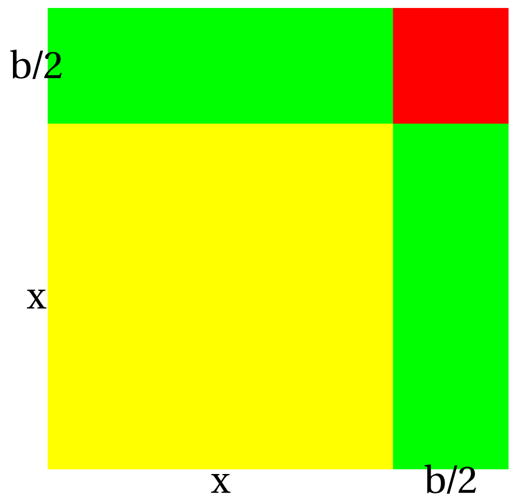
Worksheet: Completion of the Square

Problem: Solve the quadratic equation $x^2 + 10x = 61$.



The quadratic equation $x^2 + bx + c = 0$ is solved by adding $(b/2)^2$ on both sides:

$$\begin{aligned} x^2 + bx &= -c \\ x^2 + bx + (b/2)^2 &= (b/2)^2 - c \\ (x + b/2)^2 &= (b/2)^2 - c \\ x + b/2 &= \sqrt{(b/2)^2 - c} \\ x &= \pm\sqrt{(b/2)^2 - c} - b/2 \; . \end{aligned}$$

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi (780-850), born in Baghdad, was a mathematician and astronomer who is often cited as "the father of algebra". The title of his book, **Hisab al-jabr w'al-muqabala** is the origin of the name "algebra". Al-Khwarizmi introduces the solution of equations without using any symbols.

Problem: Look at the parabola $y = x^2 + 10x - 11$. We check that that $y = x^2 + 11x = 11$ has solutions which are to the left of the solutions before. Can you explain this geometrically? If not, use calculus.

I advocate to know the quadratic formula since it is often the safest way to get to the solution. But there are often short cuts

Problem: Find all the roots of $x^2 - 8x + 15$.

Here, we can guess the solutions $x = 3$ and $x = 5$ because the product of the two solutions has to be 15. This follows from the factorization

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2) + x_1 x_2 = 0 \ .$$

Problem: Find all the roots of $x^6 - 65 * x^3 + 64$.

Subject: Quadratic equations

Worksheet: Top 5 formulas?

Problem: In school geometry, there is no argument, the Pythagorean theorem is the most important result.

Which result in algebra is the most important one? There are some suggestions on the back, but maybe don't peak yet:

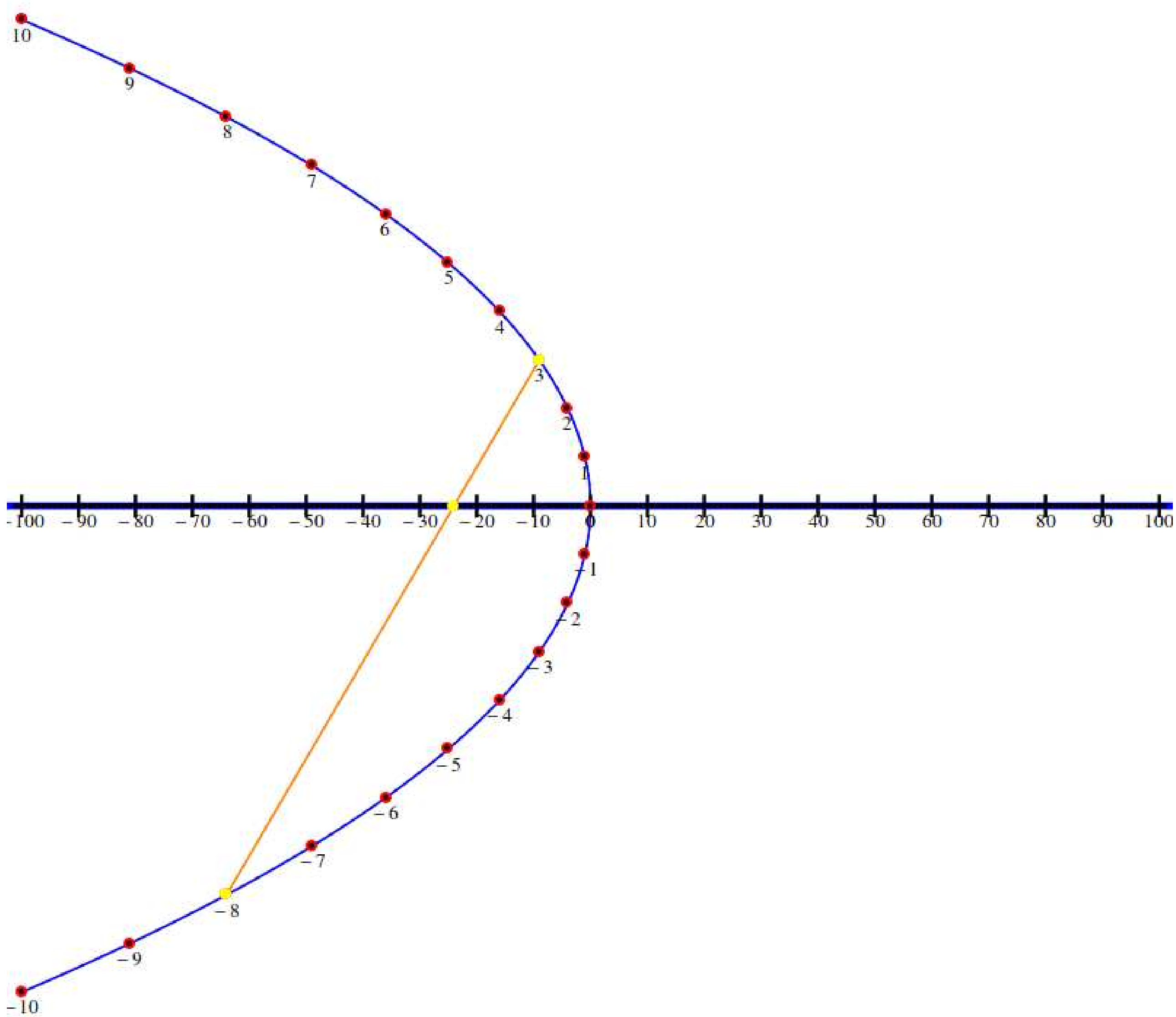
Here are some examples

- FOIL: first- outer- inner-last
- Log rule $\log(xy) = \log(x) + \log(y)$.
- Exponential rule $x^y = e^{y \log(x)}$
- The quadratic solution formula
- Add to y, go high, Add to x, go west.
- $(x-1)(x+1) = x^2 - 1$
- The associativity rule in multiplication
- The associativity rule in addition
- The distributivity rule
- $\sin^2(x) + \cos^2(x) = 1$
- $\sqrt{x}^2 = x$ but $\sqrt{x^2}$ is not x.
- We can add a constant to both sides of an equation
- When multiplying an inequality, the signs change
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$
- $1/(1/x) = x$
- $1^x = 1$ for all x
- $0 * x = 0$ for all x
- $1 * x = x$ for all x

Worksheet: Algebra with Parabola

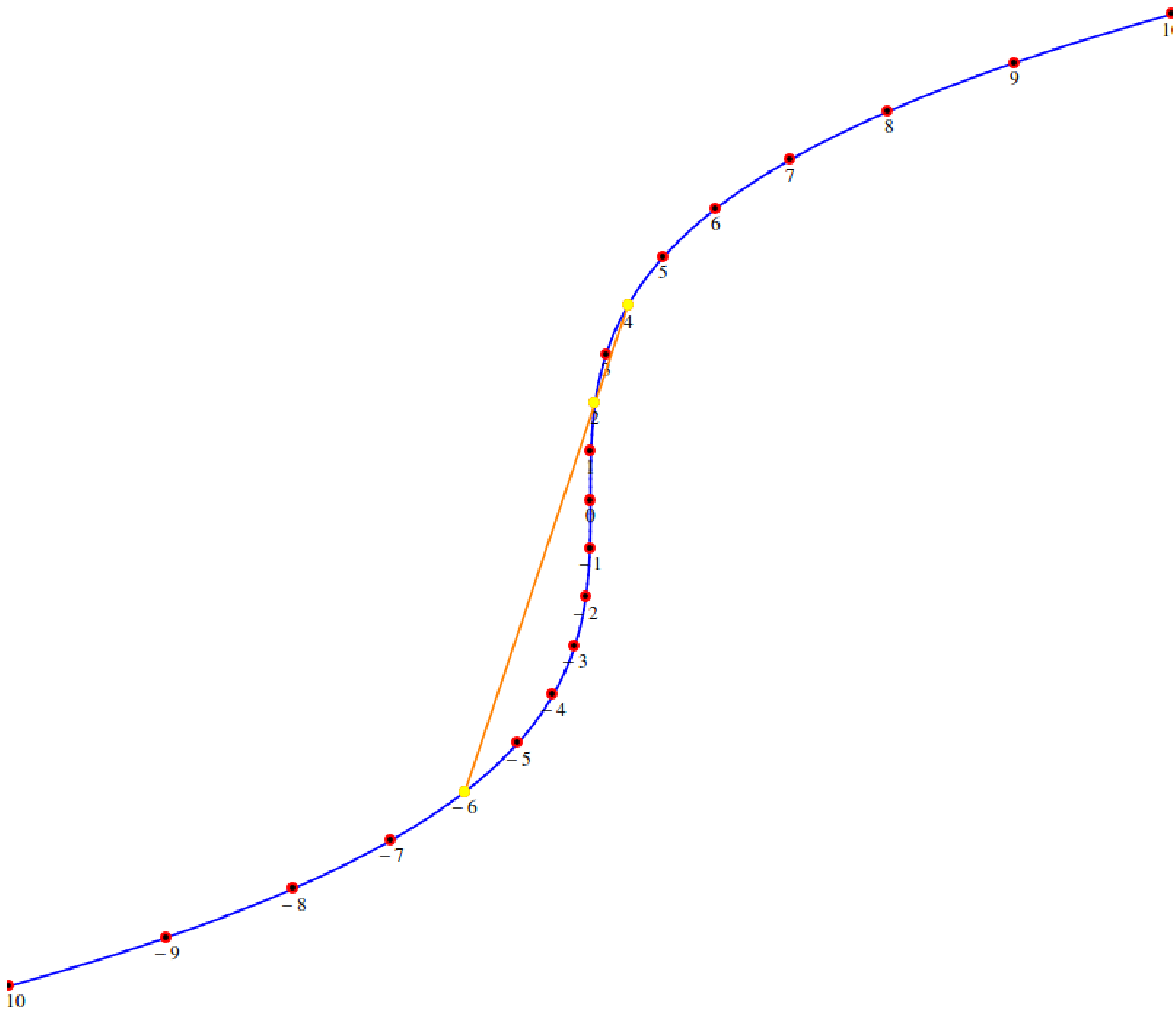
Problem: Verify the following curious property of the parabola.

The multiplication on the parabola is done by connecting two points $A = (-a^2, a), B = (-b^2, b)$ on the parabola with a line and intersect the line with the x-axes getting $C = (-ab, 0)$. The multiplication property is checked by showing that A, B, C are collinear.



Problem: Verify the following curious property of the cubic.

The addition can be realized on the cubic $x = y^3$ as follows: the line through $A = (a^3, a)$ and $B = (b^3, b)$ intersected with the cubic gives the point $(-(a+b)^3, -(a+b))$. In both cases, the group operation is realized geometrically using a line and then applying an involution.



Worksheet: A tough problem

Problem: Find the solutions to the equation $x^2 = 2^x$.

We can try $x = 2$.
Are there more solutions? Its not so easy. This example illustrates a pitfall in which teachers can step easily.

Tip: Be careful with posing problems which are too difficult for students.

We have to write $2\log(x) = x\log(2)$ and get $x/\log(x) = 2/\log(2)$. Is 2 the only solution? We can not solve for x since this is a **transcendental equation**. Indeed, there is an other solution $x = 4$. Is there an other one? **Drawing the graph** can help.

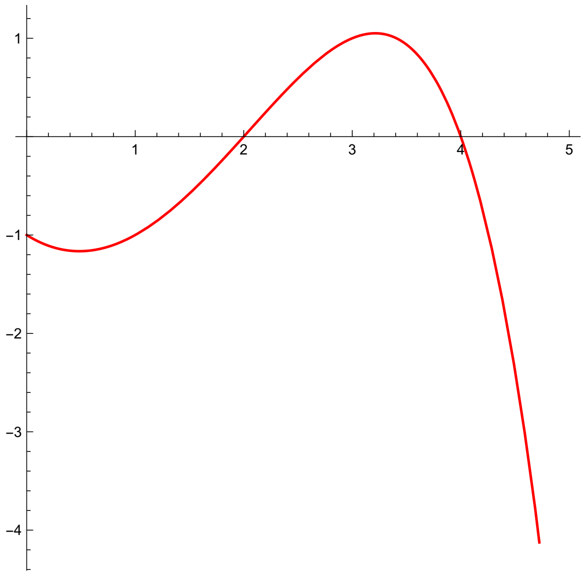


Figure 1: The graph of the function $x^2 - 2^x$. We see that there are two roots $x = 2$ and $x = 4$. They give us solutions to the equation $x^2 = 2^x$? Are these the only roots? How can we find a complete list of the roots?