow \operatorname{Ran}(\hat{P}) \le \operatorname{Ran} P^s(x)$
$1 - P^s(x_n) \to 1 - \hat{P}$
$\Rightarrow \operatorname{Ran}(1-\hat{P}) \le \operatorname{Ran}(1 - P^s(x))$

$\Big\}$ $\operatorname{Ran}\hat{P} = \operatorname{Ran} P^s(x)$ and $\hat{P} = P^s(x)$

b) suppose not.
$x_n \in \Lambda, \xi_n$
$\dfrac{\|P^s(x_n)\xi_n\|}{\|\xi_n\|} \to \infty$ normalize, so that $\|P^s(x_n)\xi_n\| = 1$

$\Rightarrow \|\xi_n\| \to 0$

Can assume $x_n \to x \quad P^s(x_n)\xi_n \to \xi \quad \|\xi\| = 1$

$\ast \quad \xi \in E^s(x)$

$\underbrace{(1 - P^s(x_n))\xi_n}_{\in E^u(x_n)} = \underbrace{\xi_n}_{\to 0} - \underbrace{P^s(x_n)\xi_n}_{\to \xi} \to -\xi$

$\Rightarrow \xi \in E^u(x) \quad$ ↯ to $E^s(x) \oplus E^u(x)$ $\quad \lrcorner$

In general: $x \mapsto P^s(x)$ k-Hölder contin. provided $Df(x)$ is.

Proof: construct operat. for appropriate space of contin. matrix valued fns. Having $P^s(x)$ as a fixed pt. ...

⚠ There exist analytic anosov diffeom, for which $P^s(x)$ is nowhere differentiable.

Remark: $\| Df^n(x)\xi \| \leq c\lambda^n \|\xi\| \qquad \xi \in E^u(x) \quad n \geq 0$

It would be nicer, if one would have $\| Df(x)\xi \| \leq \lambda \|\xi\| \qquad \xi \in E^s(x)$

but coord. dependent now.

But can remove c, by taking a norm depending on the point.

Propos: adapted metric / lyap. metric

Λ hyperbolic set for f. Then there exist a $\lambda < 1$ and a Riemanian metric with assoc. Norm $\| \; \|_x$ on M, such that for all $x \in \Lambda$,

$$\| Df(x)\xi \|_{f(x)} \leq \lambda \|\xi\|_x \qquad \xi \in E^s(x)$$

$$\| Df^{-1}(x)\xi \|_{f^{-1}(x)} \leq \lambda \|\xi\|_x \qquad \xi \in E^u(x)$$

T

Hypothesis: $\| Df^n(x)\xi \| \leq c\lambda_1^n \|\xi\| \quad \xi \in E^s(x)$
$\| Df^{-n}(x)\xi \| \leq c\lambda_1^n \|\xi\| \quad \xi \in E^u(x)$

Choose any λ $\quad \lambda_1 < \lambda < 1$
Choose λ_2 $\quad \lambda_1 < \lambda_2 < \lambda$

$$\langle \xi, \eta \rangle_x = 0 \quad \text{if} \quad \xi \in E^s(x), \eta \in E^u(x)$$

$$\langle \xi, \eta \rangle_x = \sum_{j=0}^{\infty} \frac{(Df^j(x)\xi, Df^j(x)\eta)}{\lambda_2^{2j}} \qquad \xi, \eta \in E^s(x)$$

$$\langle \xi, \eta \rangle_x = \sum_{j=0}^{\infty} \frac{(Df^{-j}(x)\xi, Df^{-j}(x)\eta)}{\lambda_2^{2j}} \qquad \xi, \eta \in E^u(x)$$

Do this only for $x \in \Lambda$.

$\langle \, , \rangle_x$ is defined well and varies continously with x

$$\langle Df(x)\xi, Df(x)\xi \rangle_{f(x)} \leq \lambda_2^2 \langle \xi, \xi \rangle_x \qquad \xi \in E^s(x)$$

$\langle \, , \rangle_x$ is only defined for $x \in \Lambda$ and is only continous!

Extend for this $\langle \, , \rangle_x$ continuously to all of M.

$\langle \, , \rangle_x = (R_x \xi, \eta)$ where $R_x \geq \varepsilon \mathbb{1} \quad \forall x \in \Lambda$

$\times$ $R_x = \varepsilon \mathbb{1} + S_x^* S_x$
with $x \mapsto S_x$ continuously in x

$S_x = \sqrt{R_x - \varepsilon \mathbb{1}}$ $\quad x \mapsto S_x$ continous

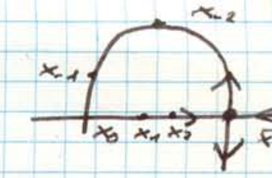
Extend matrix element S by Tietze extension thm.

Use $\times$ to extend R_x.

Approximate by a smooth $\tilde{R}_x$. Use $\tilde{R}_x$ to define $\langle \, , \rangle_x$.

For good enough approximation $\tilde{R}_x$ is an inner product for each x and $(Df\xi, Df\xi) \leq \lambda^2$...

Example $\bar{x}$ hyperbolic fixed point for f.



x_0 : point of transverse intersection of the global stable and unstable manifold.

$\Lambda := \{x_i\}_{i\in\mathbb{Z}} \cup \{\bar{x}\}$ is kompact, strictly invariant

Λ is a hyperbolic set for f.

1. Contracting and expanding vectors.

   At $\bar{x}$: just contr. and expand. vectors for $Df(\bar{x})$

   claim: Set of contr. and exp. vectors for f at x_n is tang space to $W^s(\bar{x})$, $W^u(\bar{x})$ at x_n.

   Suffices to show this for n large. (covariance)
   Reduces to local theory near hyperbolic fixed point.

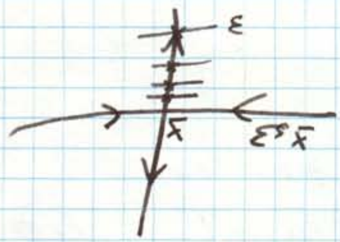
2. uniformity of contraction along homoclinic orbit.

   can ignore what happens at any finite set of x_n
   Reduces to local theory near fixed point.

   organize things so that $Df(\bar{x}) = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix}$ $\quad \|\Lambda_s\| < 1$, $\|\Lambda_u^{-1}\| < 1$

   Need to show: For n either large and positive or large and negative, if ξ is tangent to $W^s(\bar{x})$ at x_n, then $\|Df(x_n)\xi\| \le \lambda \|\xi\|$ $\quad \lambda = \|\Lambda_s\|$

λ lemma : $\quad E :=$ any linear subspace complementary to tang. space to $W^u(\bar{x})$ at x_n



$Df^{-j}(x_n) E = E_j$

$E_j \longrightarrow E^s(\bar{x})$ $\quad$ which is proved in coordinate in which $W^u = E^u$.

$Df(x_n) \approx \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix}$

Shadowing

Def let $\delta > 0$. A δ-pseudo orbit for f means a sequence of z_i in M s.t. $\quad z_i$ index by $\mathbb{Z}$

$$\|f(z_i) - z_{i+1}\| \le \delta \quad \forall i$$

„theorem"
„Shadow theorem"

Λ a hyperbolic set for the C^1 diffeomorphism f and let $\varepsilon > 0$.
Then there exists $\delta > 0$, such that every δ pseudo orbit (z_i) which stays within distance δ of Λ, there is an exact orbit $\{y_i\} = f^i(y_0)$ such that

$$\|y_i - z_i\| \le \varepsilon \quad \forall i$$

Furthermore if ε is small enough, then y_0 is unique

~~Fact~~ Can assume without loss of generality, that pseudoorbit is in Λ than simply near it.

z_i near

z_i' near z_i $z_i' \in \Lambda$. is again a pseudoorbit with a slightly bigger δ.

On the other hand: y_0 may or may not be in Λ..
Delicate question: is every $y_0 \in \Lambda$.

May as well assume, that the pseudo orbit is indexed by all of $\mathbb{Z}$.

Def $\| f^i(y_0) - z_i \| \leq \varepsilon \quad \forall i \quad : f^i(y_0)$ ε-shadows z_i

Corollary: "Expansivity"

$$\boxed{\begin{array}{l}\text{Let } \Lambda \text{ be a hyperbolic set for } f. \\ \text{There exists an } \varepsilon \text{ such that } f^i(x_0) \text{ and } f^i(\bar{x}_0) \\ \text{are exact orbits one of which stays within} \\ \text{distance } \varepsilon \text{ of } \Lambda \text{ for all } i, \text{ then } \|f^i(x_0) - f^i(\bar{x}_0)\| \geq \varepsilon \\ \text{for some } i\end{array}}$$

"expansion" constant

Formal consequ. of uniqueness.

Transverse homoclinic orbit: $\bar{x}$ hyperb. fixed point
$x_n = f^n(x_0)$ transverse homoclinic orbit
$\Lambda = \{x_n, \bar{x}\}$ hyperbolic set.

Claim: x_0 is a limit of periodic points for f.

Idea: construct periodic pseudo orbits with small enough δ, so that there is a unique shadowing orbit, the shadowing orbit must be periodic.

choose large N:

$z_0 = x_0, z_1 = x_1, \ldots, z_N = x_N$
$z_{-1} = x_{-1} \ldots \quad z_{-(N+1)} = x_{-(N+1)}$

Extend by periodicity $z_{n+2N} = z_n$. $f(z_n) = z_{n+1}$ except for n an odd multiple of N.

$$\text{f.e.} \quad \left.\begin{array}{l} z_n = z_N \quad f(z_n) = x_{N+1} \\ z_{n+1} = z_{-N+1} = x_{-N+1}\end{array}\right\} \text{both as near as I like to } \bar{x}$$

Choose ε small enough for uniqueness
Take N big enough, so that (z_n) is a δ pseudoorbit for the corresponding δ. There is then a true orbit

$$y_i = f^i(y_0) \quad \text{such that} \quad \|y_i - z_i\| \leq \varepsilon.$$

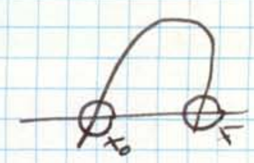
$f^{2N+i}(y_0)$ is another shadowing orbit, since $z_{i+2N} = z_i$.
$f^{2N+i}(y_0) = f^i(y_0) \Rightarrow y_0$ is periodic.

$z_0 = x_0 \quad \|x_0 - y_0\| \leq \varepsilon$
periodic point within distance ε of x_0.

Can do that for any $\varepsilon > 0$.

Proposition

$\bar{x}$ hyperbolic fixed point
x_0 transverse homoclinic point
Let $\varepsilon > 0$. For sufficiently large n there exists a compact set Ξ in the union of the balls of rad. ε about x_0 and $\bar{x}$, invariant under f^N such that $F^n \restriction \Xi$ is topological conjugate to the left shift on $\{0,1\}^{\mathbb{Z}}$



Law: Ξ is hyperbolic.

Γ $x_{(0,0)} = (x_0, x_1 \dots x_{[\frac{N}{2}]}, x_{-N+[\frac{N}{2}]+1}, \dots x_{-1})$

$x_{(0,1)} = (x_0, x_1, \dots \qquad x_{N-2}, \bar{x})$

$x_{(1,0)} = (\bar{x}, x_{-N+2}, x_{-N+3}, \dots \qquad x_{-1})$

$x_{(1,1)} = (\bar{x}, \bar{x}, \dots, \bar{x})$

Pieces of pseudoorbits of length N

Given a sequence $\underline{i} \in \{0,1\}^{\mathbb{Z}}$, make long pseudo orbit as follows.

$\dots x_{(i_{-1} i_0)} x_{(i_0 i_1)}, x_{(i_1 i_2)}, \dots = z(\underline{i})$

$z_{N_j} = \left. \begin{matrix} \bar{x} & i_j = 1 \\ x_0 & i_j = 0 \end{matrix} \right\}$

N big enough, then we have uniformly δ pseudoorbits for δ as small as we like. δ corresp. pos in propos. Make ε as small to guarantee uniqueness of shadowing.

For each $\underline{i}$ there is a true orbit $f^n(\pi(\underline{i}))$ which is unique, and ε shadows $z(\underline{i})$

$\pi : \{0,1\}^{\mathbb{Z}} \to M$

$f^{jN}(\pi(\underline{i}))$ is within distance ε of $\bar{x}$ if $i_j = 1$, x_0 if $i_j = 0$

π is injective if ε is small enough ($U x_0$ and $U(\bar{x})$ dont overlap)

$\cancel{z(\sigma \underline{i}) = f^N(z(\underline{i}))}$ $[z(\sigma \underline{i})]_n = [z(\underline{i})]_{n+N}$

$\Rightarrow f^N(\pi(\underline{i})) = \pi(\sigma(\underline{i}))$ by uniqueness of shadowing

$f^N \circ \pi = \pi \circ \sigma$

Why is π continuous?

This follows from uniqueness of shadowing:

$\underline{i}^{(n)} \to \hat{\underline{i}}$

$z(\underline{i}^{(n)})$

To show $\pi(\underline{i}^{(n)}) \to \pi(\hat{\underline{i}})$

suffices, that no subsequence converges to anything other than $\pi(\hat{i})$.

Assume: $\pi(\underline{i}^{(n)}) \to \hat{y}$

$z(\underline{i}^{(n)}) \to z(\hat{i})$

The orbit of $\hat{y}$ ε shadows $z(\hat{i})$

$z_j(\underline{i}^{(n)}) = z_j(\hat{i})$ for large enough n

$\| z_j(\underline{i}^n) - f^j(\underbrace{\pi(i^{(n)})}) \| \le \varepsilon \ \forall n$

$\| z_j(\hat{i}) - f^j(\hat{y}) \| \le \varepsilon \ \forall j$

Uniqueness: $\hat{y} = \pi(\hat{i})$ $\lrcorner$

Idea for proof: Convert construction of shadowing orbit to a fixed point probl. in Banach space.

Consider $X := \{ \text{bd sequences } \underline{z} \text{ in } M \}$

$(A\underline{z})_n = f(z_{n-1})$

$\underline{z}$ is an orbit $\iff A\underline{z} = \underline{z}$.

$\underline{z}$ is a δ pseudo orbit $\iff \| A(\underline{z}) - (\underline{z}) \|_{sup} \le \delta$

Prop: $\boxed{\begin{array}{l} B_\varrho(\underline{z}) \text{ be the closed ball of radius } \varrho \text{ with center } \underline{z} \text{ in some Banachspace } X. \ A: B_\varrho(\underline{z}) \to X, \text{ differentiable in } B_\varrho \\ \text{if there exist a lin. operat } \Gamma \text{ on } X \text{ such that } (\Gamma - 1)^{-1} \text{ is invertible and a } \varkappa > 0 \ \varkappa < 1 \text{ such that} \\ \quad \|(\Gamma-1)^{-1}\| \|(DA - \Gamma)\| \le \varkappa \quad \forall x \in B_\varrho \\ \quad \|(\Gamma-1)^{-1}\| \|A\underline{z} - \underline{z}\| \le (1-\varkappa)\varrho \\ \Rightarrow A \text{ has one and only one fixed point in } B_\varrho \end{array}}$

Γ show that $\phi : (x) \to x - (\Gamma - 1)^{-1}(A(x) - x)$ maps B_ϱ contractivly into itself $\perp$

~~Must show~~: $\boxed{\begin{array}{l} A \text{ is differentiable and} \\ (DA(x)\xi)_n = Df(x_{n-1})\xi_{n-1} \end{array}}$

Idea: Take $\Gamma = DA(\underline{z})$

Have to show $\Gamma - 1$ is invertible (with uniform bound on Λ) for $\|\Gamma - 1\|$

$\| DA(\underline{x}) - \Gamma \| = \sup_n \| Df\, x_n - Df\, z_n \| \xrightarrow{\|\underline{z}-\underline{x}\| \to 0} 0$

Take $\varkappa = \frac{1}{2}$, ϱ small enough, so that $\|DA(\underline{x}) - \Gamma\| \le \frac{1}{2\|(\Gamma - 1)^{-1}\|}$

Take δ small enough, $\left(\delta \le \frac{\varrho}{2\|(\Gamma-1)^{-1}\|} \right)$.

Lemma: $\boxed{\begin{array}{l} \exists \varrho, B \quad \underline{z} \text{ any } \beta \text{ pseudo orbit staying within } \varrho \text{ of } \Lambda \\ \text{then} \quad \|(\Gamma - 1)^{-1}\| \le B \end{array}}$

In Thm: $\varepsilon < \varrho \Rightarrow$ shadowing orbit is unique and take $\delta(\varepsilon) = \frac{\varepsilon}{2B}$.

Main idea: Consider case, where $\underline{z}$ is an orbit in Λ

$$X^s = \{ (\xi_n) : \xi_n \in E^s(z_n) \ \forall n \}$$
$$X^u = \{ (\xi_n) : \xi_n \in E^u(z_n) \ \forall n \}$$

Covariance: $Df(x)\, E^s(x) = E^s(f(x)) \Rightarrow \Gamma$ maps X^s to X^s and X^u to X^u

$$\| Df(x)\xi \| \le \lambda \|\xi\| \quad \xi \in E^s(x)$$

$$\Rightarrow \|\Gamma\| \le \lambda \text{ on } X^s$$

$$(\Gamma - 1)^{-1} = -\sum_{j=0}^{\infty} \Gamma^j \text{ on } X^s$$

$$\|(\Gamma - 1)^{-1}\| \le \frac{1}{1-\lambda} \text{ on } X^s$$

$\|Df^{-1}(x)\,\xi\| \le \lambda \|\xi\|$ on $E^u(x)$

Γ is invert on X^u $\quad \|\Gamma^{-1}\| \le \lambda$ on X^u

$(\Gamma - 1)^{-1} = \Gamma^{-1}(1-\Gamma^{-1})^{-1} = \sum_{j=1}^{\infty} \Gamma^{-j}$

$\|(\Gamma - 1)^{-1}\| \le \frac{\lambda}{1-\lambda} \le \frac{1}{1-\lambda}$ on X^u

$P^s(x)$: proj. onto $E^s(x)$ along $E^u(x)$
$P^u(x) = 1 - P^s(x)$

$D = \sup_x \max\{\|P^s(x)\|, \|P^u(x)\|\} < \infty$

P^s : Proj. onto X^s along X^u
$= P^{s(u)}(z_n)$ coord.-wise

$\|P^{s,u}\| \le D$

$\|\xi\| \le 2 \cdot \max\{\|P^s\xi\|, \|P^u\xi\|\} \le D\|\xi\|$

Have then : $\|(\Gamma - 1)^{-1}\| \le \frac{2D}{1-\lambda}$ uniform, orbit indep. estimate ■

ext to general pseudoorbits :

construct Γ_0 to which the above argument can be applied and such that $\|\Gamma - \Gamma_0\|$ is small.

Lemma : $\boxed{\text{The proj. valued fn } P^s(x) \text{ can be extended to a continuous projection valued fn defnd on a nbh. of } \Lambda}$

Γ Extend Matrix elem. continuously to $\tilde{P}(x)$
For $x \in \Lambda$, $\sigma(\tilde{P}(x)) = \{0, 1\}$
For x near Λ, $\sigma(\tilde{P}(x))$ is contained in Nhhds of $0, 1$.
P^s := spectral proj. for the part of spectrum in Nhh. of 1.

Given x, x' near Λ

$\Gamma_{x'}^{x} = P^s(x') Df(x) P^s(x) + P^u(x') Df(x) P^u(x)$

For $x \in \Lambda$, $x' = f(x)$ $\quad \Gamma_{x'}^{x} = Df(x)$
For x near Λ, x' near $f(x)$: $\Gamma_{x'}^{x} \approx Df(x)$

$\Gamma_{x'}^{x}$: Range $P^s(x) \longrightarrow$ Range of $P^s(x')$
$\quad$ - " - $\;P^u(x) \longrightarrow$ - " - $\;P^u(x')$

$x \in \Lambda$ $\quad \xi \in$ range of $P^s(x)$ and if $x' = f(x)$ then $\|\Gamma_{x'}^{x}\xi\| \le \lambda\|\xi\|$

given any $\lambda_1 > \lambda$. If x is near to Λ and x' near $f(x)$, then
$\|\Gamma_{x'}^{x}\xi\| \le \lambda_1 \|\xi\|$ $\quad \xi \in P^s(x)$

$(\Gamma_0 \xi)_n = \Gamma_{z_n}^{z_{n-1}} \xi_{n-1}$
$(\Gamma\xi)_n = Df(z_{n-1})\,\xi_{n-1}$

Take δ small enough so that $P^s(x)$ is defined and cont. on δ-nbhd of Λ, take $D \ge \sup_{x \in \delta\text{ nbh.}} \max\{\|P^s(x)\|, \|1 - P^s(x)\|\}$

choose $\lambda_1 \in (\lambda, 1)$, take δ smaller still so that
$\|\Gamma_{x'}^{x}\xi\| \le \lambda_1\|\xi\|$ $\quad \xi \in P^s(x)$ and $\|\Gamma_{x'}^{x\,-1}\xi\| \le \lambda_1\|\xi\|$ $\quad \xi \in P^u(x)$
holds for

holds for all $x \in B_\delta(\Lambda)$ and x' within dist. δ of $f(x)$

$\Rightarrow \|(\Gamma - A)^{-1}\| \leq \frac{2D}{1-\lambda_1}$

need: $\|B\| \|A^{-1}\| \leq \frac{1}{2}$, $\|(A+B)^{-1}\| \leq 2 \|A^{-1}\|$ $\Big\}$ $\|A^{-1}\| \, \|\underbrace{(1+BA^{-1})}_{\leq 2}\|$ $\leq$

For δ small enough, $d(x,\Lambda) \leq \delta$, $d(x', f(x)) \leq \delta$

$\Rightarrow \|\Gamma_{x'}^{x} - Df(x)\| \leq \frac{1-\lambda_1}{4D}$

$\|\Gamma_0 - \Gamma\| \leq \frac{1-\lambda_1}{4D}$ $\qquad$ $\|\Gamma_0 - \Gamma\| \, \|(\Gamma_0 - A)^{-1}\| \leq \frac{1}{2}$ $\|(\Gamma - A)^{-1}\|$

$\Rightarrow \|(\Gamma_0 - A)^{-1}\| \leq \frac{4D}{1-\lambda}$ $\qquad \leq \frac{4D}{1-\lambda_1}$ ∎

History: R. Bowen, Moser, Mather, Pubik: Anosov,

these estimates work to construct shadowing orbits for any $\tilde{f}$ near f.

Theorem: Λ hyperb. for f. $\varepsilon > 0$ There is a nbhd V of f in C^1 and $\delta > 0$ such that if $\underline{z}$ is a δ pseudoorbit for f staying within distance δ of Λ, ~~then there exist~~ and if $\tilde{f} \in V$ then there is a y_0 such that

$$\|\tilde{f}^n(y_0) - z_n\| \leq \varepsilon \quad \forall n$$

y_0 unique, if ε small enough

Proof: $(A(\underline{x}))_n = \tilde{f}(x_{n-1})$

$(\Gamma \xi)_n = Df(z_{n-1})\xi_n$

Suffices:

For ε suff. small (1) $\|DA(x) - \Gamma\| \leq \frac{1}{2B}$ $\quad \forall x$ in $B_\varepsilon(\underline{z})$

(2) $\|A(\underline{z}) - \underline{z}\| \leq \frac{\varepsilon}{4B}$ $\quad \underline{z}$ δ pseudoorbit.

(1) $\|DA(x) - \Gamma\| = \sup_n \|D\tilde{f}(x_n) - Df(z_n)\|$

$\leq \sup_n \|D\tilde{f}(x_n) - Df(x_n)\| +$

$\sup_n \|Df(x_n) - Df(z_n)\|$

ε suff. small: $\|Df(x) - Df(z)\| \leq \frac{1}{4B}$ $\quad \|x - z\| < \varepsilon$

V small: $\|Df(x) - D\tilde{f}(x)\| \leq \frac{1}{4B}$ $\quad \tilde{f} \in V$

(2) $\|Az - z\| = \underline{\sup_n \|\tilde{f}(z_n) - z_{n+1}\|} \leq \sup_n \|\tilde{f}(z_n) - f(z_n)\| + \underbrace{\sup_n \|f(z_n) - z_{n+1}\|}_{\leq \delta}$

$\left.\begin{array}{l} \delta < \frac{\varepsilon}{8B} \\ V \text{ small enough} \end{array}\right\}$ first term $\leq \frac{\varepsilon}{8B}$ ∎

Apply this with $\underline{z}$ an exact orbit for f in Λ.

$z_0 \in \Lambda$, $\tilde{f} \in V$

$\exists! \, \tilde{z}_0$ $\|\tilde{f}^n(\tilde{z}_0) - f^n(z_0)\| \leq \varepsilon$

Call $\tilde{z}_0 = h(z_0)$

By uniqueness: $\hat{h}(f(z_0)) = \hat{f}(\hat{h}(z_0))$
$$\hat{h} \circ f = \hat{f} \circ \hat{h} \quad \text{on } \Lambda$$

Can show: $\hat{h}$ is continous, injective $\therefore$ A homöom. onto $\hat{\Lambda}$
$\hat{f}|_{\hat{\Lambda}} = \hat{h} \circ f|_{\Lambda} \circ \hat{h}^{-1}$

Theorem (Structural stability of hyperbolic sets)

> Let Λ be a hyperbolic set for the C^1 diffeom f
> $\hat{f}$ suff. near f in C^1 topol.
> $\exists \hat{h}$ on Λ onto a compact set $\hat{\Lambda}$ invariant for $\hat{f}$ such that $\hat{f} = \hat{h} \circ f \circ \hat{h}^{-1}$ on $\hat{\Lambda}$
> $\hat{h}$ can be made as close to $1\!\!1$ as we desir.

Rem: $\hat{h}$ not diff even in the best cases.
Argument:
Λ contains some ~~pt~~ period. pt x_0
$\hat{h}(x_0)$ is a periodic pt for $\hat{f}$
if $\hat{h}$ where differentiable at x_0
then the spectrum of $\sigma(D\hat{f}^p(\hat{x}_0))$
$= \sigma(Df^p(x_0))$ is independent by chain rule...
which can be distroid by changing f

Γ only: $\hat{h}$ is cont. and inj.

(i) • inject. follows from expansivity
(ii) • continuity follow from uniqueness of shad orbit + compactness

(i) • Take ε small enough.
Any two distinct f orbits on Λ are somewhere seperated by more than ~~than~~ 2ε.

Take two dist. pts $x_0 \neq x_0'$ in Λ
Find j so that $\| f^j(x_0) - f^j(x_0') \| > 2\varepsilon$
$\| f^j(x_0') - \hat{f}^j(\hat{h}(x_0')) \| \leq \varepsilon$
$\| \hat{f}^j(\hat{h}(x_0)) - \hat{f}^j(\hat{h}(x_0')) \| > 0 \Rightarrow \hat{h}(x_0) \neq \hat{h}(x_0')$

(ii) • continuity $\left.\begin{matrix} x_n \to \bar{x} \\ \hat{h}(x_n) \to z \end{matrix}\right\} \Rightarrow z = \hat{h}(\bar{x})$

$\| f^j(x_n) - \hat{f}^j(\hat{h}(x_n)) \| \leq \varepsilon$
$\| f^j(\bar{x}) - \hat{f}^j(z) \| \leq \varepsilon \quad \forall j$
$\rightarrow z = \hat{h}(\bar{x})$ by uniqueness of shadowing. ∎

Def: $f: M \to M$ is called structural stable (with resp. to some topologie), if for every $\hat{f}$ near f there is a homöom. h with $\hat{f} = h \circ f \circ h^{-1}$

Corollary

> Suppose f is a C^1 Anosov diffeom of M, then for any $\hat{f}$ near f in C^1 topologie, there is a homöomorphism $\hat{h}: M \to M$ so that $\hat{f} = \hat{h} \circ f \circ \hat{h}^{-1}$

Γ Still to show: $\hat{h}$ of maps M onto M. $\quad h(x_0) = y_0$
Shadowing: For each $y_0 \in M$ $\exists x_0$ so that $\| f^n(x_0) - \hat{f}^n(y_0) \| \leq \varepsilon$
because if $\hat{f}$ near enough in C^0 make $\hat{f}^n(y_0)$ δ pseudoorbit for f with δ as small as desired uniformily in y_0. $\lrcorner$

~~Remarks~~ ?

Theorem

Hartmann, Grobmann linearisation theorem

> x_0 hyperbolic fixed point for the C^1 diffeom. f defined on a nbh. of x_0 in the finite dim. space E. then f admits a topological linearisation at x_0.
>
> $\exists \phi$: nbh$(0) \to$ nbh(x_0)
>
> $\phi(0) = x_0$
>
> $f = \phi \circ Df(x_0) \circ \phi^{-1}$ in a nbh. of x_0

Γ w.l.o.g. $x_0 = 0$
by multiplying the nonlin. part of f by a smooth cutoff fn $\equiv 1$ near 0 can assume f is everywhere defined and near to $Df(x_0)$ in C^1, globally on E.
Enough: $Df(x_0)$ is structurally stable on E.

An invert. hyperb. linear mapping on a finite dimensional vectorspace is very much like an Anosov diffeom.
Repeat proof for Anosov diffeom. in this simpler situation $\lrcorner$

Charakterisation of hyperbolicity as property in a Banachspace

f: C^1 diffeom. of M
Λ compact f invariant

$\mathcal{X}(\Lambda) = \{ x \mapsto \xi(x) \mid \Lambda \to E,$ continuous $\}$

space of continuous sections of tangent bundle to M defined over Λ. space of vectorfields.

$f_*(\Lambda)$: $\mathcal{X}(\Lambda) \circlearrowleft$

$\xi(x) \longmapsto Df(f^{-1}(x))\, \xi(f^{-1}(x))$

theorem Λ hyperbolic set of f iff $f_*(\Lambda)$ is a hyperbolic linear operator on $\mathcal{X}(\Lambda)$

Γ Assume Λ is hyperbolic. $E_x = E^s_x \oplus E^u(x)$
$\mathcal{X} = \mathcal{X}^s \oplus \mathcal{X}^u$ $\quad \mathcal{X}^s = \{ \xi \mid \xi(x) \in E^s(x) \ \forall x \}$

$\mathcal{X}^s$ and $\mathcal{X}^u$ are invariant subspaces of $f_*(\Lambda)$

$\xi \in \mathcal{X}^s$ $\quad (f_*^n(\xi))\, f^n(x) = Df^n(x)\, \xi(x)$ chain rule

$\| f_*^n \xi \| \leq \sup_x \| Df^n(x)\xi(x) \| \leq c\lambda^n \sup_x \| \xi(x) \| \leq c \lambda^n \| \xi \|$

$\longrightarrow \| f_*^n |_{\mathcal{X}^s} \| \leq c\lambda^n \implies \sigma(f_* |_{\mathcal{X}^s}) \subseteq \{ z \mid |z| \leq \lambda \}$

Similary $\sigma(f_* |_{\mathcal{X}^u}) \subseteq \{ z \mid |z| \geq \lambda^{-1} \} \longrightarrow f_*$ is hyperbolic.

Conversely: f_* is hyperbolic. $\implies \exists$ splitting $\mathcal{X} = \mathcal{X}^s \oplus \mathcal{X}^u$
want to show: Splitting has the pointwise form.
using the fact, that f_* acts in a pointwise way.
claim: If $\varphi(x)$ is any continuous real valued fn on Λ, then pointwise multipl. by φ maps $\mathcal{X}^s$ $(\mathcal{X}^u)$ to itself.

$\| f_*^n \xi \| \leq c\lambda^n \| \xi \|$ $\quad \xi \in \mathcal{X}^s$
$\| f_*^{-n} \xi \| \leq c\lambda^n \| \xi \|$ $\quad \xi \in \mathcal{X}^u$

Claim: $\psi(x)$ continuous real valued fn on Λ, then multiplic. by ψ maps $\mathcal{X}^s$ into $\mathcal{X}^s$ and $\mathcal{X}^u$ into $\mathcal{X}^u$

Γ $\xi \in \mathcal{X}^s \Longleftrightarrow \|f_*^n \xi\| \xrightarrow{n\to\infty} 0$

$(f_*^n \psi\xi)(f^n(x)) = Df^n(x)\ \psi(x)\ \xi(x)$

$\|f_*^n \psi\xi\| \le \|\psi\|_\infty \|f_*^n \xi\|$

$\xi \in \mathcal{X}^s \to \psi\xi \in \mathcal{X}^s$ same for $\xi \in \mathcal{X}^u$ $\lrcorner$

$\mathcal{F}^s(x) = \{\xi(x) / \underline{\xi} \in \mathcal{X}^s\}$

$\mathcal{F}^u(x) = \{\xi(x) / \underline{\xi} \in \mathcal{X}^u\}$ are linear subspaces invariant for $Df(x)$ because f_* has $\mathcal{X}^{u,s}$ invariant

$\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ span $\mathcal{E}$ $\forall x \in \Lambda$

$\underset{\mathcal{X}^s}{\underline{\xi}}, x$ given. $\|(f_*^n \underline{\xi}) f^n(x)\| = \|Df^n(x)\xi(x)\|$

$\le \|f_*^n \underline{\xi}\| \le c\lambda^n \|\underline{\xi}\| \longrightarrow 0$

$\longrightarrow$ any vector $\xi \in \mathcal{F}^s$ is a contracting vector

... $\lrcorner$ .. $\xi \in \mathcal{F}^u$ is a expanding vector

Take $\xi \in \mathcal{F}^s$ choose $\underline{\xi} \in \mathcal{X}^s$ $\underline{\xi}(x) = \xi$

By multiplying $\underline{\xi}$ ~~with~~ by an appropriate fn

$\|\xi\| = \|\underline{\xi}\|_\infty$ ← norm of vectorfield

$\|Df^n(x)\underline{\xi}(x)\| \le c\lambda^n \|\underline{\xi}(x)\|$ unif. in x

$\|Df^n(x)\xi\| \le c\lambda^n \|\xi\|$ $\forall \xi \in \mathcal{F}^s(x)$

$\|Df^{-n}(x)\xi\| \le c\lambda^n \|\xi\|$ $\forall \xi \in \mathcal{F}^u(x)$

Covariance of splitting + uniform estimates on rate of contractivity implies. $\underbrace{\|Df^n(x)\xi\| \ge c^{-1}\lambda^{-n}\|\xi\|}_{\text{apply to } n}$ $\forall \xi \in \mathcal{F}^u(x)$

$\Rightarrow \mathcal{F}^s(x) \cap \mathcal{F}^u(x) = \{0\}$

$\Rightarrow \mathcal{E} = \mathcal{F}^s(x) \oplus \mathcal{F}^u(x)$ $\forall x$ an any vector not in $\mathcal{F}^s(x)$ has it a non contracting vector $\lrcorner$

$\Rightarrow \mathcal{E}^s(x) = \mathcal{F}^s(x)$

$\mathcal{E}^u(x) = \mathcal{F}^u(x)$

Theorem: $\boxed{\text{$\Lambda$ hyperbolic set for f, then there exist a nbh U of Λ in M and a C^1 nbh V of f such that if $\tilde f \in V$ and Ξ is a compact subset of U invariant for $\tilde f$, then Ξ is a hyperbolic set for $\tilde f$}}$

Note: this implies to the imbedded subshift in a nbhd of a transverse homoclin. point.

Proof: Give a systematic procedure for constructing an operator Γ_0 on $\mathcal{X}(\Xi)$, which is manifestly hyperbolic and is near $f_*(\Xi^n)$ in the normtopology and apply operat. perturb. the

Find a contin. projection valued extension $P^s(x)$ of $P^s(x)|\Lambda$ to a comp. nbh. U_1 of Λ.

$P^u(x) = 1 - P^s(x)$

$\Gamma_{x'}^x = P^s(x')\,Df(x)\,P^s(x) + P^u(x')\,Df(x)\,P^u(x)$ maps $Rn(P^s(x))$ to $Rn(P^s(x'))$

continuous for $x, x' \in U_1$

$x \in \Lambda, x' = f(x)$
$\Rightarrow \Gamma^x_{x'} = Df(x)$

λ such that $\|Df(x)\xi\| \le \lambda \|\xi\|$ $x \in \Lambda$; $\xi \in$ Rang $P^s(x)$
$\|Df(x)\xi\| > \lambda^{-1}\|\xi\|$ $x \in \Lambda$; $\xi \in$ Rang $P^u(x)$

choose λ_1 $\lambda < \lambda_1 < 1$

There exist a subnbhd U_2 of U_1 and a positive number $d_1 > 0$ such that $\|\Gamma^x_{x'}\xi\| \le \lambda_1 \|\xi\|$ $x \in U_2$ $\xi \in$ range $P^s(x)$
and $\|x' - f(x)\| \le d_1$

similarly $\|\Gamma^x_{x'}\xi\| \ge \lambda_1^{-1}\|\xi\|$ $\xi \in$ rang $P^u(x)$

Put $D := \sup_{x \in U_2} \max \{\|P^s(x)\|, \|P^u(x)\|\}$

Claim: for any $\hat{f}$ such that $\sup_{x \in U_2} \|\hat{f}(x) - f(x)\| \le d_1$
and any kompakt
$\hat{\Lambda} \subseteq U_2$ invariant for $\hat{f}$, $(\Gamma_0 \xi)(x) = \Gamma^x_{\hat{f}(x)} \xi(x)$ is
hyperbolic and $\|(\Gamma_0 - e^{i\theta}\mathbb{1})^{-1}\| \le \frac{2D}{1-\lambda_1}$ for all real θ

If $\xi \in X^s$ $\|\Gamma_0 \xi\| = \sup_x \|\Gamma^x_{\hat{f}(x)} \xi(x)\| \le \lambda_1 \sup_x \|\xi(x)\|$
$\le \lambda_1 \|\xi\|$

$\Rightarrow \|(\Gamma_0 - \mathbb{1})^{-1}|_{X^s}\| \le \frac{1}{1-\lambda_1}$

Similarly: $\|(\Gamma_0 - \mathbb{1})^{-1}|_{X^u}\| \le \frac{1}{1-\lambda_1}$
$\rightarrow \|(\Gamma_0 - \mathbb{1})^{-1}\| \le \frac{2D}{1-\lambda_1}$

For general θ: komplexify everything
and $e^{-i\theta}\Gamma_0$ satisfies same estim. as Γ_0
$\|(e^{-i\theta}\Gamma_0 - \mathbb{1})^{-1}\| \le B$
$\|(\Gamma_0 - e^{i\theta}\mathbb{1})^{-1}\| \le B$

want to estimate $\|\Gamma_0 - \hat{f}_*\|$
If we know, that $\|\Gamma_0 - \hat{f}_*\| < \frac{1}{B}$, then $(\hat{f}_* - e^{i\theta}\mathbb{1})$
$= (\Gamma_0 - e^{i\theta}\mathbb{1}) + (\hat{f}_* - \Gamma_0)$
$= (\Gamma_0 - e^{i\theta}\mathbb{1})(\mathbb{1} + \underbrace{(\Gamma_0 - e^{i\theta}\mathbb{1})^{-1}(\hat{f}_* - \Gamma_0)}_{\|\cdot\| < 1})$

Take U_3 a subnbh of U_2 and $0 < d_2 < d_1$ such that if
$x \in U_3$ and $\|x' - f(x)\| \le d_2$
then $\|\Gamma^x_{x'} - Df(x)\| \le \frac{1}{3B}$

Take $U = U_3$: let $V = \{\hat{f} : \|\hat{f}(x) - f(x)\| \le d_2$ $\forall x \in U_3$
$\|Df(x) - D\hat{f}(x)\| \le \frac{1}{3B}$ $\forall x \in U_3\}$

$(\Gamma_0 \xi)(\hat{f}(x)) = \Gamma^x_{x'} \xi(x)$
$\hat{f}_*(\hat{f}(x)) = D\hat{f}(x)\xi(x)$
$\|\Gamma_0 - \hat{f}_*\| \le \underbrace{\sup_x \|\Gamma^x_{\hat{f}(x)} - D\hat{f}(x)\|}_{\le \frac{1}{3B}} + \underbrace{\sup_x \|D\hat{f}(x) - Df(x)\|}_{\le \frac{1}{3B}} \le \frac{2}{3B}$

Stable and unstable Manifolds for hyperbolic sets

<u>Idea</u>: Λ hyperbolic set for f

For η small enough. For any $x \in \Lambda$

$W_\eta^s(x) := \{ y : \| f^n(x) - f^n(y) \| \le \eta \ \forall n \ge 0 \}$

is a smooth manifold passing through x and tangent there to $E^s(x)$

$W_\eta^s(x)$ are not invariant but covariant:

f maps $W_\eta^s(x)$ to $W_\eta^s(f(x))$

Consider mappings $h : \Lambda \longrightarrow M$ near the identity

$$(Ah)(x) = f h (f^{-1}(x))$$

$$Ah\, f(x) = f h(x)$$

A operator on $\{ h : \Lambda \to M$ bounded near Ideal $\}$

$h = id$ is a fixed point.

A is differentiable : $\cancel{DA \lim}$ $A(h+\xi)(x) - Ah(x)$

$$= \int_0^1 Df(h(x) + t\xi(x))\, \xi(x)\, dt$$

$\longrightarrow Df(h(x))\xi(x)$ to first order

$$(DA(h)\xi)(f(x)) = Df(h(x))\, \xi(x)$$

$$DA(id) = f_*$$

A is a differentiable operator

$DA(id) = f_*(\Lambda)$ hyperbolic

Applie stable manifold theorem to the hyperbolic fixed point (id) for A.

Setting up more formally:

$X = \{ \xi : \Lambda \longrightarrow E \mid$ bounded not necess contin $\}$

$\cancel{A}$ $h(x) = x + \xi$

$(A\xi)(f(x)) = f(x + \xi(x)) - f(x)$

$DA(0) = f_*$

The contracting subspace for $DA(0) - X^s(\Lambda)$

Given any $\xi^s \in X^s$ small enough. ~~Given any~~ there is an unique $W^s(\xi^s)$ in X^u such that

$\| A^n$

Theorem

Stable manifold theorem for hyperb. sets

Λ hyperb. set for f
For any η sufficiently small $\eta > 0$

1. $\forall x_0 \in \Lambda \quad \forall \xi^s \in \mathcal{E}^s(x_0)$
$\|\xi^s\| \le \eta \Rightarrow \exists! \, \xi^u \in \mathcal{E}^u(x_0)$
$\|f^n(x_0) - f^n(x_0 + \xi^s + \xi^u)\| \le \eta \quad n > 0$

2. $f^n(x_0) - f^n(x_0 + \xi^s + \xi^u)\| \to 0$ expon.

3. For fixed x_0 $\xi \longmapsto \omega_s(x_0, \xi) = \xi^u)$ is C^r if f is $\quad \omega_s(x_0, 0) = 0 \quad D_\xi \omega_s(x_0, 0) = 0$
The ξ-deriv. of ω_s are contin fns of x_0

Γ $\quad Ah(x) = f h f^{-1} \qquad h(x) = x + \xi(x)$
X Banachspace of all bounded mapp. $\Lambda \to \mathcal{E}$ sup norm

$A\xi \; f(x) = f(x + \xi(x)) - f(x)$
$A(0) = 0 \qquad A\xi(x) = f(f^{-1}(x) + \xi f^{-1}(x)) - x$
A is C^r if f is C^r

$DA(0) = f_*$ hyperbolic

contract. subspace $\quad X^s = \{ \xi, \xi(x) \in \mathcal{E}^s(x) \; \forall x \in \Lambda \}$
Applie stable manif.-thm for fixed points to A at 0.

Def. of $\underline{\omega}$

For suff. small $\eta > 0$ for any $\xi^s \in X^s$ with $\|\xi^s\| \le \eta$ there is a unique $\xi^u \in X^u$
$[\xi_u = \underline{\omega}(\xi_s)]$ such that $\|A^n(\xi^s + \xi^u) - \underbrace{A^n(0)}_{0}\| \le \eta \quad \forall n > 0$

Easy to check: $(A^n \xi)(f^n(x)) = f^n(x + \xi(x)) - f^n(x)$
$\sup_x \|f^n(x + \xi^s(x) + \mathcal{E}^u(x)) - f^n(x)\| \le \eta \quad \forall n$

$\xi^u(x_0)$ depends in principle on the values of $\xi^s(x)$ all x
but in fact depends only on $\xi^s(x_0)$

Can consider vectorfields ξ which vanish everywhere except at x_0

$$\omega_\xi(x_0, \cdot): \underset{\underset{C^r}{\uparrow}}{\mathcal{E}^s(x_0) \longrightarrow X^s} \xrightarrow{\;\underline{\omega}\;} \underset{\underset{C^r \text{ if } f \text{ is } C^r}{\uparrow}}{X^u} \xrightarrow{\text{evaluat. at } x_0} \mathcal{E}^u(x_0) \quad \text{is } C^r$$

is C^r (linear)

To get contin. of the ξ-derivatives, read the proof again taking for X the space of continuous vector fields.

$\xi^s \to \omega_s(x, \xi^s(x))$

Differentiability in space of cont. ~~..~~ vectorfields $\Rightarrow D_\xi^j \omega_\xi(x, \xi)$ is cont. in $\xi, x, \; j \le r$ $\blacksquare$

Geometrical reformulation

$W_\eta^s(x_0) = \{ x \,/\, \|f^n(x) - f^n(x_0)\| \le \eta$ for all $n > 0 \}$
is a C^r submanifold, passing through x_0 tangent there to $\mathcal{E}^s(x_0)$
$\|f^n(x) - f^n(x_0)\| \to 0$ exponentially

$W^s(x_0) = \{ x \,/\, \|f^n(x) - f^n(x_0)\| \to 0 \quad n \to \infty \}$
$= \bigcup_n f^{-n} W_\eta^s(f^n(x_0))$

local product structure and local maximality

Prop: $\boxed{\begin{array}{l}\text{a) } \exists \eta>0\ \forall x,y\in\Lambda\ \ W^s_\eta(x)\cap W^u_\eta(y) \text{ besteht aus höchstens einem Punkt} \\ \text{b) } \exists \eta>0\ \exists \varepsilon>0\ \forall x,y\in\Lambda\ \ d(x,y)<\varepsilon\ \ W^s(x)\cap W^u(y) =: [x,y] \text{ ist eindeutig und ändert stetig mit } x \text{ und } y\end{array}}$

Def: Λ has local product structure, if $[x,y]\in\Lambda$ for $x,y\in\Lambda$

$X := W^u_\varepsilon(x_0)\cap\Lambda$
$Y = W^s_\varepsilon(x_0)\cap\Lambda$ $\quad [\cdot,\cdot] : X\times Y \to U(x_0)$

local coordinates ~~of~~ $X\times Y$ in neihorhood of x_0

Def: Λ invariant set for a homöomorphism f
Λ local maximal, if there exists $U\supseteq\Lambda$ open with
$\nexists\ \Lambda'\supsetneq\Lambda\quad \Lambda'\subseteq U$ invariant for f.

in other words: $\Lambda = \bigcap_{n=-\infty}^{\infty} f^{-n}U$ $\quad U$: isolating neighborhood.

Rem: locally maximal implies that $f^n(y_0)$ the shadowing orbit is actually in Λ

Theorem $\boxed{\Lambda \text{ locally maximal} \iff \Lambda \text{ local product structure}}$

Paradox: Λ local product structure is an intrinsic property of Λ: $\forall\eta>0\ \exists\varepsilon>0\ \forall x,y\in\Lambda\ d(x,y)<\varepsilon\ \exists z\ d(f^n(x),f^n(z))<\eta$
$d(f^{-n}(z),f^{-n}(y))<\eta$

Λ local maximal makes use of the embedding of Λ in the space X

Recurrence

$x\in M$ given, $\omega(x,f) = \{$ accum. points of x under $f\}$
$\overline{\Omega}(f) := \overline{\bigcup_{x\in M}\omega(x,f)}$. Ω limit set of M
$L(f) := \{x\in M \mid f^n(x)\to Y\}$ limit set
x is wandering if there exists a open set U around x such that $f^nU\cap U=\emptyset\ \forall n>0$

$\Omega(f) := \{$ nonwandering points $\}\ \supseteq \overline{\Omega}(f)$

x is chainrecurrent, if for every $\delta>0$ there is a δ pseudo orbit starting and ending at x / x lies on a δ pseudoorbit

$R(f) := \{$ chainrecurrent points $\}\quad \supseteq \Omega(f)$

Propos: $\boxed{x\in R(f) \Rightarrow \forall\delta>0\ \exists\ \delta \text{ pseudoorbit in } R(f)}$

f is topological mixing $\iff \forall U,V$ open $\exists n_0\ \forall n>n_0$
$f^n(V)\cap U\neq\emptyset$

$$\boxed{R(f)\supseteq\Omega(f)\supseteq\overline{\Omega}(f)\supseteq L(f)}$$

Smale decomposition theorem

Λ hyperbolic set with local product structure and periodic points dense. Then we have the following unique decomposition of Λ:

$$\Lambda = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} \Lambda_{i,j} \qquad f: \Lambda_{i,j} \to \Lambda_{i,j+1}$$

$f^{n_i} \restriction \Lambda_{i,j}$ is topolog. mixing and: $\forall x \in \Lambda_{i,j}$

$$\Lambda_{i,j} = \overline{W^u(x) \cap \Lambda} = \overline{W^s(x) \cap \Lambda}.$$

Cor $\Lambda = X$, X connected $\Rightarrow$ f topolog. mixing

Remark:
- The assumption "periodic points are dense" is necessary.
- conclusion of Smales thm and local product structure implie "periodic points dense".

The major idea is captured by the following lemma:

Lemma: f topological mixing on Λ $\Leftrightarrow$ $W^u(x) \cap \Lambda$ is dense in Λ for all periodic points $x \in \Lambda$

"$\Leftarrow$" V, U given

x periodic point: consider case where x is fixed point

η small s. that $W_\eta^u(x) \subseteq U$

$f^n(W_\eta^u(x) \cap \Lambda)$ increasing sequ. of sets expanding to $W^u(x)$ which is dense by assumption

$\Rightarrow \exists n_0 \ \forall n \geq n_0 \ f^n(W_\eta^u(x) \cap \Lambda) \cap V \neq \emptyset$

x with period p: $f^{pn+i}(W_\eta^u \cap \Lambda) \cap V \neq \emptyset \quad n \geq n_1$, $i = 0, 1 \dots p-1$

"$\Rightarrow$" $\left.\begin{array}{l} x \in \Lambda \text{ periodic (period } p) \\ W^u(x) \text{ not dense in } \Lambda \end{array}\right\}$ $U := \overline{W^u(x) \cap \Lambda}$ open ($\rightarrow$ next lemma) $\quad W := \Lambda \setminus U$ open

$f^{np}(U) = U \quad f^{np}(U) \cap W = \emptyset$

$\Rightarrow f$ not topol. mixing $\lrcorner$

Lemma: f, Λ as in theorem, $x \in \Lambda$ periodic point, $\overline{W^u(x) \cap \Lambda}$ is open

$\ulcorner$ Take δ small enough, such that $[x,y]$ is defined and lies in Λ whenever $\|x-y\| < \delta$

Show that $B_\delta(z) \subseteq \overline{W^u(x) \cap \Lambda}$ for $z \in W^u(x) \cap \Lambda$.

If this isn't true, $B_\delta(z) \setminus \overline{W^u(x) \cap \Lambda}$ is relatively open in Λ and contains therefore a periodic point y

$z_1 = [y, z] \quad \in W^u(x)$ ~~for z~~

let p = lcd(period x, period y)

$f^{np}(z_1) \to y$ because $z_1 \in W_\eta^s(y)$ $\lightning$ to $y \notin \overline{W^u(x) \cap \Lambda}$. $\lrcorner$

Lemma: $\boxed{\begin{array}{l} x,y \text{ periodic points for } f \text{ in } \Lambda \\ y \in \overline{W^u(x) \cap \Lambda} \to \overline{W^u(y) \cap \Lambda} = \overline{W^u(x) \cap \Lambda} \end{array}}$

Γ $y \in \overline{W^u(x) \cap \Lambda}$, η small enough, such that $W^u_\eta(y) \cap \Lambda \subset \overline{W^u(x) \cap \Lambda}$

$p = (\text{period } x, \text{period } y)$

$f^p(\overline{W^u(x) \cap \Lambda}) = \overline{W^u(x) \cap \Lambda}$

$\bigcup_n f^{np} W^u_\eta(y) \cap \Lambda = W^u(y) \cap \Lambda$

Claim: $x \in \overline{W^u(y) \cap \Lambda}$

Γ $\overline{W^u(y) \cap \Lambda}$ is a nbhd of y

$\exists z \in W^u(x) \cap W^u(y) \cap \Lambda$

$f^{-np}(z) \to x$ $\lrcorner$

Consequen:
$\overline{W^u(y) \cap \Lambda} = \overline{W^u(x) \cap \Lambda}$
or $\overline{W^u(y) \cap \Lambda} \cap \overline{W^u(x) \cap \Lambda} = \emptyset$ $\Leftrightarrow$ C

$\lrcorner$

end of proof of thm: look now at the collection of all closures of unstable manifolds of periodic points

compactness: only finitly many are permuted by f.

The decomposition of Λ is unique: M indecomposil.

$f^n: M \to M$ topol. mixing

$\Rightarrow$ unstable manif. of periodic pts in M are dense

$\Rightarrow$ M is a $\Lambda_{i,j}$.

Global stability

General question: $\boxed{\text{What can one say about the stability of } f \text{, if one assumes, that } L(f), \Omega(f) \text{ or } R(f) \text{ are hyperbolic?}}$

Proposition: $\boxed{L(f) \text{ hyperbolic} \to \text{period. points are dense in } L(f)}$

Γ $x \in L(f)$, y such that $x \in \omega(y)$

U small enough. nbhd of $\overline{L(f)}$ such that shadowing holds

$f^n(y) \in U$ $n > n_0$

Choose $n_2 > n_1 > n_0$ such that $\| f^{n_1}(y) - x \| < \frac{\delta}{2}$
$\| f^{n_2}(y) - x \| < \frac{\delta}{2}$

Make a pseudo orbit which stays in U:

$$f^{n_1}(y), f^{n_1+1}(y) \dots f^{n_2-1}(y)\ f^{n_1}(y)$$

This orbit is shadowed by a unique period. orbit in U (in $L(f)$ by assum.) which comes close to x. $\lrcorner$

Proposition: $\boxed{R(f) \text{ hyperbolic} \to \text{periodic points are dense in } R(f)}$

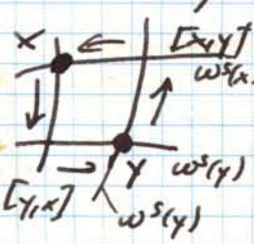
Γ use shadowing together with fact, that if $x \in R(f)$ and $\delta > 0$, there is a periodic δ pseudoorbit in $R(f)$ passing through x $\lrcorner$

! Density of periodic orbits in $\Omega(f)$ doesn't follow from hyperbolicity.

Example: Dankner, Ann. Math. 107 (1978) S. 517–533

Def f satisfies Smale's Axiom A, if $\Omega(f)$ is a hyperbolic set and periodic points are dense in $\Omega(f)$

Theorem (Smale) $\boxed{\overline{Per(f)} \text{ hyperbolic} \longrightarrow \text{local product structure}}$



Proof: Show: $[x,y] \in \Lambda = \overline{Per(f)}$
suffices: $[x,y] \in \Lambda$ for x,y periodic (just by contin. of $[\cdot,\cdot]$)
Replace f by f^p such that f^p has fixed points x, y

$\Lambda = \{x, y, \{f^n[x,y]\}, \{f^n[y,x]\}\}$ is hyperb. set

Build period. pseudoorbits of Λ with arbitrary small δ
shadowing: $\exists$ period. orbits passing arbitrary close to $[x,y]$ $\Rightarrow$ $[x,y] \in \overline{Per f}$ ∎

Corollary:
(1) $L(f)$ hyperbolic $\Rightarrow$ local product structure
(2) $\Omega(f)$ hyperbolic $\longrightarrow$ local prod. struct.
(3) Axiom A $\longrightarrow$ local prod. structure

Def: A diffeom. f of a compact manifold M is C^r structurally stable if there exists a C^r neigh.hd U of f in the space $Diff^r(M)$ such that for ~~...~~ $\forall g \in U$ $\exists h_g$ homöom. $M \to M$ with $g = h_g \circ f \circ h_g^{-1}$ and h_g can be made as near as one desires to the identity by making g near f.
"h_g is continuous at $g = f$"

Examples:
- Anosov diffeomorphism.
- Axiom A ain't structurally stable (→ later)

Def: f is C^r - Ω-stable $\Leftrightarrow$
$\exists$ C^r nbhd $V(f)$ $\forall g \in V$ $\exists h_g : \Omega(f) \to \Omega(g)$ homöom.
with $g = h_g \circ f \circ h_g^{-1}$ on $\Omega(g)$
and $h_g \xrightarrow{g \to f} 1$

Ω-stability theorem If $\Omega(f)$ is hyperbolic, then f is Ω-stable

Remind: Thm: Λ hyperbolic
$\rightarrow \exists C^1$ nbhd $V(f)$ $\forall g \in V$ $\exists h_g : \Lambda \to M$ homöom.
$g \restriction h_g(\Lambda) = h_g \circ f \circ h_g^{-1}$

Proof: Use shadowing:

Regard each orbit of f as a pseudoorbit of g

g has an invariant set Λ_g near $R(f)$ such that the action of g on $\Lambda_g \underset{\text{topq.}}{=}$ action of f on $R(f)$

Question: $\boxed{\Lambda_g \overset{?}{=} R(g)}$

$\Lambda_g \subseteq R(g)$ because period. points for g are dense in Λ_g

Want to show: $x \notin \Lambda_g \Rightarrow x \notin R(g)$

1. case : x near $R(g)$

2. case : x well away from $R(f)$

For (1) : Prop : $\boxed{\begin{array}{l} \Lambda \text{ hyperbolic for } f \\ \text{local prod. structure} \\ \Rightarrow \exists \ \text{(a) open nhhd } U \text{ of } \Lambda \text{ in } M \\ \qquad \text{(b) } C^1 \text{ nhhd } V \text{ of } f \text{ such that} \\ g \in V \Rightarrow \Lambda_g = \bigcap_{n=-\infty}^{\infty} g^{-n}(U) \text{ is closed} \\ \text{and } \exists \text{ homöomorph. } h_g : \Lambda \to \Lambda_g \quad g = h_g \circ f \circ h_g^{-1} \\ \text{on } \Lambda_g \end{array}}$

Γ g near f with invariant set Λ_g which is hyperbolic and has local prod. structure

$\Rightarrow \Lambda_g$ has isolating nhhd U_g : $\Lambda_g = \bigcap g^{-n} U_g$

Propos. says: ~~Λ_g~~ " U_g can be taken independent of g

$f^n(x)$ pseudoorbit for $g^n(x)$ is shadowed uniquely by an orbit $g^n(h_g(x))$

$x \to h_g(x)$ is a homöomorphism into M

Argum. shows: $\exists \varepsilon_0 > 0 \ \forall g \in V \ \forall x \in \Lambda \ \exists! y \in M$

$\| g^n(y) - f^n(x) \| \leq \varepsilon_0 \ \forall n \in \mathbb{N}$

and in fact: $\| g^n(y) - f^n(x) \| \leq \varepsilon \ \forall n \in \mathbb{N}$

$h_g : x \to y$ is a homöomorph.

$h_g(\Lambda)$ is invariant $\| h_g(x) - x \| \leq \varepsilon \ \forall x \in \Lambda$

Now take ε very small, choose a δ as in the orbit. shad. thm: "$\forall$ δ pseudoorbit is ε shadowed by a unique orbit"

Local product structure. shadowing orbit is in Λ

Claim $U = \delta$ nhhd of Λ will work for small enough V . . .

$\dashv$

Lemma: $f: X \circlearrowleft$ contin. X metric
$x \in X$ not chain recurrent
$\Rightarrow \exists\, U(x)$ nbhd of x in X $\exists\, C^0$ nbhd $V(f)$
such that no $y \in U_x$ can be chain recurrent for $g \in V(f)$

For any $x \notin U$ $\exists\, U_x, V_x$ as in lem. $M \setminus U$ is compact:
cover $M \setminus U$ with a finite number of U_{x_i}'s

$V = \{ C^1$ nbhd for the statement: $\exists\, h_g : \Omega(f) \to U$
$g = h_g \circ f \circ h_g^{-1}$ on $h_g(\Omega(f))$

$\cap \bigcap_i V_{x_i}$

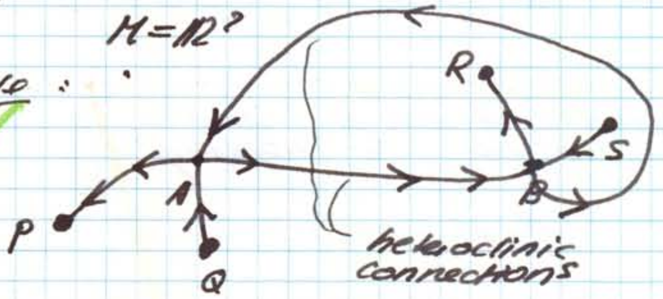
If $g \in V$ then no point of $M \setminus U$ can be in $\Omega(g)$.
Any g-orbit outside $h_g(\Omega(f))$ eventually gets into $M \setminus U$.
Chain recurrent set is invariant: $\Omega(g) = h_g(\Omega(f))$

Question: Does Axiom A implie Ω stability

No!

$M = \mathbb{R}^2$

Example:

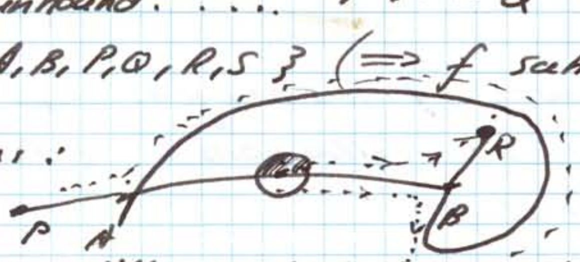


" Every orbit in the bounded component is forward asymptotic to R backward to S. "

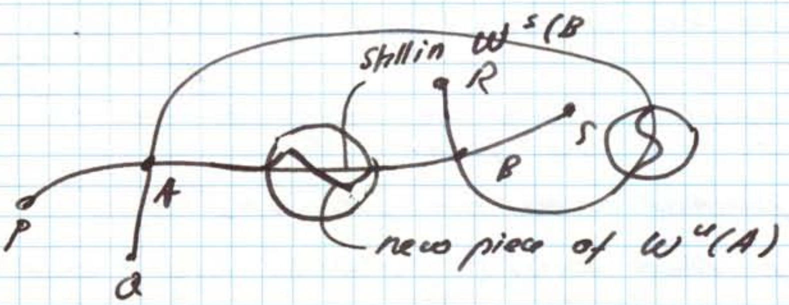
" Every orbit in the unbound. . . . P . . . Q "

Claim: $\Omega(f) = \{A, B, P, Q, R, S\}$ ($\Rightarrow f$ satisfys Axiom A)

Only heteroclinic connections:



Claim: An arbitrary little perturbation of f can brake the heteroclinic connections and replace them by transverse crossings of stable and unstable manifolds.



Can applie the argument used to prove that, if $\overline{\mathrm{Per}(f)}$ is hyperbolic has local prod. structure to show, that these transverse crossing points are limits of periodic points for the perturbed mapping.

Phenomena like this are called Ω-explosions.

conditions which rules out Ω-explosions :

f : satisfies Axiom A $\qquad \overline{L(f) \cup L(f^{-1})}$ hyperbolic

$\Omega(f) = \Omega^{(1)}(f) \cup \ldots \cup \Omega^{(m)}(f)$ Smale spectral decomposition

Make a directed graph with vertices $1, \ldots, m$
Draw an edge from i to j, if there is an orbit $f^n(x)$ in $M \setminus \Omega(f)$ such that

$$f^{-n}(x) \to \Omega^{(i)} \ (n \to \infty)$$
$$f^{n}(x) \to \Omega^{(j)} \ (n \to \infty)$$

($i = j$ is allowed)

Def $\quad f$ has the no-cycles property, if there are no cycles in this directed graph. (in particular: cycles of length 1)

Theorem :

f satisfies Axiom A and
f has the no cycles property

$\Longrightarrow \quad \Omega(f) = R(f)$

If so : f is Ω stable

If f has cycles then $R(f) \underset{\text{strict}}{\supsetneq} \Omega(f)$

To show: f has no cycles $\Rightarrow \Omega(f) = R(f)$

Technique : Filtrations. (Purely topological setting)

general set up :

X compact metric space
f : homöom. $X \to X$
$\Lambda_1, \ldots, \Lambda_m$ disjoint closed invariant sets
$\Lambda = \bigcup_{i=1}^{m} \Lambda_i \qquad \Lambda \supset L(f) \cup L(f^{-1})$

Make a directed graph : $(i \to j)$ if $\exists x \in X \setminus \Lambda$
$$f^n(x) \to \Lambda_j \quad n \to \infty$$
$$\to \Lambda_i \quad n \to -\infty$$

No cycle property : there are no cycles in the graph.

Define a partial order on $\{1, \ldots, m\}$ by

$i > j$ if there exists a path in the graph going from i to j, which can be extended to a "linear order". Can assume that the linear order is the standard order : $1 < 2 < \ldots < m$
orbits go from Λ_i to Λ_j with $j < i$.

Proposition

There exists a sequence $\emptyset = M_0 \subset M_1 \subset M_2 \ldots \subset M_m = X$ of open sets in X such that

(1) $\overline{M_i} \subseteq M_{i+1}$
(2) $\Lambda_i \subseteq M_i \setminus \overline{M_{i-1}}$
(3) $f \overline{M_i} \subset M_i$
(4) $\Lambda_i = \bigcap_{n \in \mathbb{Z}} f^{-n}(M_i \setminus \overline{M_{i-1}})$

Filtration of X

(2) is contained in (4) because every orbit outside Λ is forward asymptotic to some Λ_i and backward asymptotic to another Λ_j.

If X is a manifold, the M_i can be taken to have smooth boundary.

For a proof: Chapter II of M Shub: "Stabilité globale des systèmes Dynamiques"

Corollary: $\boxed{R(f) \subset \Lambda}$

Lemma: $\boxed{\text{If } x \notin M_i \text{ and } f(x) \in M_i \text{ then } x \text{ is not chain recurrent.}}$

If $x \in \overline{M_i} \setminus M_i$ then $f^{-1}(x) \notin \overline{M_i}$
$y = f^{-1}(x)$ $y \notin \overline{M_i}$, $f^2(y) \in M_i$

If $y_0 = y, y_1, y_2$ is a δ pseudo orbit starting at y, δ small enough, $y_2 \in M_i$

If $\delta < dist\left(f(\overline{M_i}), X \setminus M_i\right)$ then any δ pseudoorbit, which once gets in M_i stays there forever.

Any δ pseudo orbit starting at y gets into M_i in two steps and stays there forever, so y cannot be chain recurrent, so $x = f(y)$ can't be chain recurrent. $\dashv$

When is f structurally stable?

If f satisfies Axiom A ~~then every~~ with
$\Omega(f) = \Omega^{(1)}(f) \cup \ldots \cup \Omega^{(n)}(f)$

Every orbit is forward asymptotic to $\Omega(f)$.

$\Omega(f)$ has local ~~pro~~ product structure

$\Longrightarrow \exists x_+ \in \Omega(f) \; \| f^n(x) - f^n(x_+) \| \xrightarrow{n \to \infty} 0$

$f^n(x) \in W^s_\varepsilon(f^n(x_+))$ for suff. large n, so $x \in W^s(x_+)$

So every point has a stable manifold. Similar $\exists x_- \in \Omega(f) \ldots$ so $x \in W^u(x_-)$. Every orbit has a unstable manifold.

In general, the tangent spaces to $W^s(x)$ and $W^u(x)$ need not be complementary.

Example: if x_+ = attracting fixed pt., $W^s(x)$ is a nbhd of x; if x_- = repelling fixed pt, $W^u(x)$ is a nbhd of x

More subtle: heteroclinic connections

Tangent spaces are the same

$W^s(x) = W^s(B)$
$W^u(x) = W^u(A)$ } equal near x

Def: f has strong transversality, if the tangent spaces at x to $W^s(x)$ and $W^u(x)$ generate the tangent space to M at x for every x.

Theorem (Robbin-Robinson) and others

$\boxed{\begin{array}{l} \text{If } f \text{ satisfies Axiom A} \\ f \text{ is } C^1 \text{ structurally stable} \\ \Longleftarrow \; f \text{ has strong transversality} \end{array}}$

Remark: In the last two years:

Mañé and others: The converse is also true!

---

want to study internal structure of hyperbolic sets.

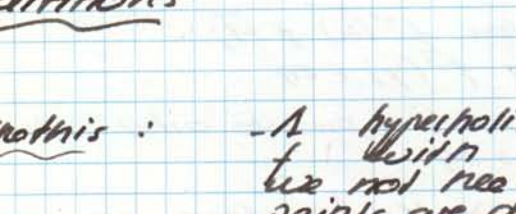
Markov partitions

Standing hypothesis: – Λ hyperbolic set for a C^1 diffeomorph. f with local product structure. we not need to assume "periodic points are dense". We're working always inside Λ forgetting the ambient space

Fix once and for all $\eta_0 > 0$ such that $W^s_{\eta_0}(x)$ and $W^u_{\eta_0}(x)$ are well behaved.

For $x \in \Lambda$, a set of canonical coordinates at x is a homöomorphism from a product space $P_0 \times Q_0 \longrightarrow U$ nbhd of x in Λ such that horizontal lines (sets of the form $P \times \{q\}$) correspond to stable manifolds $U \cap W^s_{\eta_0}(x)$. vertical lines $\{p\} \times Q$ correspond to unst. manif. $U \cap W^u_{\eta_0}(x)$

$(W^s_\delta(x) \cap \Lambda, W^u_\delta(x) \cap \Lambda) \xrightarrow{[\cdot,\cdot]}$ nbhd of x homöomorph.

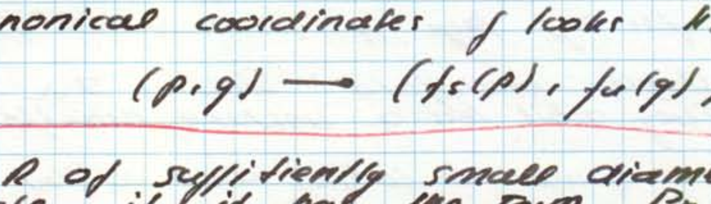


In canonical coordinates f looks like

$$(p,q) \longrightarrow (f_s(p), f_u(q))$$

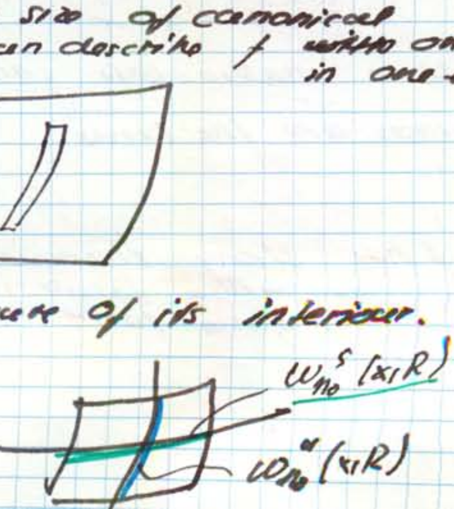
Def: A set R of sufficiently small diameter is a **rectangle** if it has the form $P \times Q$ in some set of canonical coordinates, and is closed under $[\cdot,\cdot]$

We always look at rectangles whose diameter are very small compared with the size of canonical coordinate patches, so that we can describe f ~~with~~ on the little $\square$ in one ~~set~~ coord sys.



R is **proper**, if it is the closure of its interior.

R a rectangle, $x \in R$



$W^s(x,R) := R \cap W^s_{\eta_0}(x)$

$W^u(x,R) := R \cap W^u_{\eta_0}(x)$

$R \longleftrightarrow$ canon. coord $P \times Q$

$x \longleftrightarrow (p,q)$

$W^s(x,R) \longleftrightarrow P \times \{q\}$

$W^u(x,R) \longleftrightarrow \{p\} \times Q$

Def: A **Markov Partition** of Λ is a finite covering $\{R_1 \dots R_p\}$ of Λ by proper rectangles of sufficiently small diameter such that

(1) $\quad int(R_i) \cap int(R_j) = \phi \quad i \neq j$

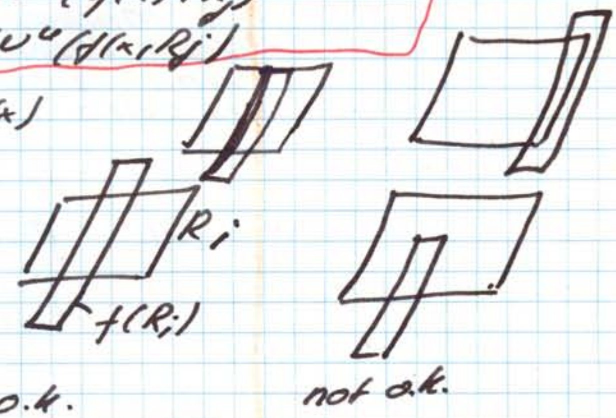
(2) $\quad x \in int(R_i)\ f(x) \in int(R_j)$, then

$$f(W^s(x,R_i)) \subset W^s(f(x),R_j)$$
$$f(W^u(x,R_i)) \supset W^u(f(x),R_j)$$

In canonical coordinates at $f(x)$

$$R_j = P_j \times Q_j$$
$$f(R_i) = P_i' \times Q_i'$$

(2) means: $\quad P_i' \subset P_j$
$\qquad\qquad\quad Q_i' \supset Q_i$



In canonical coordin. at x: $\quad R_i \longleftrightarrow P_i \times Q_i$

$x \longleftrightarrow (p,q)$

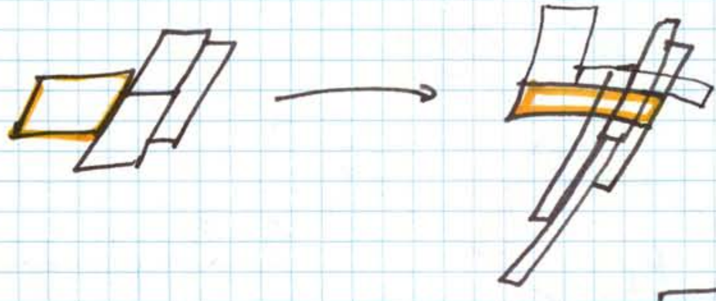
$f: (p,q) \longrightarrow (f_s(p), f_u(q))$

$W^s(x,R_i) \longleftrightarrow P_i \times \{q_0\} \xrightarrow{f} P_i' \times \{q_1\}$

where ~~f(...)~~ $f(x) \longleftrightarrow (p_1, q_1)$

$f(W^s(x,R_i)) \subset W^s(f(x),R_j)$

says $P_i' \subset P_j$



Example: Smale Horseshoe

possible Markov partitions

Theorem (Bowen) at the age of 22: $\boxed{\text{$\Lambda$ hyperbolic set with local product structure then Λ admits Markov partitions with arbitrary small diameter}}$

If Λ is a hyperbolic set with a Markov partition, then the dynamics of f in Λ is almost the same as the dynamics of a Markov shift.

Markov Matrix $A_{ij} \in M(\mathbb{Z}_2, p)$

Technical condition: $\forall i \exists j \; A_{ij} = 1$

$\Sigma(A) = \{ \underline{a} \in \{1, \dots, p\}^{\mathbb{Z}} \mid A_{a_i a_{i+1}} = 1 \}$

A specifies allowed and forbidden transitions.
A transition (from i to j) is allowed, iff $A_{ij} = 1$

$\Sigma(A)$ is shift invariant, $\Sigma(A)$ is closed in $\{1..p\}^{\mathbb{Z}}$ and hence compact.

Proposition: Let Λ be a hyperbolic set which admits a Markov partition of small enough diameter, then there exists a Markov Matrix A and a continuous $\pi : \Sigma(A) \to \Lambda$ which is onto, such that

(1) $f \circ \pi = \pi \circ \sigma$ (semiconjugacy)

(2) π is almost 1-1 in the sense that

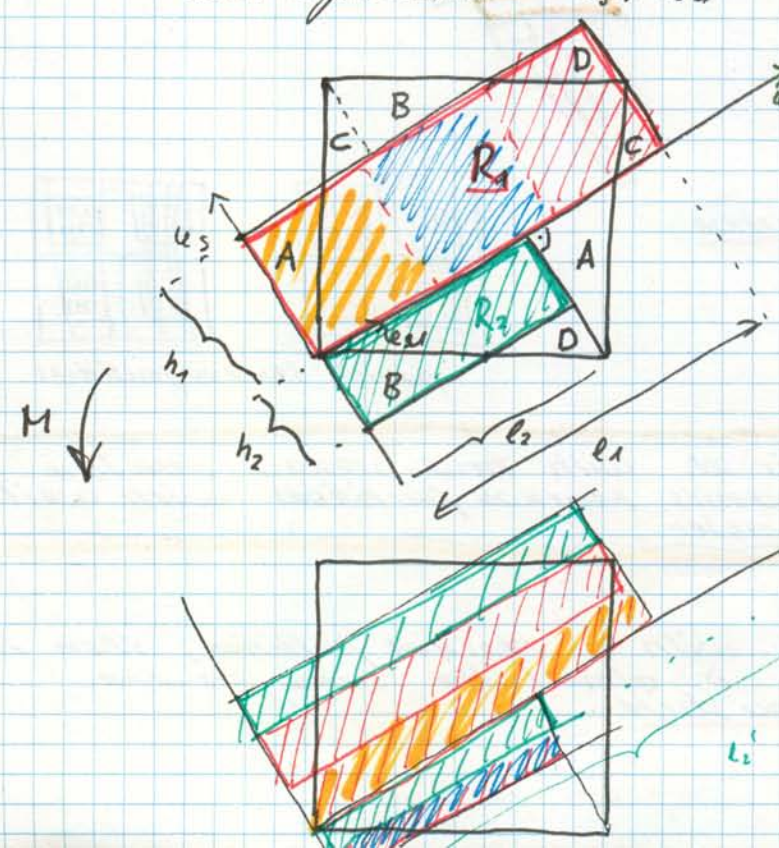
$\{ x \in \Lambda, \pi^{-1}\{x\}$ contains more than one pt $\}$ is contained in a countable union of closed nowhere dense set.

(3) $\# \pi^{-1}\{x\}$ is uniformly bounded.

Idea: $\begin{cases} \{R_1 \dots R_p\} & A_{ij} = 1 \text{ if } f(R_i^\circ) \cap R_j^\circ \neq \emptyset \\ \text{Map } \underline{a} \xrightarrow{\pi} x & \text{s.that } f^j(x) \in R_{a_j} \; \forall j \end{cases}$

Example: Markov partition for the cat map $M : \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ due to Adler and Weiss.

Eigenvalues: $\lambda^{\pm} = \frac{\pm\sqrt{5} + 3}{2}$
with Eigenvektors: e_s, e_u



$R_1 \cup R_2$ is a fundamental domain

- R_1 and R_2 are similar d.h. $\frac{h_1}{l_1} = \frac{h_2}{l_2}$
- $\frac{l_2}{l_1} = \lambda^+ - 2$
- $\frac{l_1}{l_2} = \lambda^+ - 1$ "goldener Schnitt"

by comput.

$l_2' = \lambda \cdot l_2 = l_2\lambda - l_2$
$= l_2(\lambda - 1)$
$= l_1$

$l_1' =$

Remark: . and iterate so that partition becomes arbitrary small until thing become nice.

. general construction for Markov part. for arbitrary toral automorphisms.

. original development: measure theoretic isomorphism between (M, ν_{eb}) and a simple Markov chain

. In higher dimension, it's known that Markov partitions of this kind don't exist.

. Connection with continued fraction

Proof of the Propos.

$\{R_1, \dots, R_p\}$ Markov partition of small enough diameter

$A_{ij} = \begin{cases} 1 & \text{if } \mathring{R}_i \cap f^{-1}(\mathring{R}_j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$

$\forall \underline{i} \in \Sigma(A)$ given $\exists! \; x \in \Lambda \quad f^j(x) \in R_{i_j} \; \forall j$

call $x = \pi(\underline{i})$

$\Delta^{(-r,s)}(\underline{i}) := \{x \in \Lambda \mid f^j(x) \in \mathring{R}_{i_j} \text{ for } j \in \{-r, \dots, s\}\}$

Lemma $\boxed{\text{Each } \Delta^{(-r,s)}(\underline{i}) \text{ is nonempty and as } r,s \to \infty \text{ with } \underline{i} \text{ fixed, } \operatorname{diam}(\Delta^{(-r,s)}(\underline{i})) \to 0}$

⌐ later ⌐

$\bigcap_{r,s} \overline{\Delta^{(-r,s)}(\underline{i})}$ is nonempty (by compactness) and contains exactly one pt called $\pi(\underline{i})$

It's immediate, that $f(\pi(\underline{i})) = \pi(\sigma \underline{i})$

continuity of π follows from the fact, that $\operatorname{diam}(\overline{\Delta^{(-r,s)}}) \to 0$

$U := \bigcup_{i=1}^{p} \mathring{R}_i$ is dense and open

$Z := \bigcap_{n \in \mathbb{Z}} f^{-n}(U)$ is a count. intersection of dense open sets

Take $x \in Z$. For each $j \in \mathbb{Z}$ $f^j(x) \in \mathring{R}_{i_j}$ for a unique $i(j)$. The sequence $(i_j) = \underline{i} \in \Sigma(A)$

and therefore $\pi(\underline{i}) = x$ and $\pi(\Sigma(A)) \supset Z$

Z is dense, π is contin. $\Sigma(A)$ compact

$\Rightarrow \pi(\Sigma(A)) = \Lambda$

$f^j(x) \notin R_k$ for $k \neq i_j$ because $\underbrace{\overline{\mathring{R}_k}}_{R_k} \cap \mathring{R}_{i_j} = \emptyset$

If $x = \pi(\underline{i})$, then necessarily $f^j(x) \in R_{i_j} \quad \forall j$

If $x \in Z$, x has only one preimage under π ⌐

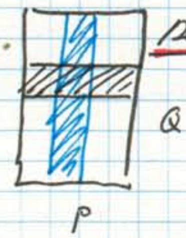
Prop $\boxed{\# \pi^{-1}(x) \leq p^2}$

Proof of the lemma :

Consider first $r=0$ (forward orbits)

$$\Delta^{(0,s)}(\underline{i}) = \{x \in \mathring{A}_{i_0} : f^{j}x \in \mathring{R}_{i_{j+1}} \quad 0 \le j \le s-1\}$$

$$= \mathring{R}_{i_0} \cap f^{-1}(\Delta^{(0,s-1)}(\sigma \underline{i}))$$



Def : $R = P \times Q$ is a rectangle. An s-band means a subrectangle of the form $P \times Q'$ $Q' \subset Q$
A u-band : $P' \times Q$ $P' \subset Q$

lemma (#) If $\mathring{R}_i \cap f^{-1}(\mathring{R}_j) \ne \emptyset$ and if Δ is a nonempty s-band in $\mathring{R}_j$, then $\mathring{R}_i \cap f^{-1}(\Delta)$ is a nonempty s-band in $\mathring{R}_i$.

lemma # + induction $\Delta^{(0,s)}(i)$ has the form $P \times Q'$ Q' non empty
same argument for f^{-1}

$$\Delta^{(-r,0)}(i) = P' \times Q \,; \quad P' \text{ nonempty}$$

$$\Delta^{(-r,s)}(i) = P \times Q' \,; \quad Q' \text{ nonempty}$$

$\bigcap_{r,s} \Delta^{(-r,s)}(i)$ is nonempty (compactness)

$$x \in \bigcap_{r,s} \Delta^{(-r,s)}(i) \longrightarrow f^{j}(x) \in \mathring{R}_{i_j} = R_{i_j} \quad \forall j$$

If $\max(\operatorname{diam}(R_i))$ is an expansivity constant for $f|\Lambda$, then there can be only one $x \in \bigcap_{r,s} \Delta^{(-r,s)}(i)$

Again by compactness $\operatorname{diam}(\Delta^{(-r,s)}(\underline{i})) \to 0$ $\lrcorner$

Proof of lemma # : [Write everything in canonical coordinates.]

take local coordinates $\mathring{R}_i$ & $\mathring{R}_j$ other local coordinates

$$f(p,q) \longleftrightarrow (f_s(p), f_u(q))$$

$$\left.\begin{aligned} \mathring{R}_j &= P_j \times Q_j \\ \mathring{R}_i &= P_i \times Q_i \end{aligned}\right\} \quad \begin{aligned} f_s(P_i) &\subseteq P_j \\ f_u(Q_i) &\supset Q_j \end{aligned}$$

$$\Delta = P_j \times Q_j'$$

$$Q_j' \subset Q_j$$

$$f_u^{-1}(Q_j') \subset f_u^{-1}(Q_j) \subseteq Q_i$$

$$\Rightarrow f^{-1}(\Delta) = f_s^{-1}(P_j) \times f_u^{-1}(Q_j')$$

$$f^{-1}(\Delta) \cap \mathring{R}_i = P_i \times f_u^{-1}(Q_j') \text{ is a nonempty s-band} \lrcorner$$

Proposition (with notations as above) $\{R_1, \dots, R_p\}$ Markov partition, π.
If $\operatorname{diam}(R\text{'s})$ is small enough, then no point x has more than p^2 preimages under π

lemma : $\underline{i}, \underline{i}' \in \Sigma(A)$
$\pi(\underline{i}) = \pi(\underline{i}')$
Also assume $i_n = i_n'$, $i_m = i_m'$ $n < m$
Then $i_j = i_j'$ for $n \le j \le m$

lemma implies proposition: Suppose x has more than p^2 preimages.
For n big enough, x has more than p^2 preimages pairwise distinguishable for $-n \le j \le n$.
There are only p^2 possibilities for $(\underline{i}_n, i_n)$. ~~some pair of these~~
If there are more than p^2 preimages, then some pair of this preimages has to agree both at $-n$ and at n.
By lemma, have to agree for $-n \le j \le n$.
contradiction choice of n.

Proof of lemma: Assume $n = 0$

$$\Delta^{(-r,s)}(\underline{i}) = \bigcap_{-r \le j \le s} f^{-j}(\mathring{R}_{i_j})$$

There exist $x : f^j(x) \in \mathring{R}_{i_j} \quad 0 \le j \le m$
$y : f^j(y) \in \mathring{R}_{i'_j} \quad 0 \le j \le m$

$\pi(\underline{i}) = \pi(\underline{i}') \Rightarrow R_{i_j}$ and $R_{i'_j}$ intersect for each j

$f^j(\pi(\underline{i})) \in R_{i_j} \quad \forall j$

$i_0 = i_0' \quad (x,y) \in \mathring{R}_{i_0}$

$[x,y] \in \mathring{R}_{i_0}$
distance from $f^j(x)$ to $f^j(y)$ remains small for $0 \le j \le m$
so $[f^j(x), f^j(y)]$ is defined for $0 \le j \le m$

once I know these are all defined, we can argue by induction on j that

$$[f^j(x), f^j(y)] = f^j([x,y])$$

$$[x,y] \in \underbrace{W^s_{\eta_0}(x) \cap \mathring{R}_{i_0}}_{W^s(x, \mathring{R}_{i_0})}$$

$\mathring{R}_{i_0} \cap f^{-1}(\mathring{R}_{i_1})$ is nonempty.

$f(W^s(x), R_{i_0}) \subset W^s(f(x), R_{i_1})$ by defin

claim: $f(W^s(x), \mathring{R}_{i_0}) \subset W^s(f(x), \mathring{R}_{i_1})$

$\Gamma R_i = P_i \times Q_i \quad R_j = P_j \times Q_j$

$f : (p,q) \longrightarrow (f_s(p), f_u(q))$

$f_s P_i \subset P_j$

f_s is a local homöomorphism. $f_s \mathring{P}_i = \mathring{P}_j$ $\lrcorner$ claim

* In general: $f^j([x,y]) \in \mathring{R}_{i_j} \quad 0 \le j \le m$

Do the same thing for f^{-1} in place of f working down from top.

$\mathring{R}_k \cap \mathring{R}_l = \emptyset \quad im' = im \Rightarrow [f^m(x), f^m(y)] \in \mathring{R}_{im}$

$\Rightarrow i'_j = i_j \quad f^j([x,y]) \in \underbrace{f^{-(m-j)}([f^m(x), f^m(y)])}_{\in W^u(f^j(y), \mathring{R}_{i'_j})}$

$0 \le j \le m$

qed * $\Rightarrow f^j([x,y]) \in \mathring{R}_{i_j} \quad 0 \le j \le m$

Proof of Bowen's theorem

standing smallness conditions: η_0 small, so that $W^u_{\eta_0}(x)$ is well behaved and η_0 is an expansivity constant for $f|\Lambda$

$d_0 < \eta_0/2$: there are canonical coordin everywhere covering ball of radius d_0
Rectangle have diameter $\ll d_0$

1. Step: Make a covering, which is like a Markov partition but allowing arbitrary overlaps.
Idea: Reverse proof of preceding thm:
Cover Λ by a finite number of small balls whose centers $z_1 \dots z_p$ define a Markov matrix A_{ij} by $A_{ij} = \{ 1$ dist $\|f(z_i) - z_j\|$ is small enough
For any $\underline{i} \in \Sigma(A)$, $\{z_{i_j}\}$ is a pseudo orbit with small δ.
Map $\underline{i} \xrightarrow{\theta}$ unique true orbit in Λ which ε shadows this pseudoorbit. $f \circ \theta = \theta \circ \sigma$
Arrange choices, so that θ is surjective.
Then the Markov like covering = covering by the images of the sets $\theta\{\underline{i} : i_0 = k\} = T_k$. $W^s(x, T_k) := W^s_{\eta_0}(x) \cap T_k$

Choose $\varepsilon < d_0$, diam (Markov partition)
Take $\delta < \varepsilon$ small enough, so that every δ pseudoorbit in Λ is ε shadowed by a unique true orbit in Λ.

Choose $\gamma < \frac{\delta}{2}$ so that $\|x_1 - x_2\| < \gamma \implies \|f(x_1) - f(x_2)\| < \delta$

Choose $\{z_1, \dots, z_p\}$ so that the balls of radius γ about the z_i cover Λ.
Put $A_{ij} = 1$, if $\|f(z_i) - z_j\| < \delta$
For any $\underline{i} \in \Sigma(A)$ the sequence (z_{i_j}) is a δ pseudo orbit which is therefore ε-shadowed by a unique true orbit $f^j(\theta(\underline{i}))$ defines $\theta: \Sigma(A) \to \Lambda$

Lemma: (1) $\boxed{f \circ \theta = \theta \circ \sigma}$
(2) θ is surjective
(3) θ is continuous

Γ (1) immediate
(2) Take $x \in \Lambda$ $\forall j \exists i_j$ so that $\|f^j(x) - z_{i_j}\| < \gamma$
If $(i_j) \in \Sigma(A) \implies \theta(\underline{i}) = x$
(i_j) To show: $(i_j) \in \Sigma(A)$
$$\|f(z_{i_j}) - z_{i_{j+1}}\| \le \underbrace{\|f(z_{i_j}) - f(f^j(x))\|}_{< \delta/2} + \underbrace{\|f^{j+1}(x) - z_{i_{j+1}}\|}_{< \gamma < \frac{\delta}{2}}$$
$< \delta$ i.e. $A_{i_j i_{j+1}} = 1$

(3) continuity: suffices by compactness:
$$\underbrace{\underline{i}^{(n)} \to \underline{i} \quad \text{with } \theta(\underline{i}^{(n)}) \to x}_{\implies \theta(\underline{i}) = x}$$
$\|f^j(x) - z_{i_j}\| \le \varepsilon$
$\|f^j(\theta(\underline{i})) - z_{i_j}\| \le \varepsilon$ $\Big\}$ $\|f^j(x) - f^j(\theta(\underline{i}))\| \le 2\varepsilon$
2ε is an expansiv. extension
$\implies x = \theta(\underline{i})$ $\lrcorner$

For each $k = 1, 2, \dots p$ put $T_k = \theta\{\underline{i} \in \Sigma(A)\ i_0 = k\}$ which is kompakt
The T_k's cover Λ. contin.

Lemma: $\boxed{\begin{array}{l} x \in \theta(\underline{i})\ x' \in \theta(\underline{i}') \\ (1)\ i_j = i'_j\ j \ge 0 \implies x' \in W^s_{\eta_0}(x) \\ (2)\ i_j = i'_j\ j \le 0 \implies x \in W^u_{\eta_0}(x) \end{array}}$

Γ (1) $i_j = i'_j$ for $j \ge 0$
$\|f^j(x') - z_{i_j}\| < \varepsilon$
$\|f^j(x) - z_{i_j}\| < \varepsilon$ $\Big\}$ $\|f^j(x) - f^j(x')\| \le 2\varepsilon$ $j \ge 0$
(2) the same $\lrcorner$

Notation: $\underline{i} \in \Sigma(A)$ $W^s_0(\underline{i}) = \{\underline{i}' : i'_j = i_j$ for $j \ge 0\}$
$W^u_0(\underline{i}) = \{\underline{i}' : i'_j = i_j$ for $j \le 0\}$

Analoges of stable and unstable manifolds for irregu. space

$\underline{i}, \underline{i}' \in \Sigma(A)$ $i_0 = i'_0$
define $[\underline{i}, \underline{i}']_j = \begin{cases} i_j & j \ge 0 \\ i'_j & j \le 0 \end{cases} = W^s_0(\underline{i}) \cap W^u_0(\underline{i}')$
is again in $\Sigma(A)$

$\implies$ $T_k = \theta(\underline{i}, i_0 = k)$ is a rectangle because $\{\underline{i}\ i_0 = k\}$ is a rectangle

Lemma: (1) $[\underline{i}, \underline{i}'] \in \Sigma(A)$
(2) $\theta[\underline{i}, \underline{i}'] = [\theta(\underline{i}), \theta(\underline{i}')]$
Bracket homomorphism

Γ $[\underline{i}, \underline{i}'] \in W^s_0(\underline{i})$
$\theta[\underline{i}, \underline{i}'] \in W^s_{\eta_0}(\theta(\underline{i})) \cap W^u_{\eta_0}(\theta(\underline{i}'))$ $\lrcorner$

both $\underline{i}'' = [\underline{i}, \underline{i}']$, $\underline{i}'' \in W^s_0(\underline{i})$

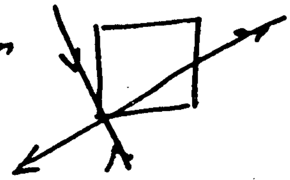
Lemma: $\underline{i} \in \Sigma(\underline{i})$, $x = \theta(\underline{i})$ $k = i_0$ $x \in T_k$
~~maps $W^s_0(\underline{i})$ onto~~
$W^s(x, T_k) = \theta(W^s_0(\underline{i}))$
$W^u(x, T_k) = \theta(W^u_0(\underline{i}))$

Γ $\underline{i}' \in W^s_0(\underline{i})$ $i'_j = i_j$ $j \ge 0$ $i'_0 = k$
$\theta(\underline{i}') \in T_k \cap W^s_{\eta_0}(x) = W^s(x, T_k)$
$\theta(W^s_0(\underline{i})) \subset W^s(x, T_k)$
$x' \in W^s(x, T_k)$ $x' = \theta(\underline{i}')$ $i'_0 = k$ $x' \in T_k$

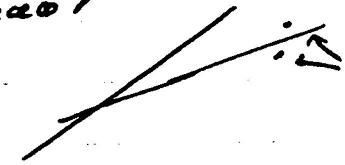
Concrete construction of Markov partitions for hyperbolic linear automorphisms of $\mathbb{T}^2$:

$F \in Sl(2,\mathbb{Z})$ hyperbolic, assume eigenvalues > 0
induces action on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

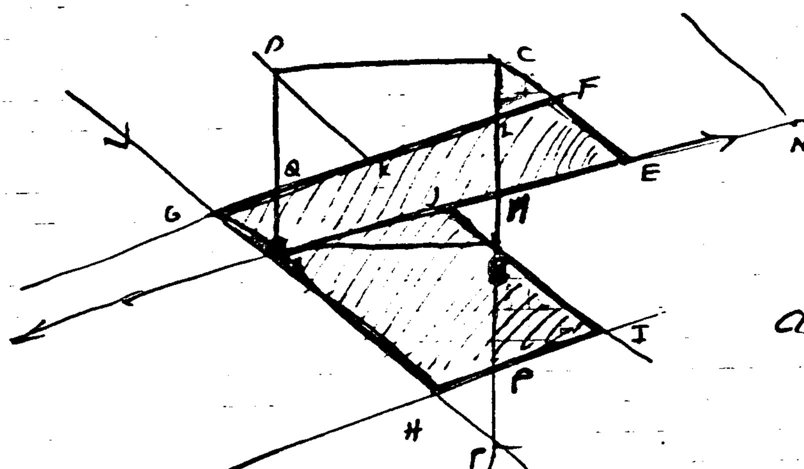
without loss of generality. Take Form



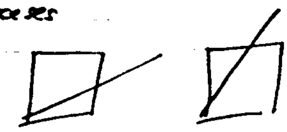
Else make coord transformation to a integer basis which is adequate



take this as integer basis continued fraction expansion for the slope of eigenvector



Half of the cases



Claim: [rectangle] is a fundam. domain

Γ $BIP = CFL$
$CEH = DKQ$
$DKLC = AHPB$
$GAQ = IBH$ $\lrcorner$

The stable boundary is $\equiv \overline{GH}$ ~~un~~ for projection

Multiplying this segment by F maps it into itself.

<u>Conclusion</u>: The long rectangle mapped under F stretched, projected on torus. After finite number of mappings F and translations

→ last piece of image of each of P_i, P_j runs all the way across whereever P_i is in.

For expanding direction take F^{-1}.

For the 3-torus $\mathbb{T}^3$, can do the same construction Markov partitions exist by general theory. ~~They never~~ But the rectangles making them up <u>never</u> have smooth boundaries! (Bowen)

<u>Summarize</u>: $\Lambda, f,$ Markov matrix A
$\theta : \Sigma(A) \to \Lambda$ continuous, surjective Bracket homomorphism

$f \circ \theta = \theta \circ \sigma$

$\Sigma(A) = \{ i \mid A_{i_j i_{j+1}} = 1 \}$

$C_k = \{ i \in \Sigma(A) : i_0 = k \}$ "k-Zylinder"

~~[illegible]~~ $W_\delta^s(i) = \{ i' : i_j' = i_j \quad j \geq 0 \}$
$W_\delta^u(i) = \{ i' : i_j' = i_j \quad j \leq 0 \}$

If i and i' are in the same C_k, then
$[i, i']_j := i_j \quad j \geq 0 \quad \in \Sigma(A)$ unique point of intersection

$\mathcal{W}^{-1}(f(C_k)) \neq (\Sigma)$

$T_k = \theta(C_k)$

$x \in T_k = \theta(\underline{i})$ $\underline{i} \in C_k$ i.e. $i_0 = k$

Have shown: θ maps $W_0^s(\underline{i})$ onto $W^s(x, T_k)$
θ maps $W_0^u(\underline{i})$ onto $W^u(x, T_k)$

The organisation of the T_k's is almost as nice as a markov partition.

Take a pair of indeces k, l such that $A_{kl} = 1$.

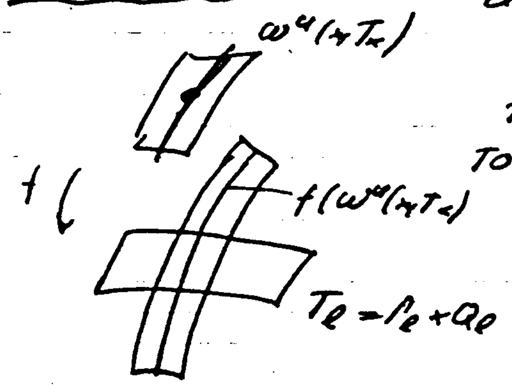
$T_k \cap f^{-1} T_l \neq \phi$

because it contains the image of any $\underline{i}$ with $i_0 = k$, $i_1 = l$.

$$\left.\begin{aligned} &f(T_k) \cap T_l \neq \phi \\ &f(T_k) = Q_k' \times P_k' \\ &T_l = Q_l \times P_l \end{aligned}\right\} \text{canonical coord. around } T_l$$

Lemma: $\boxed{\begin{aligned} Q_k' &\subset Q_l \\ P_k' &\supset P_l \end{aligned}}$

semimarkov property



Because $y \in W^s(x, T_k)$

Put $y := \theta(\sigma(\underline{i}'))$

$f(y) \in T_l$

$\Rightarrow y \in W^s(f(x), T_l)$

⌈ Lift up to $\Sigma(A)$ where it's trivial, project down again:

$x \in T_k$

$f(W^u(x, T_k)) = \{p\} \times Q_k'$

To show: $p \in P_l$ and $Q_k' \supset Q_l$

- Choose $\underline{i}$, with $\theta(\underline{i}) = x$, $i_0 = k$
- Then choose $\underline{i}'$ in $W_0^u(\underline{i})$ with $\underline{i}'(1) = l$.
This is possible because $A_{kl} = 1$

Claim: $f(W^u(x, T_k))$
$= f(\theta_* W_0^u(\underline{i}))$
$= f\theta\, W_0^u(\underline{i}) = \theta\sigma W_0^u(\underline{i}')$
$\supset \theta\, W_0^u(\sigma \underline{i}')$
$= W^u(y, R_l)$

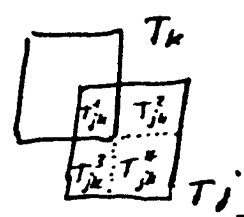
$\Rightarrow \{p\} \times Q_k' \supset \{p\} \times Q_l$

$Q_k' \supset Q_l$ ⌋

Defectiveness: Overlapping possible of this semimarkov partition.

We will make a Markov partition by cutting up the T_k's: Geometrie of rectangles.

~~Let (j,k) so that~~
Let (j,k) so that $T_j \cap T_k \neq \phi$
($j = k$ allowed)



$T_j = P_j \times Q_j$
$T_k = P_k \times Q_k$
$T_{jk}^{(1)} = (P_j \cap P_k) \times (Q_j \cap Q_k) = T_j \cap T_k$
$T_{jk}^{(2)} = (P_j \setminus P_k) \times (Q_j \cap Q_k)$
$T_{jk}^{(3)} = (P_j \cap P_k) \times (Q_j \setminus Q_k)$
$T_{jk}^{(4)} = (P_j \setminus P_k) \times (Q_j \setminus Q_k)$

The $T_{jk}^{(n)}$ are rectangles, mutually disjoint. They cover T_j. Alternative description

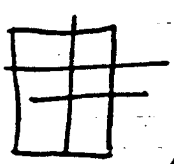
$$T_{jk}^{(1)} = \{x \in T_j : W_{\delta_0}^s(x) \cap T_k \neq \emptyset, \quad W_{\delta_0}^u(x) \cap T_k \neq \emptyset\}$$

(The other cases similarly).

$$Z := \Lambda \setminus \bigcup_{jkn} \partial T_{jk}^{(n)}$$

Lemma

- Z is an open dense subset of Λ
- If R is any rectangle, then $R \setminus Z$ can be covered by a finite union of closed rectangles $\bar{P}_k \times \bar{Q}_k$, where for each k at least one of P_k, Q_k is nowhere dense



Γ Topological lemmas:

For any rectangle
$\partial(P \times Q) = (\partial P) \times Q \cup (P \times \partial Q) \cup \partial P$

$\partial(X \cap Y) \subset \partial X \cup \partial Y$
$\partial(X \setminus Y) \subset \partial X \cup \partial Y$
X closed $\Rightarrow \partial X$ nowhere dense

$$\partial T_{jk}^{(n)} \subset (\partial P_j \cup \partial P_k) \times Q_j \cup \{P_j \times (\partial Q_j \cup \partial Q_k)\}$$ ⌟

$x \in Z$ and $x \in T_{jk}^{(n)} \Rightarrow x \in \mathring{T}_{jk}^{(n)}$

$x \in Z \Rightarrow x \in$ some $T_j \Rightarrow x \in T_{jk}^{(n)} \Rightarrow x \in$ some $\mathring{T}_{jk}^{(n)}$

Define for $x \in Z$ $R(x)$ = intersection of all $\mathring{T}_{jk}^{(n)}$'s containing x is an open rectangle.
$\{R(\cdot)\}$ form a finite set of open rectangles.

Show: The closures of the distinct $R(x)$'s form a Markov partition.

Description of $R(x)$: $x \in Z$
For each j so that $x \in T_j$ and each k ~~so that [illegible]~~ $T_j \cap T_k \neq \emptyset$ there is a unique $n = n(x,j,k)$ so that $x \in T_{jk}^{(n)}$ $\quad x \in Z \Rightarrow x \in \mathring{T}_{jk}^{(n)}$

$$R(x) = \bigcap_{j : x \in T_j} \left\{ \bigcap_{k: T_j \cap T_k \neq \emptyset} \mathring{T}_{jk}^{\,n(x,j,k)} \right\}$$

Lemma $\boxed{x, y \in Z \quad R(x) \cap R(y) \neq \emptyset \implies R(x) = R(y)}$

Γ It suffices to consider the case, where $y \in R(x)$.
The general case: Take $z \in R(x) \cap R(y) \cap Z$
special case: $R(x) = R(z) = R(y)$

$y \in R(x) \implies y \in T_{jk}^{(n)}$ which contain x
Need to show: y is not in any other $T_{jk}^{(n)}$.
To show: j's s.t. $x \in T_j$ are the same as the j's s.t. $y \in T_j$

$y \in R(x) \subset$ intersection of all T_j's containing x
$x \in T_j \implies y \in T_j$

$y \in T_k$;
For any j s.t. $x \in T_j$, $y \in R(x) \implies T_k \cap T_j \neq \emptyset$
In fact $y \in T_j \cap T_k = T_{jk}^{(1)}$
Simply choose some such j. $y \in T_{jk}^{(1)}$. $y \in R(x) \subset T_{jk}^{(n)}$
for some n.
This $n(x, j, k)$ must be 1.
$\implies x \in T_{jk}^{(1)} = T_j \cap T_k \subset T_k \implies x \in T_k$ ~~$\#$~~

$$R(x) = \bigcap_{j : x \in T_j} \left\{ \bigcap_{k : T_j \cap T_k \neq \emptyset} \overset{\circ}{T}_{jk}{}^{n(x, j, k)} \right\}$$

$$R(y) = \bigcap_{j : x \in T_j} \left\{ \bigcap_{k : T_j \cap T_k = \emptyset} \overset{\circ}{T}_{jk}{}^{n(y, j, k)} \right\}$$

If $n(x, j, k) \neq n(y, j, k)$ for some j and k than
$R(x) \cap R(y) = \emptyset$ $\lightning$

Therefor $n(x, j, k) = n(y, j, k)$ $\forall j, k \implies R(x) = R(y)$ $\lrcorner$

Let $\{R(x_1), \ldots, R(x_p)\}$ be an enumeration of distinct $R(x)$.

$R_i := \overline{R(x_i)}$

Want to show, that $\{R_1, \ldots, R_p\}$ is a Markov partition.

$R_i \supset R(x_i)$

$\bigcup_i R(x_i) \supset Z$ is dense $\implies \bigcup R_i = \Lambda$ because $\bigcup R_i$ is closed.

$\overset{\circ}{R}_i \supset R(x_i)$

$\overset{\circ}{R}_i$ is dense in $R_i \implies R_i$ is a proper rectangle

$$\emptyset = R(x_i) \cap R(x_j) \overset{\text{because } R(x_j) \text{ is open}}{=} \overline{R(x_i)} \cap R(x_j) = \overset{\circ}{R}_i \cap R(x_j) = \overset{\circ}{R}_i \cap \overline{R(x_j)}$$
$$= \overset{\circ}{R}_i \cap \overset{\circ}{R}_j$$

To prove the R_i are a Markov partition we have to proof:

- $R_i \cap f^{-1}(R_j) \neq \emptyset$ and if $x \in R_i \cap f^{-1}(R_j)$

(1) • $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$

(2) • $f(W^u(x, R_i)) \supset W^u(f(x), R_j)$

The proof of (2) is the proof of (1) with f replaced by f^{-1} and W^s replaced by W^u.

Only give the proof for (1).

$f(R_i) = P_i' \times Q_i'$

$R_j = P_j \times Q_j$

$f(x) = (p, q)$

$f(W^s(x, R_i)) = P_i' \times \{q\}$

$W^s(f(x), R_j) = P_j \times \{q\}$

Statement $\Leftrightarrow P_i' \subset P_j$

Because sets involved are automatically rectangels, it suffices to show, that for one $x \in R_i$ such that

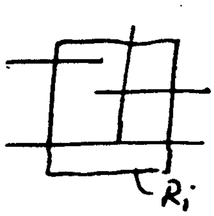
$$f(x) \in \mathring{R}_j$$

we have $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$.

Want to choose x judiciously:

$x \in \underbrace{\mathring{R}_i \cap f^{-1}(\mathring{R}_j)}_{\text{open}} \cap \underbrace{Z \cap f^{-1}(Z)}_{\text{open dense}} \neq \emptyset$

Claim: Can choose x, so that $W^s(x, R_i) \cap Z \cap f^{-1}(Z)$ is dense in $W^s(x, R_i)$



$R_i \setminus (Z \cup f^{-1}(Z)) \subset P_i \times \hat{Q} \cup \hat{P} \times Q_i$

$\hat{P}_i \times Q_i$ where $\hat{Q}$ and $\hat{P}$ are nowhere dense

Choose x $\underset{(p,q)}{}$, so that ~~p is not in $\hat{P}$~~ $q \notin \hat{Q}$.

Then $W^s(x, R_i) = P_i \times \{q\}$

$W^s(x, R_i) \cap Z \cap f^{-1}(Z) \supset \underbrace{(P_i - \hat{P})}_{\text{dense in } P_i} \times \{q\}$

x is now chosen.

Now we have to proof, that

$$f(W^s(x, R_i) \cap Z \cap f^{-1}(Z)) \subset W^s(f(x), R_j)$$

then the image of the closure will be containd also.

enough

$$f(W^s(x, R_i) \cap Z \cap f^{-1}(Z)) \subset R_j$$

Lemma: $\boxed{\begin{array}{l}\text{If } x, y \in Z \cap f^{-1}(Z), \text{ if } R(x) = R(y) \ (\Leftrightarrow y \in R(x)) \\ \text{and if } y \in W^s_{R_0}(x) \\ \text{then } R(f(x)) = R(f(y))\end{array}}$

Proof: First step: Show, that $f(y)$ is in the same T_j's as $f(x)$ is.

Suffices by symmetrie of hypothesis in x & y

$$f(x) \in T_j \Rightarrow f(y) \in T_j$$

Choose $\underline{i}$, such that $\theta(\underline{i}) = x$; $i_1 = j$

Write k for i_0

$A_{kj} = 1$

$x \in T_k$

$R(y) = R(x) \Rightarrow y \in T_k$

$y \in W_{\eta_0}^s(x)$

$y \in W^s(x, T_k)$

$\exists \underline{i}' \in W_0^s(\underline{i})$ s.t. $\theta(\underline{i}') = y$

$i_1' = i_1 = i \Rightarrow f(y) = \theta(\sigma \underline{i}') \in T_j$

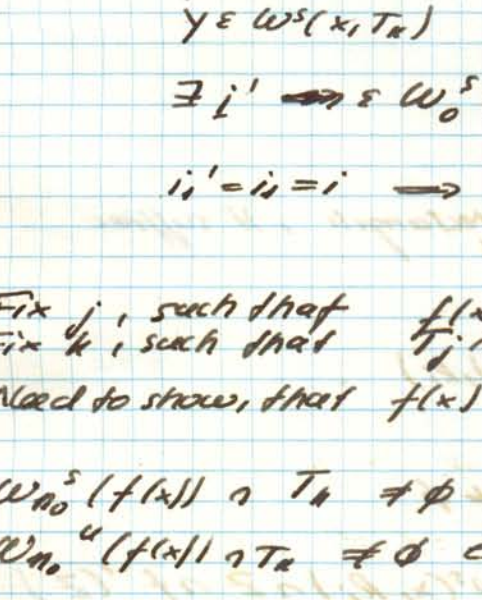
Fix j, such that $f(x) \in T_j \Leftrightarrow f(y) \in T_j$

Fix k, such that $T_j \cap T_k \neq \phi$

Need to show, that $f(x)$ is in the same $T_{jk}^{(n)}$ as $f(x)$ is.

- $W_{\eta_0}^s(f(x)) \cap T_k \neq \phi \Leftrightarrow W_{\eta_0}^s(f(y)) \cap T_k \neq \phi$ ← immediate
- $W_{\eta_0}^u(f(x)) \cap T_k \neq \phi \Leftrightarrow W_{\eta_0}^u(f(y)) \cap T_k \neq \phi$

For the second: Suffices to proof, that $W_{\eta_0}^u(f(x)) \cap T_k \neq \phi$
$\Rightarrow W_{\eta_0}^u(f(y)) \cap T_k \neq \phi$



$W_{\eta_0}^u(f(x)) \cap T_k \cap T_j \neq \phi$

Take $f(z) \in W_{\eta_0}^u(f(x))$

$f(z') = [f(z), f(y)]$

$\boxed{f(z') \in T_k}$

$x = \theta(\underline{i})$ with $i_1 = j$ $\quad i_0 = r$

$z = \theta(\underline{i}')$ with $i_1' = k$ $\quad i_0' = s$

$f(z) \in W^u(f(x), T_i) \xrightarrow{\text{semi-Markovprop.}} z \in W^u(x, T_r)$

$z \in T_s$

$W^u(x, T_r) \cap T_s \neq \phi$

Because $R(x) = R(y)$, $W_{\eta_0}^u(y) \cap T_s \neq \phi$

Take $z'' \in W_{\eta_0}^u(y) \cap T_s$

Claim: $z' := [z, z''] \in T_s$ since $z, z'' \in T_s$

Also $z' \in W_{\eta_0}^u(y)$

$z' = [z, y]$ $\qquad f(z') = [f(z), f(y)]$

By semimarkovproperty : $z' \in W^s(z, T_s)$
$\Rightarrow f(z') \in W^s(f(z), T_u)$
In particular $\boxed{f(z') \in T_u}$

Zeta functions of hyperbolic sets

~~Let~~ $f : X \circlearrowleft$

$N_p(f) = \#$ fixed points of f^p

Assume : $N_p(f)$ is finite for all p

$$\zeta(f)(z) = \exp\left(\sum_{p=1}^{\infty} \frac{N_p(f)}{p} z^p\right)$$

$P_q = \#$ of periodic cycles of period q
$q \cdot P_q = \#$ of periodic points of period q

$$N_p = \sum_{q|p} q \cdot P_q = p \cdot P_p + \sum_{\substack{q<p \\ q|p}} q \cdot P_q$$

Möbius Inversion formula

Nice Situation : $\zeta(z)$ is rational

If $\zeta(z)$ is rational, $\zeta(z) = \dfrac{\prod_{i=1}^{r} (1-\lambda_i z)}{\prod_{j=1}^{s} (1-\mu_j z)}$

$$\log \zeta(z) = \sum_{i=1}^{r} \log(1-\lambda_i z) - \sum_{j=1}^{s} \log(1-\mu_j z)$$

$$\log \zeta(z) = \sum_{p=1}^{\infty} \frac{N_p}{p} z^p$$

$$\sum N_p z^{p-1} = \sum_{i=1}^{r} \left(\frac{-\lambda_i}{1-\lambda_i z}\right) - \sum_{j=1}^{s} \left(\frac{-\mu_j}{1-\mu_j z}\right) = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{s} \mu_j^{n+1} - \sum_{i=1}^{r} \lambda_i^{n+1}\right) z^n$$

$$\rightarrow \boxed{N_p = \sum_{j=1}^{s} \mu_j^p - \sum_{i=1}^{r} \lambda_i^p} \quad \text{For } p = 1, 2, \dots$$

$$\Updownarrow$$

$$\boxed{\zeta(z) = \frac{\prod_{i=1}^{r} (1-\lambda_i z)}{\prod_{j=1}^{s} (1-\mu_j z)}}$$

Theorem Guckenheimer Manning

> Λ hyperbolic set for f with local product structure
> then
> $\zeta(f|_\Lambda)$ is rational

Mannig's proof goes via symbolic dynamics.
would be trivial if can make π be 1-1 above all periodic pts.

If A is a Markov Matrix, then $N_p(\sigma) = \#$ of A-admissible periodic sequ
σ : left shift on $\Sigma(A)$

$N_p(\sigma)$ = # of A-admissible periodic sequences of period dividing p
$= tr(A^p)$ (Exercise)

spectrum of $A = \{\mu_1, \dots, \mu_\omega\}$ including multiplicity.

$tr\, A^p = \sum_{j=1}^{\omega} \mu_j^p$ As above, no A's.

Guckenheimer : Lipschitz fixpoints

Hyperbolic Attractors

$f : X \circlearrowleft$ continuous

Def $\Lambda \subseteq X$ an attracting set, if Λ is closed, invariant $f(\Lambda) = \Lambda$ and there exists a nbh. U of Λ, such that

$$f^n U \longrightarrow \Lambda$$

in the sense that, if V is any nbhd of Λ, then $f^n(U) \subseteq V$ for all suffi. large n.

Can assume : $fU \subset U$

If Λ has a compact nbhd,

$\boxed{\Lambda \text{ is attracting}} \iff \boxed{\exists \text{ compact Nbhd. } U \text{ of } \Lambda \text{ s.that } fU \subset U \text{ and } \bigcap_n f^n U = \Lambda}$

$B(\Lambda)$ basin of attraction of Λ $= \{x : f^n(x) \xrightarrow{n \to \infty} \Lambda\}$
$= \{x : f^n(x) \in U \text{ for some } n \geq 0\}$
$= \bigcup_{n=0}^{\infty} f^{-n} U$

$B(\Lambda)$ is open.

Def : A hyperbolic attractor for a C^1 diffeom. f is a set which is both hyperbolic and attracting.

Proposition : Λ hyperbolic set for f. Then equivalent are

(1) Λ is a hyperbolic attractor
(2) $\exists$ arbitrary small nbhd's U of Λ such that $f(U) \subset U$
(3) For all sufficiently $\eta > 0$
$\bigcup_{x \in \Lambda} W^s_\eta(x)$ is a nbhd of Λ

If this hold

(4) $\forall x \in \Lambda$ $W^u(x) \subset \Lambda$
(5) Λ has local product structure

Proof $(1) \to (2)$ immediate
$(4) \to (5)$ immediate

a) $(2) \Rightarrow (4)$
b) $(2)+(4) \Rightarrow (3)$
c) $(3) \Rightarrow (1)$

a) $(2) \Rightarrow (4)$

$U : f(U) \subseteq U$
$x \in \Lambda \quad y \in W^u(x)$

$\| f^{-n}(x) - f^{-n}(y) \| \to 0$

$f^{-n}(y) \in U \quad \exists n \geq 0$

$y = f^n(f^{-n}(y)) \in U$

since U can be taken to be an arbitrarily small nbhd of Λ, $W^u(x) \subset \Lambda$.

b) $(2)+(4) \Rightarrow (3)$

Much easier: Λ local product structure, any orbit $\{f^n(y)\}_{n \in \mathbb{N}}$ which stays near enough to Λ for all n, is in $W^s(x)$ for some $x \in \Lambda$. (shadowing).

For any $\eta > 0$ $\exists \delta > 0$ such that $f^n(y) \in \delta$ nbhd of Λ for all $n \geq 0$, then $y \in W^s_\eta(x)$ for some $x \in \Lambda$.

Take U such that $fU \subset U$; $U \subset \delta$ nbhd of Λ

$y \in U \to f^n(y) \in U \quad \forall n \geq 0$

$\Rightarrow y \in W_\eta^s(x)$.

$U \subseteq \bigcup_{x \in \Lambda} W_\eta^s(x)$

, $\bigcup_{x \in \Lambda} W_\eta^s(x)$ is a nbhd of Λ.

c) $(3) \Rightarrow (1)$

Take small $\eta > 0$.

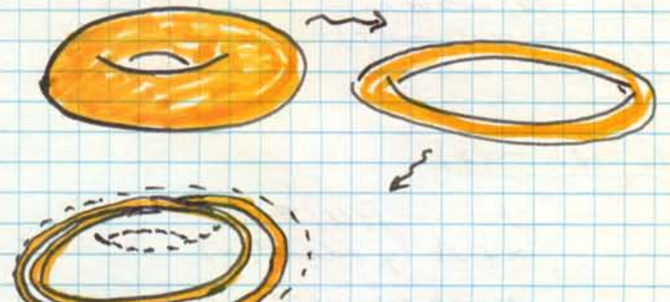
$f^n W^s_\eta(x) \subset W^s_{\lambda^n \eta}(f^n(x)) \qquad 0 < \lambda < 1$

$f^n \underbrace{\left(\bigcup_{x \in \Lambda} W_\eta^s(x) \right)}_{\text{is an attracting set and has defin. properties}} \subset \lambda^n \eta$ nbhd of Λ ∎

Remark: $(4) \Leftrightarrow (1) \Leftrightarrow 2 \Leftrightarrow (3)$ is known to be true.

Example: Solenoid

In $\mathbb{R}^3$ (not possible in $\mathbb{R}^2$)



T: solid torus

$f: T \to T$

$f(T)$ is a thin torus wrapping around the hole twice.

$\underbrace{f^n T}_{\text{wrapping around } 2^n \text{ times}} \subset f^{n-1} T \subset \dots \subset fT \subset T$

$\Lambda := \bigcap_{n=0}^{\infty} f^n(T)$ is made up of lines of zero thickness winding infinitely often around the hole in T.

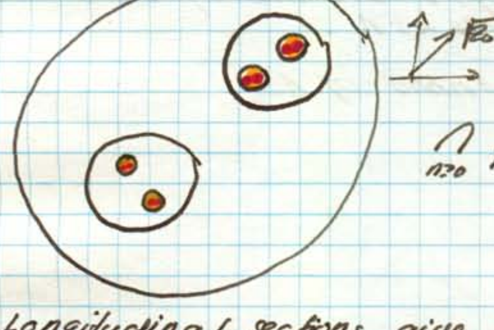
There are uncountably many lines!

Explicit formulas: $T = S^1 \times D^1 = \{z \mid |z| = 1\} \times \{w \mid |w| \le 1\}$
$= \{(z, w) \mid |z| = 1, |w| \le 1\}$

$f(z, w) = (z^2, \frac{1}{2} z + \frac{1}{4} w)$

A transverse section means an intersection with $\{z = z_0\}$

$fT \cap \{z = z_0\} = \{(z_0, \frac{1}{2}\sqrt{z_0} + \frac{1}{4} w)\} \cup \{(z_0, -\frac{1}{2}\sqrt{z_0} + \frac{1}{4} w)\}$



$\bigcap_{n \ge 0} f^n T \cap \{z = z_0\}$ is a Cantor set.

Exercise: where comes a ~~z~~ in ~~startin~~ transverse section starting point back in the section

Longitudinal sections give smooth curves:

$z = e^{i\theta}$ $\sqrt{z} = e^{i\theta/2}$ $z_0 \ne 1$ fixed

$(i_0, i_1, i_2, \dots)$ sequence of ± 1
gives a point in the Cantor set $\frac{\xi_0}{2} + \frac{1}{2}\frac{\xi_1}{4} + \frac{1}{2}\frac{\xi_2}{16} + \dots$
$= \xi(z, i)$

$\xi_0 = i_0 \sqrt{z_0}$
$\xi_1 = i_1 \sqrt{\xi_0}$
$\xi_2 = i_2 \sqrt{\xi_1}$
$\vdots$

By changing θ, the point changes analytically:
$\xi(z, i)$ is a smooth function of z on the circle cut at 1.

w direction is invariant and contracting:
the stable manifolds = transverse sections $\{z = z_0\}$
are two dimensional.
The expanding direction: Tangent to the longitudinal curves.
$\Rightarrow$ longitudinal curves are the unstable manifolds

Markov partitions and symbolic dynamics for the solenoid

$R_0 = \{(e^{i\theta}, w) \mid 0 \le \theta \le \pi\} \cap \Lambda$
$R_1 = \{(e^{i\theta}, w \mid \pi \le \theta \le 2\pi\} \cap \Lambda$

is a Markov partition

Markov matrix : $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\Sigma(A) = \{0,1\}^{\mathbb{Z}}$

$\pi : \{0,1\}^{\mathbb{Z}} \to \Lambda$ onto

$\pi(\underline{i}) =$ the x such that $f^{j}(x) \in R_{i_j} \ \forall j$

The z component of $\pi(\underline{i})$ is determined just by the $(i_0, i_1, \dots)$ i.e. by future of $\underline{i}$

correspondence : $z = e^{i\theta}$

$$\frac{\theta}{\pi} = \sum_{j=0}^{\infty} \frac{i_j}{2^j} \qquad (i_0, i_1, \dots) \text{ binary expansion of } \frac{\theta}{2\pi}$$

Once the z component is fixed, the w component is a fn of $(i_{-1}, i_{-2}, \dots)$

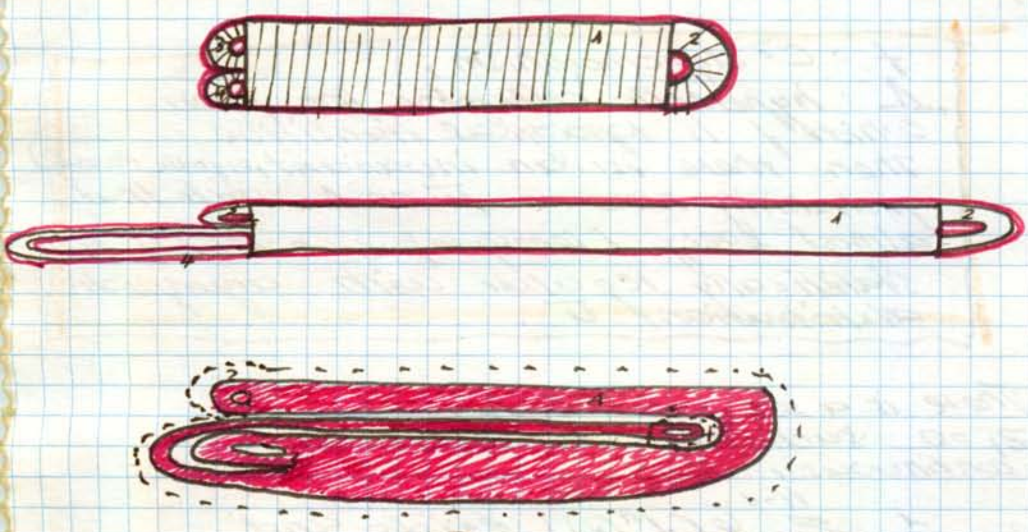
$(i_{-1}, i_{-2}, \dots) \to w$ is a homeomorphism

Non injectness of π : Nonuniqueness of binary expansion : $.100\dots = .0111\dots$

Remark : Smale horseshoe : Also a projection. The same symbolic dynamics. where it was ~~a~~ injectiveness.

Exercise : $\boxed{\begin{array}{l} \cdot \text{ Compute } \zeta \text{ Funktion} \\ \cdot \text{ Proof statements} \\ \cdot \text{ Other Markov partitions finer.} \end{array}}$ $e^{\frac{1}{1-z}}$

Example : Plykin attractor in two dimension



Can't start building in $\mathbb{R}^2$ hyperbolic attractor beginning with a region with 2 holes

Exercise : ζ la funktion

X topol. space $f : X \circlearrowleft$ continuous
$x \in X$ $\{f^n(x) : n > 0\}$ is compact

$\varphi : X \to \mathbb{R}$ continuous "Observable"

$\frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ f^n(x)$ average,

Def : Say : x is statistically regular if for every continuous φ, the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ f^n(x) \text{ exists}$$

If this limit exists, $\varphi \mapsto \lim \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ f^n(x)$ is a normalized positiv. linear fn on the space of cont. fn's, i.e. it is a measure. Call it $\bar{\mu}_x$ the asymptotic distribution of orbit of x

$$\int \varphi \, d\bar{\mu}_x = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ f^n(x) \quad \forall \varphi$$

$\bar{\mu}_x$ is an invariant measure of f: $\bar{\mu}_x(E)=\bar{\mu}_x(f^{-1}E)$
$\forall E$ measurable.

(X,μ) measurable space with probability measure μ
$f: X \to X$ measurable leaving μ invariant.

Birkhoff Pointwise ergodic theorem:

$\forall$ integrable φ, the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi\circ f^n = \bar{\varphi}$
exists a.e. and $\int \bar{\varphi}\, d\mu = \int \varphi\, d\mu$

Twist: The set of x's, which are statistically regular is a set of measure 1 for every invariant probability measure.

An invariant measure is ergodic, if E is measurable, such that $f^{-1}(E)=E$ implies: $\mu(E)=0$ or $\mu(X\setminus E)=0$, or
the the only invariant measurable functions are constants

If μ is ergodic, then almost all x's are statistically regular with asymptotic distribution μ: $\int \varphi\, d\bar{\mu}_x = \bar{\varphi}(x) \underset{\mu\text{-a.e.}}{=} \int \bar{\varphi}\, d\mu = \int \varphi\, d\mu$

~~whe~~ If we have two different ergodic measures, then
$\mu_1 \{x \mid x$ statist. regular with respect to $\mu_1\} = 1$ $\quad \mu_2(\ldots)=0$
$\mu_2 \{x \mid x$ statist regular " " " to $\mu_2\} = 1$ $\quad \mu_1(\ldots)=0$

Ruelle-Bowen-Ergodic theorem

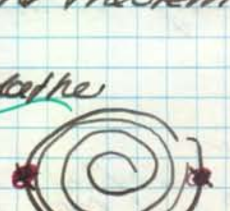
f: C^2 diffeomorphism,
Λ hyperbolic attractor for f on which f is topological transitive.
Then there is an invariant, ergodic (Poincaré) probability measure $\bar{\mu}$ on Λ, such that almost every (Lebesgue) $x \in B(\Lambda)$ is statistically regular with asymptotic distribution $\bar{\mu}$.

This says: There is a set $Z \subset B(\Lambda)$
$\mu_{Leb}(Z)=0$ such that $x \in B(\Lambda)\setminus Z$ and
φ is continuous, then
$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi(f^n(x))$ exists and
is independent of x.

Role of Lebesgue measure: Used to formulate precisely the intuitive notion of an negligible set of initial conditions.

(If Lebesgue measure were invariant and ergodic then the theorem would follow from the Birkhoff thm.

Example of Hathe



- No orbit inside is statistically regular.

Fixedpoints $\quad$ Point spends 99% near fixed points.

Proof after some remarks:

1. It suffices to show: For each continuous φ, there is a Z_φ of Lebesgue meas. 0, such that $\lim_{N\to\infty} \frac{\sum \varphi f^n(x)}{N}$ exists and is const. on $B(\Lambda) \setminus Z_\varphi$.

   We can take than $Z_{indiff} = \bigcup Z_{\varphi_n}$, where the φ_n's are a countable dense set in $C(M)$.

2. Can replace $B(\Lambda)$ by an arbitrary nbhd of Λ.
Every orbit in $B(\Lambda)$ eventually gets and stays as near as desired to Λ. Limits are independent of any finite number of pts on the orbit. If set of bad points intersects a nbhd of Λ in a set of zero Lebesgue measure, then it has zero Lebesgue measure because Lebesgue measure is quasiinvariant under diffeomorphism. (preimage of a set of measure zero has measure zero.

$\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi of^n(x)$ is strictly invariant. If it exists for x, than it exists for $f(x)$ and $f^{-1}(x)$ und values are the same.

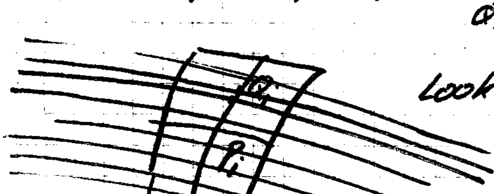
Limit exists for $\varphi \Longleftrightarrow$ It exists for φf^n $n \in \mathbb{N}$.

f^n maps $W_\eta^s(z)$ into $W_{\lambda^n \eta}^s(f^n(z))$

n large: $\varphi o f^n$ is almost constant on local stable manifolds

To see the idea: Suppose that φ is constant on local stable manifolds.

Put down a Markov partition $R_1 \dots R_p$ covering Λ.
$R_i = P_i \times Q_i$ P_i is a little piece of stable manifold.
Q_i piece of unstable manifold.



Look at $\# \bigcup_{x \in Q} W_\eta^s(x)$ $\eta >> \text{diam } R_i$
contains $\bigcup_{x \in R_i} W_{\eta'}^s(x)$ $\eta' < \eta$

$\bigcup_i W_\eta^s(x) \supset \bigcup_{x \in \Lambda} W_{\eta'}^s(x)$ nbhd of Λ.

By assumption: Funktion φ is a funktion on Q_i.
Project Lebesgue measure on $\bigcup_{x \in Q_i} W_\eta^s(x)$ along W_η^s's onto Q_i.

This is a measure on Q_i. This is a finite measure on $\coprod_{i=1}^{p} Q_i$ disjoint union

Symbolic dynamics: A = Markov matrix associated with $\{R_1, \dots R_p\}$

$\pi: \Sigma(A) \longrightarrow \Lambda$ onto, almost 1-1

$\pi(i^{(1)})$ and $\pi(i^{(2)})$ are in the same local stable manifold iff $i_j^{(1)} = i_j^{(2)}$ $j \geq 0$.

$\pi^+: \Sigma_+(A) \xrightarrow{\text{onto}} \coprod_{i=1}^{p} Q_i$ $\qquad \Sigma_+(A) - \{ i$ defined for $j \geq 0$, $A_{i_j i_{j+1}} = 1 \ \forall j \geq 0 \}$

$\pi(\underline{i}) = x$ i_0 tells me which R_i x is in.
$(i_j)_{j\geq 0}$ tells me which point of Q_{i_0} corresponds to x
$(i_j)_{j \leq 0}$ tells me which point of the P_i component of x.

Lift everything to $\Sigma_+(A)$
φ lifts to $\underbrace{\varphi o \pi_+}_{\text{called } \varphi \text{ again}}$, want to lift the measure.

This is nonfunctorial.
We do it by:

$$C_{i_0 \cdots i_{n-1}} = \{ \underline{i} : i_j = \bar{i}_j \text{ for } j = 0, \dots, n-1 \}$$

Zylinder sets in $\Sigma_A(A)$

Construct measure λ on $\Sigma_A(A)$ by $\lambda(C_{i_0 \cdots i_{n-1}}) =$ projected Lebesgue measure of $\pi_A(C_{i_0 \cdots i_{n-1}})$

Need to know, that the measure of the overlaps is zero. (one sees that at the end of the argument)

Once everything has been lifted.

Want to show $\lim\limits_{N \to \infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} \varphi \circ \sigma^n(\underline{i})$

exists and is constant λ almost everywhere on $\Sigma_A(A)$.

Big ideas:

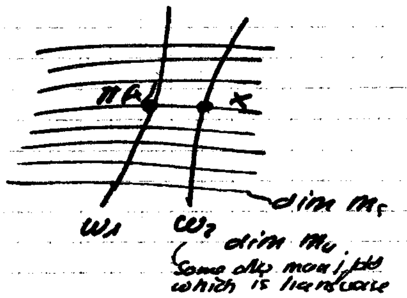
1. ~~## #~~ Show, that λ is equivalent to an invariant ergodic measure μ on $\Sigma_A(A)$ (same sets of measure zero.)
   Then the existence and constancy of time average follows from Birkoff.

2. μ is the equilibrium measure (Gibbs measure for an auxilliary statistical mechanical system built of $\Lambda, f, \{R_1 \cdots R_p\}$.

dimension: $m = m_s + m_u$

Technical difficulties:

Absolute continuity of contracting foliation



The stable manifolds fill up a whole neighborhood

Any point x on W_2 is on some local stable manifold which intersects W_1 somewhere. Call intersection point $\pi(x)$

$\pi : W_2 \to W_1$

π is continuous (even Hölder contin.) but in general it needn't be differentiable.

However: π is absolutely continuous:

π maps Lebesgue measure on W_2 to a measure equivalent to Lebesgue measure on W_1. Furthermore, the Radon-Nykodym derivative is Hölder continuous.

x_1 coordinates for W_1
x_2 " " W_2

Then there is a strictly positive Hölder contin. fn g on W_1 such that

$$\int_{E \subset W_1} g(x_1)\, dx_1 = \int_{\pi^{-1}(E)} dx_2 .$$

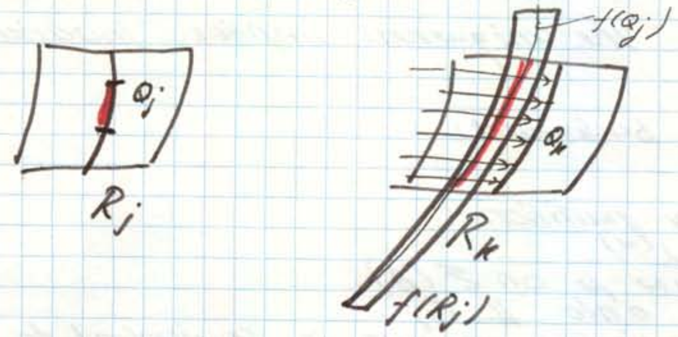
If π were differentiable then g would be the Jacobian of π.
$g(x) = J(\pi^{-1}(x))$

Consequence: The projected Lebesgue measure is equivalent to ordinary Lebesgue measure on Q_i - subset of some local unstable manifold.

Assume, that λ is the lift of ordinary Lebesgue measure on Q_i.

Determining lifted measure:

Take (jk) with $A_{jk} = 1$.



f_{jk} : subset of $Q_j \to Q_k$

f_j then projection along stable manifolds.

f_{jk} is absolutely continuous. with respect to Lebesgue measure

Given a sequence $(i_0 \cdots i_{n-1})$ which is admissible, then

$$\lambda(C_{i_0 \cdots i_{n-1}}) = |\Delta(i_0 \cdots, i_{n-1})|$$

where $\Delta(i_0, \cdots, i_{n-1}) = \{ x \in Q_{i_0} \mid f^j(x) \in R_{i_j} \text{ for } j = 0, 1, \cdots, n-1 \}$ (Lebesgue measure)

$$\Delta(i_0 \cdots i_{n-1}) = f_{i_0 i_1}^{-1} \Delta(i_1, \cdots, i_{n-1})$$

Lebesgue measure $|\Delta(i_0 \cdots i_{n-1})|$ = average of Jacobian of $f_{i_0 i_1}^{-1}$ over $\Delta(i_1 \cdots i_{n-1})$ $\times$ $|\Delta(i_1 \cdots i_{n-1})|$

If n is large, $\Delta(i_1 \cdots i_{n-1})$ is small and the Jacobian is essentially constant on it.

$J^{(u)}(i_0, \ldots)$ $\quad \underline{i} \in \Sigma_+(A)$
= Jacobian of $f_{i_0 i_1}$ at $\pi(\underline{i})$ (which is Hölder continuous)

$$|\Delta(i_0 \cdots i_{n-1})| \simeq \frac{1}{J^u(\underline{i})} |\Delta(i_1 \cdots i_{n-1})|$$

$J^{(u)}(i_0, i_1, \ldots)$ depends exponentially little on $(i_n, i_{n+1}, \ldots)$

i.e. $\exists \varepsilon > 0$ $\quad \dfrac{J^{(u)}(\underline{i}^{(1)})}{J^{(u)}(\underline{i}^{(2)})} \leq e^{-\varepsilon n}$ if $i_j^{(1)} = i_j^{(2)}$ $j = 0, \ldots n-1$

$$|\Delta(i_0, \cdots, i_{n-1})| \approx \prod_{j=0}^{n-1} \frac{1}{J^{(u)}(i_j, i_{j+1}, \cdots)}$$

The error of this approxim. is bounded uniformly in n.

Want to make an invariant measure equivalent to λ:
Idea:

$$\mu(\{\underline{i}\}) \qquad \underline{i} \in \Sigma(A) \quad \underline{\text{not}}\ \Sigma_+(A)$$

$$\propto \prod_{j=-\infty}^{\infty} \frac{1}{J^{(u)}(\sigma^j \underline{i})}$$ which is translation invariant

(proportional)

But this product doesn't make any sense.
In statistical mechanics (of 1 dim lattice systems)

$$\mu(\{\underline{i}\}) \propto e^{-\beta H(\underline{i})}$$ (infinite configuration)

Make a $\underline{\text{local}}$ version of this prescription

$$\frac{\mu(\{\underline{i}^{(1)}\})}{\mu(\{\underline{i}^{(2)}\})} = e^{-\beta \{ H(\underline{i}^{(1)}) - H(\underline{i}^{(2)}) \}}$$

$\underline{i}^{(1)}, \underline{i}^{(2)}$ differ only in finitely many places

This is the notion of Gibbs state, which is unique for 1 dim systems with rapidly decaying interaction.

Invent a statist. mechanical system so that $e^{-\beta H(\underline{i})} = \prod_{j=-\infty}^{\infty} \frac{1}{J^{(u)}(\sigma^j \underline{i})}$

(Formally)

Apply unique Gibbs state to this system:

Gibbs state $= \mu$. The uniqueness implies invariance and ergodicity of μ.

Actual proof goes backwards:

- Choose Markov partition.
- Construct $J^{(u)}$
- Build measure μ on $\Sigma(A)$
- Project this onto ΠQ_i
- Show that projected measure is equivalent to the projected Lebesgue measure.
- Apply formal scheme.

The theorem requires f to be C^2. It's needed for the ε-continuity of projection along stable manifolds.

Another thing which comes out:

Proposition

$\boxed{\text{$f$ C^2 diffeomorphism. Λ a hyperbolic set for f which has local product structure but which is not an attractor (Ex. Smale horseshoe) then $\bigcup_{x \in \Lambda} W^s(x)$ has Lebesgue measure zero.}}$

"This result is definitely false for $f \in C^2$. Can find "fat horseshoes"." ($f \in C^1$)

Corollary

$\boxed{\text{$f \in C^2$ and satisfies Axiom A, then almost all orbits (Lebesgue) converge to hyperbolic attractors.}}$

$$\Omega(f) = \bigcup_{i=1}^{p} \Omega^{(i)}(f)$$

topol. transit. and hyperbol.:

Remarks:

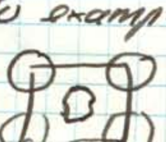
Hyperbolic attractors (other than fixed points and periodic orbits) do not seem to turn up very often in applications.

Another kind of example: Lorentz attractor.

Set of 3 coupled quadratic eq ordinary differential equation. (Approxim. for two dimensional convection) For appropriate parameter values, this system has an attractor with uniform expansion and contraction along well defined directions, but these directions have singularities. (Produced by having orbits pass very near a hyperbolic fixed point.

"Hyperbolicity with singularities"

Another example: Sinai billiards



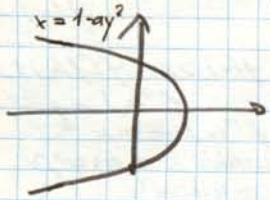
In a three dim state space

Singularities produced by collisions at zero angle

The main analytical ideas about par hyperb. systems can be adapted to such systems by an ad hoc way.

(shadowing tends to fail)

Even more frequently occur things like Henon attractor



$$T : (x,y) \mapsto (1 - ax^2 + by, x) \qquad b \text{ small}$$

$$DT : \begin{pmatrix} -2ax & b \\ 1 & 0 \end{pmatrix}$$

Argument: It can't be a hyperbolic attractor if intuition is alright.

Why can't this be a hyperbolic attractor?

Stable manifolds, if exist have to be approximately vertical.

unstable manifolds have to lie along the curve $\{x = 1 - ay^2$

Stable and unstable manifolds have to be tangent to each other.

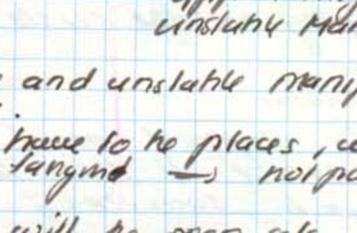
There have to be places, where stable and unstable manifolds have to be tangent $\rightarrow$ not possible that it's a hyperbolic set.

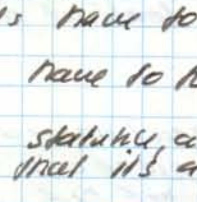
There will be open sets, where systematic expansion and contraction fails.

Things seem to be fundamentally more complicated than hyperbolicity.

Differentiable intervall mappings with turning point (smooth folding in the simplest form)

Simple Model: $f_r \quad x \mapsto 4rx(1-x)$ where r is a parameter $0 \le r \le 1$. Then f_r maps $[0,1]$ to itself.





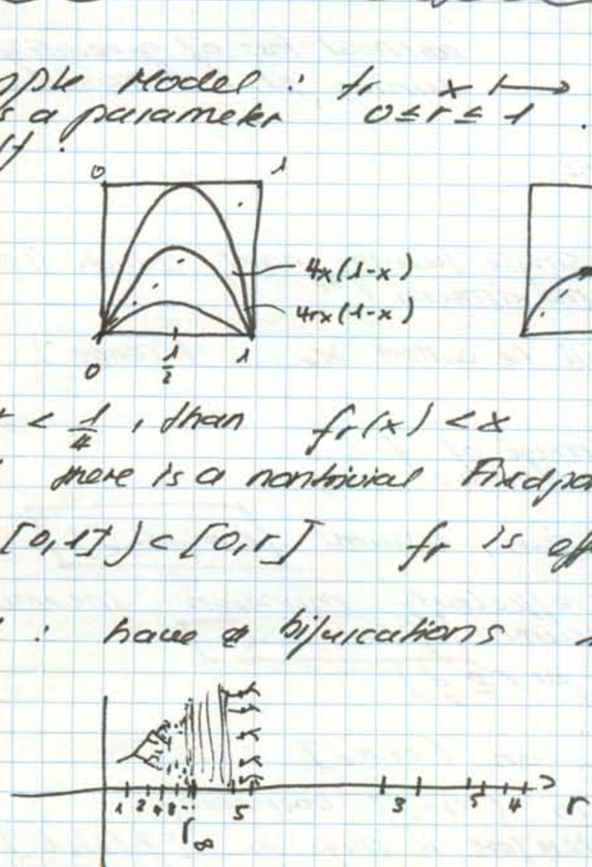
If $r < \frac{1}{4}$, than $f_r(x) < x \quad x \in [0,1] \qquad f_r^{\,n}(x) \to 0$

$r > \frac{1}{4}$ there is a nontrivial Fixedpoint. For $r < \frac{1}{2}$

$f_r([0,1]) \subset [0,r] \qquad f_r$ is effectively monotonic.

$r > \frac{1}{2}$: have a bifurcations $1 \to 2 \to 4 \to 8 \to \ldots$

$r = r_\infty < 1$



"windows" Range of parameters, where f_r has an attracting cycle of indicated period.

Qualitatively, this bifurcation history is universal.

For $g_r(x) = r\sin(\pi x)$ you see the same structure of periodic windows.

If $f_r(x)$ 

~~because~~ get another sequence of windows.

• Monotone mappings produce uninteresting dynamics

$f : [a,b] \to$ continuous nondecreasing

$\boxed{\text{For any } x \in [a,b] \quad f^n(x) \to \text{fixed pt.}}$

[2 possib. $f(x) > x$
$f(x) = x$ ✓

$f(x) > x \quad f(f(x)) \geq f(x)$
$\therefore \; f^n(x) \geq f^{n-1}(x)$
$f^n(x) \to x^*$
which is a fixed pt. by continuity]

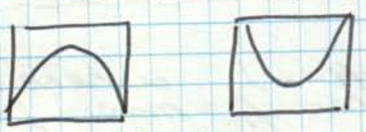
f ~~non~~ decreasing
$\to f^2$ nondecreasing
Every orbit of f is asymptotic to either a fixed point or a cycle of period 2.

Another escape: Look at a mapping on the circle.
~~Notation~~

Notation: f: real valued continuous fn defined on $[a,b]$.
f is piecewise monotone, if there exist $x_0 = a < x_1 < x_2 < \ldots < x_\ell = b$ such that f is <u>strictly</u> monotone on each $[x_{i-1}, x_i]$.
not essential

$\bar{x}$ is called a <u>turning point</u> of f, if f is not monoton on any nbd. of $\bar{x}$.
can assume x_i's $1 \leq i \leq \ell-1$ are turning points.
Restriction of f to $[x_{i-1}, x_i]$ is a <u>lap</u> of f.
ℓ = number of laps.

we consider case $\ell = 2$.



equiv. by reparametris. of the intervals

without loss of generality: turning pt. is a maximum.

f is <u>unimodal</u> $\Leftrightarrow$ "single turning point which is a maximum"

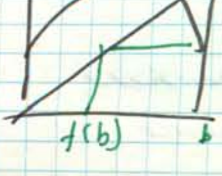
The turning point will be called x_c. (centre)

$[a, f(x_c)]$ contains image of f
All orbits are in here in one step.

without loss of generality can assume $\boxed{b = f(x_c)}$

If $f(x_c) \leq x_c$ f is effectively monoton increasing.
So we may as well assume $\boxed{f(x_c) > x_c}$
(Ex. quadratic family for $r \geq \frac{1}{2}$)

<u>Nice situation</u>: $f(x) > x$ on $(a, x_c]$
then every orbit starting in $(a, x_c]$ eventually gets into $[x_c, b]$, therefore it stays in $[f(b), b]$.



$f(b)$ $\quad b$

~~Can as~~ If $f(b) \geq x_c$, f i.e. the essential part of $f = f|_{[f(b),b]}$ is monoton decreasing.
(Ex. quadrat. family $\frac{1}{2} < r < \frac{1+\sqrt{5}}{4}$)

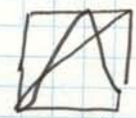
So, can assume $\boxed{a = f(b)}$

Nice situat- not true $\Rightarrow$ f has 1 fixed point in $[a, x_c]$ let x be the monotone on this interval. $f(x) > x$ for $x_f < x \le x_c$ is invariant and f is. If $f(b) \ge x_f$ can split motion in two indep. parts. on $[a, x_f]$ trivial and on $[x_f, b]$ where analy. applies

Standardizing the interval:

Linear change of variables with $x_c \to 0$
$b = f(x_c) \to 1$

if $f(b) < x_f$



pathological situation rule it out

quadratic family. $x \mapsto 1 - \mu x^2$
$\mu = 4r(r - 1/2)$
$[-1, 1]$ is invariant.

Can usually make f be even by making a nonlinear change of variables.

Explaining periodic windows:

- What ~~we~~ has to be explained is the occurence of attracting cycles. Much cheaper, In more generality it's easier not to worry on the attract. an looking just on cycles.

Sarkovsky (theorem) ordering of cycles periods

- x_0 is periodic with period p
$f^p(x_0) = x_0$. cycle is attracting, if $|(f^p)'(x_0)| < 1$
$x_i = f^i(x_0)$
$(f^p)'(x_0) = \prod_{i=0}^{p-1} f'(x_i)$

If $f'(x_0) = 0$, the cycle is certainly attracting.
Say, the cycle is <u>superstable</u> in this case

Intuitively: only exceptional f's have superstable cycles, but perturbing an f with a superstable cycle gives a $\tilde{f}$, which has an attracting cycle of same period near the original superstable cycle.

$\mu \mapsto f_\mu$ C^1 continuous

For μ_0 f_{μ_0} admits a superstable cycle of period p. then for μ in some nbhd of μ, f_μ admits an attracting cycle of period p.

Second element:

- topological rather than C^1
- using unimodal

Idea: Produces parameter values, where $f_{\mu_0}^p(x_c) = x_c$ because: f differentiable and the turning pt. is periodic than the cycle ~~of~~ turning point is attracting.

~~Examples~~ Ex: $f_\mu(x) = 1 - \mu x^2$ $0 < \mu \le 2$
$x_c = 0$
$0 < \mu < 1$: $f_\mu[-1, 1] \to [1-\mu, 1]$

$f_\mu^p(0) > 0$ for $p = 1, 2, 3, \dots$

$\mu = 2$: $f_2(0) = 1$
$f_2(1) = -1$ $\Rightarrow f_2(-1) = -1$ $p \ge 2$
$f_2(-1) = -1$

look at $\mu \mapsto f_\mu^p(0)$

$\mu \longmapsto f_\mu^{\,p}(0) \Big\}$ is positive for $0<\mu<1$
$\qquad\qquad = -1$ for $\mu = 2$

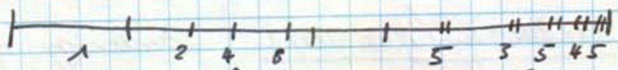
So for some $\mu \in (1,2)$ must have $f_\mu^{\,p}(0)=0$
0 is periodic with period dividing p.
$\Longrightarrow$ infinitely many periodic windows.

Remarks: μ_p = largest μ such that $f_\mu^{\,p}(0)=0$, than
$f_{\mu_p}^{\,p+1}(0) = f_{\mu_p}(0) = 1 > 0$

$f_\mu^{\,p+1}(0) = 0$ has a root between $\mu_p \leq 2$

so $\mu_{p+1} > \mu_p$
and the period of 0 under f_{μ_p} is exactly p.
$\Longrightarrow$ there is at least one periodic window with each period ≥ 2.

To show, that there are many periodic windows, look at the possible orderings of cycles of period p.



2 4-periodic cycles

4.1. First cycle on the right:

3 5-periodic cycles

4.2 on the left:

want understand combinatorial structure.

The argument above works perfectly well for $1-\mu|x|$
Exercise: 4.2 isn't possible for the family $1-\mu|x|$

Itineraries: symbolic dynamics

Take f unimodal on $[a,b]$
$f(x_c) = b$
Define for $x \in [a,b]$

$$I_i(x,f) = \begin{cases} l & \text{if } f^i(x) < x_c \\ c & \text{if } f^i(x) = x_c \\ r & \text{if } f^i(x) > x_c \end{cases}$$

If a c occurs: $I_n(x,f) = c$, than
$I_{n+j}(x,f) = I_j(x_c,f)$
Convention: Itinerary of x under f
$-\ \underline{I}(x,f) = $ sequence $I_0 I_1 \dots$
is truncated after first C, if there is one.

Abstract itinerary: $\left.\begin{array}{l} - \text{ Infinite sequence of } L\text{'s and } R\text{'s} \\ - \text{ Finite (or empty) of } L\text{'s and } R\text{'s followed by } C \end{array}\right\} \mathcal{A}$

Ex: $\underline{I}(x_c,f) = C$

Observation: $\underline{I}(x_1,f) \neq \underline{I}(x_2,f)$, than can determine whether $x_1 < x_2$ simply from the two itineraries in an f independent way.

If $I_0(x_1,f) \neq I_0(x_2,f)$: $L < C < R$ clear.

If $I_0(x_1,f) = I_0(x_2,f)$ (then it cannot be C)

case 1 : $I_0(x_1,f) = L$
$x_1 < x_2 \iff f(x_1) < f(x_2)$ because f is increasing on $[a, x_c]$
$\underline{I}(f(x),f) = \sigma \underline{I}(x,f)$ $x \neq x_c$
shift : "take away first term"

case 2 : $I_0(x_1,f) = L$
$x_1 < x_2 \iff f(x_1) > f(x_2)$ because f is decreasing on $[x_c, b]$

There is therefore an ordering of abstract itineraries :

$\underline{I}^{(1)} \neq \underline{I}^{(2)}$ n = first index such that $I_n^{(1)} \neq I_n^{(2)}$

$I^{(1)} < I^{(2)}$ if
- The number of R's in the initial segment $I_0^{(1)} \dots I_{n-1}^{(1)}$ is even and $I_n^{(1)} < I_n^{(2)}$ ($L<C<R$ ordering)
- The number of R's in the initial segment $I_0^{(1)} \dots I_{n-1}^{(1)}$ is odd and $I_n^{(1)} > I_n^{(2)}$

Proposition : $\boxed{\text{If } \underline{I}(x_1,f) \neq \underline{I}(x_2,f) \text{ then } x_1 < x_2 \iff \underline{I}(x_1,f) < \underline{I}(x_2,f)}$

Digression : Reformulation

Remark : This ordering is a total ordering called Twisted lexicographic ordering .

Reformulation : Given $\underline{I}$, make a corresponding sequence $\underline{\varepsilon}$ of $\pm 1, 0$ as follows :

$$\varepsilon(R) = -1$$
$$\varepsilon(C) = 0$$
$$\varepsilon(L) = +1$$

$\varepsilon_0 = \varepsilon(I_0)$
$\varepsilon_n = \varepsilon_{n-1} \cdot \varepsilon(I_n)$ = "Parity of number of R's in the sequ. $I_0 \dots I_n$"

have here ordinary lexicographic ordering .

$\underline{I} \to \underline{\varepsilon}$ is a bijection

$\varepsilon_n(x,f) = +1$ if f^n is increasing at x
$\qquad = -1$ if f^n is decreasing at x

$\underline{I} \longrightarrow \underline{\varepsilon}$ carries periodic sequ. to sequences .
If applied to a periodic sequence with an odd number of R's in a period , the image sequence has double the original period .
correspondence between period of $\underline{I}(x,f)$ and period of x (assuming x periodic) is defective anyway .

Ex : $1-\mu x^2$ $\mu = 1{\cdot}0$ has an attracting cycle of period 2
only itinerary are $\begin{cases} R^\infty & \text{which has period 1 .} \\ C \\ LR^\infty \end{cases}$

Def kneading sequence for f := $\underline{k}(f)$ is
the itinerary of $f(x_c) = b$.
Is a topological invariant of f.

Rem under other hypothesis : $\underline{k}(f) = \underline{k}(g) \Leftrightarrow f \sim_{top\,equi} g$

Def: $\underline{k}(f)$ is a finite sequence $\Leftrightarrow$ x_c is periodic
say then f is superstable. (name justified in the differentiable case)

Questions: Which abstract itineraries actually occur as kneading sequences?

Necessary conditions: $\boxed{f(x_c) = b > x_c}$ $k_0(f) = R$
For any n, $f^n(b) \in [a,b]$, $f^n(b) \le b$,
$I(f^n(b)) \le I(b)$
$\sigma^n \underline{k}(f) \le \underline{k}(f)$

Def: $\underline{J}$ abstract itinerary is shift maximal, if $\sigma^n(\underline{J}) \le \underline{J}$ for all $n <$ length of $\underline{J}$.

So Any kneading sequence is shift maximal

Remark: The only shift maximal sequence of the form $RR\ldots$ is R^∞
Γ $R^nL \ldots$ cannot be shift maximal
$\sigma^{n-1}(R^nL\ldots) = RL\ldots > RR\ldots$ $\lightning$ shift maximality.
$R^nC\ldots$ cannot be shift-maximal $(n>1)$

Observation: • the smallest shift maximal sequence beginning with R is $R^\infty = k(f_\mu)$ $0<\mu<1$ $f_\mu(x) = 1-\mu x^2$
• The biggest shift maximal sequence is
$RL\,L^\infty = RL^\infty$ largest sequ. starting with R
It's shift maximal
$= k(f_\mu)$ for $\mu = 2$.

Intermediate value theorem

$$\boxed{\begin{array}{l}\text{Let } \underline{J} \text{ be a shift maximal sequence begin with } R \text{ and let } \mu \mapsto f_\mu \text{ be a continuous} \\ \text{family of mappings from } [\mu_0, \mu_1] \text{ to the space of continuously unimodal mappings on } [a,b] \\ \text{equipped with } C^1 \text{ topology.} \\ \text{Assume } \underline{J} \text{ lies between } \underline{k}(f_{\mu_0}) \text{ and } \underline{k}(f_{\mu_1}) \\ \text{Then there exist } \bar{\mu} \in [\mu_0, \mu_1] \text{, such that} \\ \underline{k}(f_{\bar{\mu}}) = \underline{J}\end{array}}$$

Corollary

$\mu \mapsto f_\mu$ C^1
$k(f_{\mu_0}) = R^\infty$ $k(f_{\mu_1}) = RL^\infty$.
Then every shift maximal sequence beginning with R occurs as the kneading sequence of some member of the family.

For any finite sequence $\underline{J}$, let M_J be the largest μ such that $\underline{k}(f_\mu) = \underline{J}$

$J \mapsto M_J$ is order preserving.

Can ~~..~~ choose M_J around μ_J, on which f_μ has an attracting cycle of period $|\underline{J}|$.

Can arrange things, such that x_c is attracted to this cycle.

Corollary: If $J^{(1)} < J^{(2)}$, then every μ in $M_{J^{(1)}}$ < every μ in $M_{J^{(2)}}$
$\underline{J} \to M_{\underline{J}}$.

Say: $\mu \mapsto f_\mu$ is monotonic, if $\mu \mapsto \underline{k}(f_\mu)$ is.

Until recently question: Do there exist nontrivial monotone families?

Hard to decide, if $\underline{k}(f_\mu) > \underline{k}(f_{\mu'})$?

Theorem (Milnor, Thurston, Sullivan, Douady-Hubbard)

The quadratic family is monotone

Proof of the Int. Value thm: Strategy: Imitate proof of ordin. IV theorem.

Try to show: $\{f : k(f) < \underline{J}\}$, $\{f : k(f) > \underline{J}\}$ $\Big\}$ C^1 open

If so: $\{\mu : \underline{k}(f_\mu) > \underline{J}\}$, $\{\mu : \underline{k}(f_\mu) < \underline{J}\}$ $\Big\}$ disjoint open non empty in $[\mu_0, \mu_1]$
$\to$ union cannot be $[\mu_0, \mu_1]$

connectivity of $[\mu_0, \mu_1] \to$
$\exists$ μ not in either of them.

~~Kioukori~~

Main step is done in

Lemma

Let $\underline{J}$ be a non periodic shift maximal sequence beginning with R. Then $\{f : \underline{k}(f) < \underline{J}\}$ and $\{f : \underline{k}(f) > \underline{J}\}$ are C^1 open

Proof : Fix $\underline{J}$
Take $\bar{f}$ such that $\underline{k}(\bar{f}) < \underline{J}$.
Want to have a whole nbhd, where this is true.

Find first n, such that $k_n(\bar{f}) \neq J_n$.
If $k_n(\bar{f}) \neq C$, there is a whole C^1 nbhd U of $\bar{f}$
Such that $k_i(f) = k_i(\bar{f})$ $i = 0, 1, \ldots, n$ $f \in U$
Every f in U has $\underline{k}(f) < \underline{J}$.

Only case giving trouble is $\underline{k}(f) = \underline{A}C$ and where is an initial segment of $\underline{J}$.

Lemma 1 : $\underline{k}(\bar{f}) = \underline{A}C$
$\longrightarrow$ ~~then~~ there is a C^1 nbhd of $\bar{f}$, such that every $f \in U$ has $\underline{k}(f)$ equal to one of $(\underline{A}R)^\infty, \underline{A}C, (\underline{A}L)^\infty$.

Lemma 2 : $\underline{A}C$ is a finite (shift maximal) sequence
$\underline{J}$ shift maximal
$\underline{A}C < \underline{J} \Longrightarrow (\underline{A}R)^\infty \leq \underline{J}$ and $(\underline{A}L)^\infty \leq \underline{J}$
$\underline{A}C > \underline{J} \Longrightarrow (AR)^\infty \geq J$ and $(AL)^\infty \geq \underline{J}$

Proof of lemma 1 : p = lenght of $\underline{A}C$
$\bar{f}^p(x_c) = x_c$
$\bar{f}^n(x_c) \neq x_c$ for $n < p$.

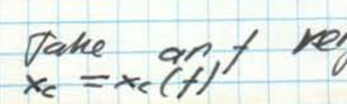
Take an f very near to $\bar{f}$.
$x_c = x_c(f)$

- $x_c \approx \tilde{x}_c$
- $f^p(x_c) \approx x_c$

if $f^p(x_c) = x_c$ $\underline{k}(f) = \underline{A}C$

Near enough to $\bar{f}$ $f^n(x_c) - x_c$ has same sign as $\bar{f}^n(x_c) - x_c$.
For definiteness, suppose $f^p(x_c) = x_c + d$ $d > 0$
$(f^p)'(x_c) = 0$
$(f^p)'(x)$ small on $J_d = (x_c, x_c + 2d)$

f^p maps x_c to the midpoint of the interval



$|(f^p)'(x)| < \frac{1}{2}$ on J
$f^p : J_d \longrightarrow \mathring{J}_d$

$f^{2p}(x_c) = f^p(\underbrace{f^p(x_c)}_{\in J_d}) \in J_d$

$\Longrightarrow f^{2p}(x_c) > x_c$
sign of $f^{2p}(x_c) - x_c$ = sign of $f^p(x_c) - x_c$
sign of $f^{np}(x_c) - x_c$ = sign of $f^p(x_c) - x_c$

Proof of lemma 2 :

Consider seperatly cases, where the number of R's in $\underline{A}$ is odd or even.

Suppose $\underline{A}$ odd.

$L < C < R$

$AR \ldots < AC < AL \ldots$ $\quad$ $\underline{A}$ odd.

$\underline{A}L < AC < AR \ldots$ $\quad$ $\underline{A}$ even.

$\underset{even}{AX} < AC < \underset{odd}{AX}$

$(AR)^\infty$
$(AL)^\infty$ $\quad$ one of these is $< AC < J$

The one I do have to worry about has an odd number of $\underline{R}$'s.

Write this one as $\underline{A}X$.

$\boxed{(\underline{A}X)^\infty \le \underline{J}}$

$A_0 \ldots A_{p-2}\ X \ldots \overset{?}{\le} J_0 \ldots J_{p-2} \ldots$

$A_0 \ldots A_{p-2}\ C \ldots \le J_0 \ldots J_{p-2}\ J_{p-1}$

If $A_0 \ldots A_{p-2}$ is not an initial segment of $\underline{J}$, then order of $(AX)^\infty$ relative to $\underline{J}$ is the same as the ordering of $\underline{A}C$ relative to $\underline{J}$.

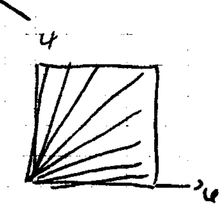
Can assume: $J_0 = A_0 \cdots J_{p-2} = A_{p-2}$

~~Claim: we must also have $J_{p-1} = X$.~~

~~If $J_{p-1} \ne X$, then $J_0 \ldots J_{p-2} J_{p-1}$ is even.~~

~~$J_0 \ldots J_{p-2} J_{p-1} < J_0 \ldots J_{p-2}\ C$~~

$A_0 \ldots A_{p-2}$



$f(e^{i\varphi}, e^{i\psi})$

Claim: Must also have $J_{p-1} = X$.

$\underline{J} < \underline{A}C$

contrary to assumption.

Proof unless $J = \underline{A}X\bar{J}$

Shift maximality says, that $\bar{J} \le J \not\le B\bar{J}$

But if $\bar{J}$ does not have AX as an initial segment, then $(AX)^\infty < J$

If $(AC) \ge \bar{J}$

$(\underset{\uparrow \text{odd number of } R\text{'s}}{A}X)AC \le \bar{J}AX\bar{J} = J$

To show $(\underline{A}X)^\infty \le \underline{J}$

Contradictionary: $\underline{A}C \le \underline{J} < (\underline{A}X)^\infty$
$\Longrightarrow A$ is an initial segment for J

Also: next symbol must be X

$AC > J = \underset{odd}{\underline{A}X}\bar{J} < (AX)(AX)^\infty \longrightarrow \bar{J} > (AX)^\infty_{*}$

$*$ but by shift maxim.

$\bar{J} \le J = AX\bar{J}$

$(AX)^\infty > J \ge \bar{J} > (AX)^\infty$

$\uparrow$ contradiction

Assume

$J < AC$

To show $J \le \underset{even}{(AX)}^\infty$

Assume contrary

$(AX)^\infty < J \not< AC$

J must have A as init. segm.

Next symbol must be X

$J = AX\bar{J} > (AX)(AX)^\infty$

Cancelling doesn't reverse the equality. $\bar{J} > (AX)^\infty$

By shift maxim

$AX\bar{J} \ge \bar{J} > AX^\infty$

Claim: This implies:

$J = AX^\infty$

$\Rightarrow AX = AX\bar{J}$

Have shown: $\{f \mid k(f) > J\}$
$\{f \mid k(f) < J\}$ are C^1 open

unless $J = (AR)^\infty, (AL)^\infty$

$\mu \longmapsto f_\mu$ C^1 continuous

J shift maximal
strictly between $k(f_{\mu_0})$ and $k(f_{\mu_1})$

$\exists \mu^1 \Rightarrow k(f_{\mu}) = J$

Proved unless $J = (AR)^\infty$ or $(AL)^\infty$

To deal with these exceptional sequences.
For definiteness assume

- $k(f_{\mu_0}) < k(f_{\mu_1})$
- $J = (AR)^\infty$
- $(AR)^\infty > AC$

$\exists \bar{\mu} \Rightarrow k(f_{\bar{\mu}}) = AC$
largest

For $\mu > \bar{\mu}$, must have $k(f_\mu) > AC$
(uses part of IVT already proved.

by lemma 1 μ must have $k(f_\mu) = AC$ or $k(f_\mu) = (AR)^\infty$
or $k(f_\mu) = (AL)^\infty$

For any μ sufficiently near to but $> \bar{\mu}$: $k(f_\mu) = (AR)^\infty$ ∎

Order structure of finite kneading sequences

Proposition :
- $\boxed{\text{AC shift maximal} \Rightarrow (AR)^\infty \text{ and } (AL)^\infty \text{ are shift maximal}}$
- AX^∞ shift maximal
  AC shift maximal $\Leftarrow$ ~~either~~ period of $(AX)^\infty = p$ (=: length of AC
  (or period of $(AX)^\infty = \frac{p}{2}$ and number of R's per period is ~~even~~ odd.)

Proposition : AC shift maximal

$X = L, R$ according as # of R's in A is odd even
(Ax odd)

$\check{X} = \begin{cases} R & \text{if } X = L \\ L & \text{if } X = R \end{cases}$

$(A\check{X})^\infty < AC < AX^\infty = AX(AX)^\infty < AXAC$
$< AX(A\check{X})^\infty = (AXA\check{X})(AXA\check{X})^\infty < AXA\check{X}AX.AC$

No other shift maximal sequence lies between any successive pairs.

Observation: Period-windows with period $p, 2p, 4p, \ldots 2^n p$

AC shift maximal $\Rightarrow$ Period of $(Ax)^\infty = p$ or $\frac{p}{2}$
$\frac{p}{2}$ impossible because we have an odd number of R's in Ax.

$(AxAx)^\infty$ has period $2p$ and odd # of R's per period
$\Rightarrow AxAC$ is shift maximal

Normally: Expect $p \to 2p \to 4p \to$
goes by successive period doubling bifurc.
$\longrightarrow$ Negativity of Schwarzian derivative.

Counting:

n_p = # of finite shift maximal sequences of length p corresponds to finite window

$\longrightarrow (AR)^\infty$ and $(AL)^\infty$ are shift maximal

$(Ax)^\infty_{\text{odd}}$ Period of this sequence is p

Conversely: If $(Ax)^\infty$ is a shift maximal, periodic sequence with Ax odd
$\Rightarrow AC$ is shift maximal

1) So we have a bijection between

$$\underset{\substack{\text{finite}\\ \text{length } p}}{AC} \longleftrightarrow \underset{\substack{\text{odd}\\ \text{period } p}}{(Ax)^\infty}$$

2) If J is a periodic sequence of period p, then exactly one of

$J, \sigma J, \sigma^2 J, \dots, \sigma^{p-1} J$ is shift maximal (take the biggest one)

$\Rightarrow$ # of shift maximal sequences, period p, odd number of R's
$= 1/p \cdot$ # of periodic sequences of period p with an odd number of R's per period.

~~$n_p\ 2^{p-1}$~~

3) The total number of sequences J, with $\sigma^p J = J$, odd number of R's in each subsequence of length p. is 2^{p-1}.
Choose $J_0, \dots, J_{p-2}$ in $\{L, R\}$ arbitrarily
J_{p-1} fixed by parity. Repeat periodically.

4) Each of these sequences has some period q with divides p
Furthermore it must have an odd # of R's per period.
, must have $\frac{p}{q}$ odd

$$2^{p-1} = \sum_{q:\, \frac{p}{q} \text{ odd}} q \cdot n_q < p \cdot n_p + \sum_{\substack{q<p \\ \frac{p}{q} \text{ odd}}} q \cdot n_q$$

$n_1 = 1$

$2^{2-1} = 2 = 2 \cdot n_2 \Rightarrow n_2 = 1$

$2^{3-1} = 4 = 3n_3 + 1n_1 \Rightarrow n_3 = 1$

$2^{4-1} = 8 = 4n_4 \qquad n_4 = 2$

$2^{5-1} = 16 = 5n_5 + 1n_1 \quad n_5 = 3$

$2^{6-1} = 32 = 6n_6 + 2n_3 \qquad n_6 = 5$

p odd prime: $n_p = \dfrac{2^{p-1}-1}{p}$

p power of 2 $\quad n_p = \dfrac{2^{p-1}}{p}$

observation: $p \cdot n_p \le 2^{p-1}$

r = smallest odd divisor of p.

$$2^{p-1} \ge p n_p \ge 2^{p-1} - \sum_{j=1}^{p/r} 2^{j-1} = 2^{p-1} - (2^{p/r} - 1)$$

$\left(q \le \frac{p}{r} \quad r \text{ smallest odd divisor} \right)$

$$2^{p-1} \ge p \cdot n_p \ge 2^{p-1} - \underbrace{[2^{p/r} - 1]}_{\text{much smaller than}} \qquad \le 2^{p/3}$$

$$\boxed{n_p \sim \frac{2^{p-1}}{p}}$$

$f \in \mathcal{C}^3 \quad f'(x_0) \neq 0$

$$(Sf)(x_0) = \left(\frac{f'''(x_0)}{f'(x_0)}\right) - \frac{3}{2}\left(\frac{f''(x_0)}{f'(x_0)}\right)^2$$

Schwarzian derivative

19. Jhd.
theory of conformal mappings
Poincaré, ...)

- useful in theory of conformal mappin.
- useful in theory of iterations of interval mappings
(D. Singer, Guckenheimer, Sullivan...)

Lemma 1

$$\boxed{\begin{gathered} S(f g)(x) = S(f)\, g(x) \left[g'(x)\right]^2 + Sg(x) \\ \frac{d^2}{dx^2} (f'(x))^{-1/2} = -\frac{1}{2} \, |f'(x)|^{-1/2} \, Sf(x) \end{gathered}}$$

Cor $\quad Sf < 0$ and $Sg < 0 \Rightarrow Sf \circ g < 0$
$\Rightarrow Sf^n < 0$

$Sf < 0 \Rightarrow \dfrac{1}{|f'(x)|}$ convex on $f'(x) \neq 0$

$\Rightarrow |f'(x)|$ has no non zero local minimum

Singers thm:

$f \in C^3$ piecewise monotone $f:[a,b] \Rightarrow$
$Sf \le 0$ every attracting periodic cycle attracts a critical pt. or a or b (if $Sf = 0$ for only finitely many points where $f'(x)=0$)

Remark: Don't need C^3, $f \in C^1$ suffices ~~and~~ $\frac{1}{\sqrt{|f'(x)|}}$ convex

$f(x_0) = x_0$
x_0 attracting from the left, if $\exists \varepsilon > 0$ $\forall x \in (x_0 - \varepsilon, x_0) \Rightarrow f^n(x) \to x_0$
$\Longleftrightarrow \exists \varepsilon > 0$ $f(x) > x$ $(x_0 - \varepsilon, x_0)$

x_0 attracting from the right $f^n(x) \to x_0$ for $x \in (x_0, x_0 + \varepsilon)$

semiattracting left or right attracting

Remark: Can replace attracting by semiattracting if $f'(x_0) = 1$

Corollary

f unimodal, $Sf \le 0$, $f(x_c) = b$
either $f(a) = f(b)$ or $a = f(b)$
Then f can have no more than one attracting cycle

Rem: Complex functions $\exists$ analog theorems Idea: $Sf \ge 0 \Longleftrightarrow$ univalent functions

We have shown that, if x_0 is an attracting periodic pt of period p then there $\exists$ either a critic. pt. or an end pt. in the domain of direct attraction for x_0 with respect to f^p.

Corollary

f unimodal, $Sf \le 0$ $f(x) > x$ for $x \in (a, x_c)$

f has an attracting or semi attracting periodic cycle

$\Updownarrow$

$k(f)$ is periodic or finite

Rem: Situat.  RL^{∞} kneading seq. is not periodic

Proof: "$\Rightarrow$" Assume $\exists$ attracting cycle. $k(f)$ is not finite
To proof: $k(f)$ is periodic

Notation: x_0 pt on attracting cycle with period p $f^p(x_0) = x_0$
$f^j(x_0) =: x_j$
J_j = connect. component of x_j in the domain of attraction of x_j for f^p

J_j disjoint intervals for $j = 0, \dots, p-1$

If $x_c \notin J_j$ $\forall j$, look at the right most J_j.
Singer thm $\Rightarrow$ b is in the right most J_j
$\Rightarrow I(b) = k(f)$ is periodic

Otherwise can assume $x_c \in J_0$

for $1 \le j \le p-1$, x_c cannot be between $f^j(x_c)$ and $f^j(x_0)$
because both of these are in J_j because $J_j \supset c\, J_0$
For $j = p$ x_c cannot be between $f^p(x_0) = x_0$ and $f^p(x_c)$
f^p only one critical pt in J_0 so f^p is monoton from c to x_0

case: - Increasing:

$f^p(x_c)$ has to be strictly between x_c and x_0.

. Decreasing:

$f^p(x_c)$ is on the opposite side of x_0 from x_c.

For $j = p+1, \dots 2p-1$ again x_c is not between $f^j(x_0)$ and $f^j(x_c)$

$j = 2p$ increasing case...

For $j > 1$ x_c cannot be between $f^j(x_0)$ and $f^j(x_c)$

That means: $\underline{I}(f(x_0)) = \underline{I}(f(x_c) = b) =: k(f)$ which is periodic.

"$\Leftarrow$" Don't need Schwarz. deriv.

Assume $k(f)$ is periodic with period p.

$J = \{x \mid I(x) = I(b) = k(f)\} \neq \emptyset$

This is an interval, because of the ordering.

It's mapped into itself by f^p

Contains p and $f^p(b)$.

$f^p(b) \neq b$ because x_c is the only preimage of b. and $k(f)$ is not finite

So J is a non trivial interval.

f^p is monotone on J because if x is a critical point for $f^p \Longleftrightarrow I(x)$ has a C before the p^{th} place.

Lemma: $\boxed{\begin{array}{l} f \text{ a monotone increasing mapping } J \circlearrowleft \\ \text{Either } f^2 = Id \text{ on } J \text{ or} \\ f \text{ has an attracting or semi attracting fixed pt.} \end{array}}$

Pf: suppose $f(\bar{x}) > \bar{x}$ for some $\bar{x} \in J$. $J = (\alpha, \beta)$

$f(\beta) \leq \beta$.

Let x_0 be the least fixed point for f between $\bar{x}$ and β

$x_0 < \bar{x}$ $f(x) > x$ $x \in [\bar{x}, x_0)$ $\Rightarrow \bar{x}_0$ is attracting from the left.

Similarly: $f(\bar{x}) < \bar{x}$ for some $\bar{x}$, then f has a fixed pt attracting from the right

Thm

Guckenheimer
Masiurewitz

$\boxed{\begin{array}{l} f \text{ unimodal} \\ Sf < 0 \\ f(x) > x \quad x \in (a, x_c) \\ k(f) \text{ not periodic} \\ \text{Then any distinct points have distinct itinerarys, any two such } f\text{'s with the} \\ \text{same } k(f) \text{ are topologically conjugate} \end{array}}$

$\Updownarrow$ Preimages of x_c are dense

Roughly speaking the $k(f)$ determines, which itinerarys can occur. This is strictly true if distinct pts have distinct itinerarys.

Preimag. dense $\quad \overset{}{\underset{x_1 \quad x_2}{\vdash\!\!-\!\!\dashv}} \quad I(x_1) = I(x_2) \Longleftrightarrow$ It never that x_c lies in the closed interval with end pts $f^j(x_2)$ and $f^j(x_1)$

Given an itinerary $\underline{I}$.
Does $\underline{I}$ occur for f?

If yes, then $\sigma^n \underline{I} \le k(f) \quad \forall n$
$I(a) \le \underline{I} \le I(b)$

Show that $\{x \mid I(x) < \underline{I}\}$ and $\{x \mid I(x) > \underline{I}\}$ are open,
then $\underline{I}$ occurs. This is easy if $k(f)$ is ~~finit~~ not finite
and $\sigma^n(\underline{I}) < k(f) \quad \forall n$

$k(f_1) = k(f_2)$
$I(x, f_1) = I(x', f_2)$
$h : x \to x'$
Easy : h homeo $f_2 h = h f_1$

Thm : If $Sf \le 0$ and $k(f)$ periodic
then $\{x \mid f^n(x)$ does not converge to the semi attracting periodic cycle $\}$ has Lebesgue measure zero