29 September 29, 2021

OLIVER KNILL
DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
ONE OXFORD STREET
CAMBRIDGE, MA 02138, USA.
knill@math.harvard.edu

Professor Knill,

About three years ago, I saw a problem on the internet.

$$6 \div 2(1+2) =$$

Since I had a strong background in mathematics and was taught that multiplication is performed before division, I knew immediately that the answer was 1.

Wrong.!

How can that be. There must be some mistake. So, I checked further and was startled to see the even split between those that believed the answer to be 1 and those that saw the answer as 9.

I wanted to know why the divergence on the presumably pure science of mathematics. That crusade led me to your paper, Ambiguous PEMDAS.

I went full cycle. The answer is 1. The equation is ambiguous, The teaching of mathematics changed for the current world. The answer is 9. And finally, after a bit of an epiphany, the answer is 1 and we have been teaching our children wrong. The remainder of this communication is why I believe that the current teaching of PEMDAS Left to Right is wrong and contrary to the natural order of mathematics.

# PEMDAS is Wrong

First, there is no directionality with numbers and counting. Any imposed directionality is a limitation that distorts numbers and counting.

Second, It is commonly held that there are four basic operations in mathematics: Addition; Subtraction; Multiplication; and Division. I contend that there is really only one operation in Mathematics and that operation is Addition.

Third, the Order of Operations seems to have some area of interpretation based on PEMDAS Left to Right and what I was taught. I believe the Order of Operations is natural and not subject to interpretation:

| What I was taught | PEMDAS Left to RIGHT |
| --- | --- |
| Parenthesis | Parenthesis |
| Exponents | Exponent |
| Multiplication | Multiplication or Division L-R |
| Division | Addition or Subtraction L-R |
| Addition | |
| Subtraction | |

DIRECTIONALITY

1 + 1 + 1 = 3

| | | |
| --- | --- | --- |
| 1 | +1 | |
| 1 | +1 | +1 |
| 1 | =3 | |
| --- | | |
| 3 | | |

It doesn't matter in which direction you count, where you begin or where you end, the answer is still the same. This is stated and accepted in the Associative and Commutative properties of numbers.

## OPERATIONS

I contend that the only operation in Mathematics is Addition. All of the other presumed operations are no more than extensions of the operation of Addition.

Rationale below:

## ADDITION

Addition is the baseline of Operations in Mathematics. It is cumulative counting.

## MULTIPLICATION

Multiplication is a natural extension of Addition.

$2 + 2 = 2 \times 2$ $2 + 2 + 2 = 3 \times 2$ $2 + 2 + 2 + 2 = 4 \times 2$ etc.

$(2 + 2) = 2(2)$ $(2 + 2 + 2) = 3(2)$ $(2 + 2 + 2 + 2) = 4(2)$ etc.

Multiplication is nothing more than the Grouping and Addition of "Like" terms identified with a shorthand nomenclature e.g., 2(2).

## EXPONENTS

Exponents are an extension of Multiplication (and therefore of Addition also) and follow a similar path:

$(2 + 2) = (2(2)) = 2^2$

$((2 + 2) + (2 + 2)) = 2(2(2)) = 2^3$

$(((2 + 2) + (2 + 2)) + ((2 + 2) + (2 + 2))) = 2(2(2(2))) = 2^4$

Exponents are nothing more that the Grouping of Multiplication (Addition) of "Like" terms identified with a shorthand nomenclature e.g., $2^3$.

Note: The parentheses, though not necessary for the calculations, show the grouping which elevates Addition to Multiplication to Exponents.

SUBTRACTION

Subtraction is treated as an entirely separate operation from Addition. Because of that Subtraction imposes directional limitations on the fundamental operation of Mathematics. However, Subtraction really isn't a separate operation. Subtraction is nothing more than the addition of negative numbers.

$4 - 2 = 2$

Is really

$4 + (-2) = 2$

And

$4 - (2 - 1) = 3$

Is really

$4 + (-2) + 1 = 3$

With the proper identification of the negative numbers, full Associative and Commutative Properties are restored, and arbitrary directionality limitations of treating Subtraction as an operation are removed. Subtraction isn't an operation, it's an identifier of negative numbers.

DIVISION

Division by definition is the act of separating and dividing into equal parts.  Division is treated as an entirely separate operation from Multiplication. Because of that Division imposes directional limitations on the fundamental operation of Mathematics. However, Division really isn't a separate operation nor is Division a Grouping symbol.  Division is nothing more than the addition of fractional numbers.

$$3 \div 4 = \tfrac{3}{4} = 3(\tfrac{1}{4}) = (\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4})$$

With the proper identification of the fractional numbers, full Associative and Commutative Properties are restored, and arbitrary directionality limitations of treating Division as an operation are removed. Division isn't an operation, it's an identifier of fractional numbers.

## GROUPING

It should be noted here that grouping was employed as a method of prioritization long before operations and even numerals were created.

Early on, slashes were used to group and count

𝍸 𝍸 |||

Parenthesis were subsequently used to Group:

| Mine | Yours | Theirs |
| --- | --- | --- |
| (111111) | (11111) | (111111) |

Therefore, the natural Order of Operations evolution was based on Grouping:

Parenthesis
Exponents
Multiplication
Addition

Subtraction and Division are merely identifiers of negative and fractional numbers used in Addition and Multiplication. If Subtraction and Division are treated as identifiers within Addition and Multiplication, and not operations, then they pose no ambiguity.

The answer to the Equation is:

$6 \div 2(1+2) = 1$

There is no ambiguity.

$2(1 + 2) = 6$

It is a value represented by two factors Grouped by Multiplication and not to be arbitrarily separated by the division obelus preceding it.

For the Equation to equal 9, it should have been written:

$(6 \div 2)(1+2) = 9$

Regards

Louis Peregino