Pitfalls in teaching and learning calculus

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Abstract

Students studying calculus often make the same mistakes. Similarly, teachers teaching calculus have patterns of mistakes. It is much easier to tell how things should **not** be done then how things should be done. I collect in this document some "no-go's". This is a living document and is expected to be updated and adapted in the future.

General teacher mistakes

Lack of preparation. Even routine calculus problems need to be rehearsed.

Examples: a teacher takes a random 3x3 matrix and goes onto the task to find the eigenvalues of this matrix. Row reduction leads to mess, the characteristic polynomial has no apparent root.

Improvisation. Rely on improvisation skills for lectures.

Example: In a multivariable calculus course, an arc length computation which does not work like for a Lissajou curve $\vec{r}(t) = (\cos(5t), \sin(7t))$.

Difficulty. Assign too difficult problems as homework.

Example: Find the polytop with 12 edges with fixed surface area and maximal volume using Lagrange extremization.

Polititical correctness Use politically incorrect problems or jokes.

Example: show slides with sexist joke. Also politically biased statements can be tricky. There are students of almost any political orientation in the classroom.

Lecture too long. The flow of lectures at the college is interrupted. Students come too late to the next class. Everybody has to wait.

Example: the teacher discusses with students after class in the class room, while students of the next class have to wait. The next class gets delayed.

Too late. Come too late to class. After cleaning the black board, collecting homework.

Example: At Harvard, classes start 7 minutes after the hour.

No interaction. Have no interaction or activities during class. Students turn off.

Example: Tuesday-Thursday calculus classes at Harvard are 90 minutes. This is a long time to keep the attention span.

Wrong lecture. Cover an other topic than the other sections. The students can not do the homework and later do not know the material for the course wide exam.

Example: Basic definitions which are necessary for the homework are not introduced.

Outfit. Improper dressing can be a distraction. While in educational environments, almost everything goes, there are limits, like avoiding coming barefoot. Rule of thumb: have at least a shirt on ...

Example: For male teachers: check trousers after the bathroom. The range of acceptable clothing or hairstyle certainly also depends on the personality of the teacher and students are quite tolerant. But clothing issues should not become a distraction or an embarressment.

Untested technology. The use of technology unfamiliar to the teacher can eat away precious classroom time.

Example: it is annoying to watch somebody fiddling with the computer, the overhead projector or trying to fix a line of Mathematica code in front of the class.

Misuse of technology. Each technology has its own advantages. It is often difficult to transfer one to an other.

Examples: 1) make slides by scanning a handwritten paper and show it with the overhead projector. 2) take a PDF version of a text and pull it into a power point presentation 3) use the overhead projector to sketch a figure.

Exceptions. Focus too early on exceptions.

Example: "the infimum of a set is not always smaller then the supremum of a set". State exceptional examples too early: like examples, where $f_{xy} \neq f_{yx}$. For nonmathematicians, such counter examples may be amusing but have also a pathological touch.

Uninspiring. Showing lack of preparation and enthusiasm. Why should people at all come to class?

Example: A teacher copies his notes word by word onto the blackboard. Worse: a teacher copies problems from the book directly onto the blackboard.

Ignore student questions. A teacher brushes over a student question. Better: possibly repeat the question and answer carefully. If the question is not clear, restate it. If the answer is not known, refer to the next hour, or give it as a challenge to the class.

Example: Student: "Could you please explain the last step again". Teacher: "I'm sorry, we do not have time for that." or "You should have payed more attention to what I just sayd".

Mumble. Instead of useing loud and clear voice and speaking to the class, the teacher mumbles to the blackboard.

Example: A long computation is performed on the blackboard. The class sees only the back of the teacher. Worse, the teachers body covers the written part.

Giving away solutions. Providing full solutions to problems instead of hints to problems. the solution. Instead, one can give hints, still leaving the task of solving the problem to the student.

Example: Compute $\int_0^{2\pi} \sin^4(x)\, dx$. Hint 1: Use $\sin^2(x) - 1 = \cos^2(x)$ Hint 2: use this only for one $\sin^2$ and leave the other $\sin^2$. Hint 3: use then a trigonometric identity.

Mistakes by Course-head

Lack of guidance. Syllabus, homework problems, homework solutions, practice problems, quizzes are not available ahead of time.

Examples: The homework assignment is only fixed a day before class. Because a lecture has to be tailored to make the homework useful, the section leaders can not prepared early enough.

Too much guidance. Too much advise like detailed prescribed lecture scripts can kill the innovation of the individual teacher. Some feel just becoming a narrator of a script. Sharing advise, tricks and experience is good but teaching is also a creative act.

Examples: Individual lecture-plans make often only sense for the teacher who prepares the lecture. They often do not mention things which are obvious. The most useless "lecture plan" is a list of the section titles which are covered in the book. Better: point out possible pitfalls, student questions, common misconceptions.

Distributing the responsibility. Organization tasks can be shared with the section leaders but ultimately, the course head has the responsibility.

Examples: A section leader is assigned to proctor a midterm and does not show up. Because midterms can not be repeated easily (schedule, holidays, conflicts, lecture hall reservation), this is a disaster.

Use of CAs. CAs have about 15 hours per week. Some volunteer to do more and are happy to share more responsibility.

Examples: CAs have to grade midterms or even the final exam. CAs have to take their own midterms and exams, usually just at the time of the calculus midterms or exams. There also seems to be a Harvard policy that undergraduates are not allowed to grade exams. This makes sense because some CAs will have to grade the exams of their own friends.

Change the syllabus. In service courses like calculus, there are some things which have to be taught and which other departments require to have included. While there is some variation of the syllabus possible, it should be fixed before the semester starts. The syllabus should have flexibility in the time frame like one or two hours to "catch up".

Examples: Some section falls behind and all other sections adapt to the pace of the slower section. It can happen that classroom time is spent with valuable discussion time and not everything can be covered. In such a case, one has to ask the students to read the rest in the book. It is better to anticipate that the hour does not suffice for all the material, reduce the complexity of the examples or ask students to preread.

Mathematical teacher mistakes

Ambiguous notation. Use ambiguous notation.

Example: let $\mathbf{v}$ and $\mathbf{w}$ be vectors in space. Define the parametrized surface $r(u, v) = u\mathbf{v} + v\mathbf{w}$. Even so the difference between vectors and real numbers is done by taking bold face letter, such features can get lost on the blackboard.

Uncommon notation. Use uncommon notation.

Example: let α, β, γ be points in the plane and let P, Q, R denote the line segments connecting them.

Unusual twists. Unusual orientation of coordinate system.

Example: use a left handed orientation e_1, e_3, e_2.

Incorrect definitions. Give incorrect definitions of mathematical concepts.

Example: like a vector is an object with length and orientation.

Overkill in rigor. Adapt the rigor to the class.

Example: a $\epsilon - \delta$ definition of continuity is ok for mathematicians and physisics, but not appreciated by biochem majors for example. Eulers definition x close to y, then $f(x)$ close to $f(y)$ is more intuitive.

Too difficult problems Give too difficult problems without a hint.

Example: Calculate $\int_0^{\pi/2} \log(\sin(x)) \quad dx$. There is an easy solution to this problem, but it is not easy to find. 99 percent of all students would get very frustrated when assigned this problem.

Incorrect theorems State incorrect theorems.

Example: all extrema of $f(x, y)$ under the constraint $g(x, y) = c$ are obtained at points, where $\nabla f = \lambda \nabla g$. (This is incorrectly stated as a theorem in many text books.)

Course assistant mistakes

Not coming to class. It is not only to see what material is covered, the CA should also get a feel, what how students cope.

Example: Real life example: A course assistant worked for the second year. Feeling he knows the material enough, fails to come to lectures. He got fired.

Not grading in time. Bringing homework back too late is the biggest turn off.

Example: Real life and worst case scenario: a course assistant does not grade for several weeks and is out of town. The papers pile up. It becomes impossible for one person to do it. The teaching fellows spend an afternoon grading the papers.

General student mistakes

Do not come to lectures.

It is possible to learn mathematics by reading the book. However, the brain does not keep the information as reliably. Having seen and heard the material makes it stick better. In a time, where distraction (phone, Internet, friends etc.) is tempting, it needs stamina to make up the learning time away from class.

Do not speak up. If something is not clear.

Most of the time, if something is not clear to one student, it is not clear to the entire class.

Copy homework from someone else.

Mathematics is doing. The process of struggling with a homework problem is important too.

Come unprepared to class.

The most effective use of class room time is to have already partly absorbed the material before hand.

Cheat during exams.

Sometimes, this happens unconsciously. A student uses a calculator in an exam, where none are allowed.

Mathematical student mistakes

Improper use of substitution.

Example: $\int_0^{3\pi/4} \sin^2(x)\cos(x) \quad dx = \int_0^{\sin 3\pi/4} u^2 \quad du = \int_0^{1/\sqrt{2}} u^2 \quad du$, where $u = \sin(x), du = \cos(x)\, dx$.

Improper handling of indefinite integrals.

Example: $\int_{-\pi}^{\pi} 1/x \quad dx = 0$ so that $\int_0^{\pi} 1/x\, dx = 0$.

Matrix algebra issues:

$(A+B)^2 = A^2 + 2AB + B^2$.

Wrong linearity of determinant:

$\det(2A) = 2\det(A)$.

Too linear Inappropriate linearity:

$\sqrt{x+y} = \sqrt{x} + \sqrt{y}, e^{x+y} = e^x + e^y, (x+y)^2 = x^2 + y^2$. $1/(x+y) = 1/x + 1/y$.

Too commutative Inappropriate commutativity of composition:

$\log\sqrt{x} = \sqrt{\log(x)}, \sin(5x) = 5\sin(x)$.

A sin. A student "sin" as a joke.

$\frac{\sin x}{x} = \sin$.

Inappropriate cancellations.

$((x+1)+5x^2)/(x+1) = 1 + 5x^2/(x+1)$. $\frac{3+x}{x} = 3 + x/x = 3 + 1 = 4$.

Unclear brackets.

Example: $3x/5 = 3/5x = 3/5x = 3/(5x)$.

Unit confusion.

Example: $1\$ = 100c = (10c)^2 = (\$0.1\$)^2 = 0.01\$ = 1c$. (source: Eric Schechter, 2004)

Division by zero.

Example: $x(x^2-1) = x$ implies $x^2 = 2$ and so $x = \pm\sqrt{2}$.

Square root errors

A famous example: $a = b$ implies $a^2 = ab$ so that $a^2 + a^2 = a^2 + ab$ or $2a^2 = a^2 + ab$. The manipulation $a^2 - 2ab = a^2 + ab - 2ab$ gives $2a^2 - 2ab = a^2 - ab$. Writing this as $2(a^2 - ab) = 1(a^2 - ab)$ and canceling $(a^2 - ab)$ from both sides gives $1 = 2$.

Signs in inequalities.

Example: $-a < -3$ implies $a < 3$.

Misconception about constants.

Example: $\frac{d}{dx} x^x = x x^{x-1}$.

Integration by parts. Forget one integral.

Example: $\int x\cos(x)\, dx = x\sin(x) - \sin(x)$.

Polar coordinates. The inversion formula for polar coordinates is not complete.

Example: Given the point $(-1,-1)$. Find the polar coordinates. We have $r = \sqrt{2}$ and $\theta = \arctan(-1/-1) = \arctan(1) = \pi/4$.

An SAT pitfall. True facts for integers are extrapolated to all real numbers.

Example: This is a common trap for SAT exams: because $0 \leq 0^2, 1 \leq 1^2, 2 \leq 2^2, 3 \leq 3^2$ one concludes $x \leq x^2$.

Sloppiness about signs. Especially in connection with the chain rule.

Example: $\int dx/(1-x) = \log|1-x| + C$.

Literature:

- Eric Schechter, "The most common errors in undergraduate mathematics education" http://atlas.math.vanderbilt.edu/ schectex/commerrs
- http://tutorial.math.lamar.edu/AllBrowsers/CommonErrors/CalculusErrors.asp
- Andy Engelward, "do's and don'ts for Classroom style", talk at Bok Center Fall Teaching Orientation Sept. 14th, 2004