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The energy of a simplicial complex

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ABSTRACT

A finite abstract simplicial complex G defines a matrix L, where $L(x, y) = 1$ if two simplicies x, y in G intersect and where $L(x, y) = 0$ if they don't. This matrix is always unimodular so that the inverse $g = L^{-1}$ has integer entries $g(x, y)$. In analogy to Laplacians on Euclidean spaces, these Green function entries define a potential energy between two simplices x, y. We prove that the total energy $E(G) = \sum_{x,y} g(x, y)$ is equal to the Euler characteristic $\chi(G)$ of G and that the number of positive minus the number of negative eigenvalues of L is equal to $\chi(G)$.

1. The theorems

1.1. A finite set G of non-empty sets which is closed under the operation of taking finite non-empty subsets is called a **finite abstract simplicial complex**. The n elements in G are called **simplices** or **faces**, the $n \times n$ matrix L satisfying $L(x, y) = 1$ if x and y intersect and $L(x, y) = 0$ else is the **connection matrix** of G. Define $\dim(x) = |x| - 1$, where $|x|$ is the cardinality of x. If $\omega(x) = (-1)^{\dim(x)}$, then $\chi(G) = \sum_{x \in G} \omega(x)$ is the **Euler characteristic** of G. A multiplicative analog of $\chi(G)$ is the **Fermi characteristic** $\phi(G) = \prod_{x \in G} \omega(x) \in \{-1, 1\}$.

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1.2.

Theorem 1 *(Unimodularity theorem [35] (2016)).* $\det(L) = \phi(G)$*.*

1.3. It follows from the Cramer formula in linear algebra that the inverse matrix $g = L^{-1}$ has integer entries $g(x, y)$. We can think of $g(x, y)$ as the **potential energy** between the simplices x and y. The number $E(G) = \sum_{x,y \in G} g(x, y)$ is the **total energy** of G.

Theorem 2 *(Energy theorem [37] (2017)).* $E(G) = \chi(G)$*.*

1.4. Let $p(G)$ be the number of positive eigenvalues of L and let $n(G)$ be the number of negative eigenvalues of L. Let $b(G)$ denote the number of even dimensional simplices in G and $f(G)$ the number of odd dimensional simplices in G. If $f_k(G)$ is the number of elements in G with cardinality $k+1$, then $(f_0, f_1, \cdots, f_d)$ is the **f-vector** of G and $\chi(G) = \sum_{k \text{ even}} f_k - \sum_{k \text{ odd}} f_k = b(G) - f(G)$.

Theorem 3 *(Hearing Euler characteristic [42], (2018)). We have* $b(G) = p(G)$ *and* $f(G) = n(G)$*. Therefore,* $\chi(G) = p(G) - n(G)$*.*

1.5. It follows from $|\det(L)| = 1$ that $\mathrm{tr}(\log(|L|)) = 0$ so that

$$\chi(G) = \frac{2}{i\pi} \mathrm{tr}(\log(iL))$$

if the branch $\arg(\log(z)) \in [-\pi, \pi)$ is chosen for $z \in \mathbb{C}$. This writes $\chi(G)$ as the **logarithmic potential** of $z = 0$ under influence of the charged spectral set $\sigma(iL)$ on the imaginary axes in the complex plane $\mathbb{C}$. This again reinforces that $\chi(G)$ is some sort of energy.

1.6. Let $W^+(x) = \{y \in G, x \subset y\}$ denote the **star** of x. It is just a set of non-empty sets and not necessarily a simplicial complex; but the same definition $\sum_{x \in A} \omega(x)$ defines an Euler characteristic for a subset A of G. The Green function entries are explicitly known:

Theorem 4 *(Green star formula [37] (2017)). The Green function entries are* $g(x, y) = \omega(x)\omega(y)\chi(W^+(x) \cap W^+(y))$*.*

1.7. The potential energies are **local**, of bounded range, unlike in Euclidean spaces, where potentials are long range. Simplicial complexes have a natural hyperbolic structure for which the star $W^+(x)$ is the unstable manifold of a gradient vector field of the dimension functional. The stable manifold $W^-(x) = \{y \in G, y \subset x\}$ is the simplicial complex generated by x and $\chi(W^-(x) \cap W^-(y)) = L(x, y)$. As $\omega(x) = \chi(W^-(x) \cap W^+(x)) = \chi(\{x\})$, both L and g have matrix entries given as the Euler characteristic of "homoclinic" or "heteroclinic points" of a hyperbolic dynamical system.

1.8. The disjoint union of complexes defines an additive monoid structure on the set of simplicial complexes in which the empty complex $G = \{\}$ is the neutral element. It can be Grothendieck completed to become a group. The group operation $G+H$ results in the direct sum of matrices $L(G+H) = L(G) \oplus L(H)$. An element in the completed group is naturally described by its connection Laplacian L if we postulate that $L(-G) = -L(G)$. The Cartesian product of two simplicial complexes is not a simplicial complex in general but it carries a natural exterior derivative d defining a Hodge Laplacian $H(G) = (d+d^*)^2$ which is a direct sum of form-Laplacians $H_k(G)$ for which by Hodge, the nullity of the kernel are the Betti numbers $b_k(G) = \ker(H_k(G))$ and for which the Künneth formula holds. On the full ring $\mathcal{G}$ the Poincaré map $p(G) = b_0 t + b_1 t^2 + \cdots + b_n t^n$ is a ring homomorphism from the ring $\mathcal{G}$ generated by simplicial complexes to polynomials in one variable. The spectrum of the Hodge Laplacian is not compatible in general with Cartesian multiplication. It is however compatible with the connection Laplacian:

Theorem 5 *(Tensor algebra representation [40], (2017)). The map $G \to L(G)$ represents the ring $\mathcal{G}$ in a tensor algebra of finite dimensional invertible matrices.*

1.9. The connection Laplacian $L(G)$ acts on the same Hilbert space as the Hodge Laplacian $H(G) = (d + d^*)^2$. The multiplication of complexes defines a strong product for the corresponding connection graphs and the **tensor product** of the connection Laplacians $L(G)$. It follows that when adding two simplicial complexes G and H, the spectrum of $L(G + H)$ is the union of the spectra as for "independent quantum mechanical processes". When multiplying two complexes, then the spectra of L multiply $\sigma(L(G \times H)) = \sigma(L(G))\sigma(L(H)))$ and are mathematically described in the same way as multi-particle states are written in physics.

2. Examples

2.1. There are various ways to build simplicial complexes: any finite set of finite non-empty sets A **generates** a complex $G = \{x \subset y \mid x \neq \emptyset, y \in A\}$. Given a complex G, the **k-skeleton** of G is the set of subsets of G of dimension $\leq k$. A finite simple graph (V, E) generates the **Whitney complex** $G = \{x \subset V \mid \forall a, b \in x, (a, b) \in E\}$, where the simplices are the vertex sets $V(K)$ of complete subgraphs K of (V, E). The Whitney complex is also known under the name **clique complex**. Its dual is the **independence complex**, in which the simplices are the independent sub-sets of V. An other example of a complex defined on a graph is the **graphic matroid** $G = \{x \subset E \mid x$ generates a non-empty forest in $(V, E)\}$.[1]

[1] As for simplicial complexes, there are definitions of matroids in which $x \subset E$ is required to be non-empty (like [65]) and definitions, where $x = \emptyset$ is included. We comment on this at the end.

2.2. **Example 1)** The set of sets $A = \{(1,2,3),(2,3,4)\}$ generates the complex $G = \{(1),(2),(3),(4),(1,2),(1,3),(2,3),(2,4),(3,4),(1,2,3),(2,3,4)\}$ which is the Whitney complex of the diamond graph. The f-vector is $(4,5,2)$, the Euler characteristic $\chi(G) = 4-5+2 = 1$, the Fermi characteristic $(+1)^4(-1)^5(+1)^2 = -1$ which agrees with the determinant of L and the determinant of $g = L^{-1}$. Written out the later is

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix} .$$

2.3. **Example 2)** Not every complex is a Whitney complex. The one-dimensional complex generated by $A = \{\{1,2\}, \{2,3\}, \{3,1\}\}$ is

$$G = C_3 = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\} .$$

It is the 1-skeleton complex C_3 of the triangle K_3. While K_3 is the Whitney complex of a graph, the complex C_3 is not. Its f-vector is $(3,3)$ with Euler characteristic $\chi(G) = 3-3 = 0$ and Fermi characteristic $\phi(G) = (+1)^3(-1)^3 = -1$. Here are the matrices:

$$L = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, g = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix} .$$

We check $\det(L) = -1$ and $\sum_{x,y \in G} g(x,y) = 0$.

2.4. **Example 3)** If $G = \{\{1,2\},\{1\},\{2\}\}, H = \{\{1,2\},\{2,3\},\{1\},\{2\},\{3\}\}$ then

$$L(G) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, L(H) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} ,$$

with $\det(L(G)) = -1$ and $\det(L(H)) = 1$. The eigenvalues of $L(G)$ are $\{1+\sqrt{2}, 1, 1-\sqrt{2}\}$, the eigenvalues of $L(H)$ are

$$\sigma(L(H)) = \left\{ \frac{1}{2}\left(3+\sqrt{13}\right), \frac{1}{2}\left(1+\sqrt{5}\right), 1, \frac{1}{2}\left(1-\sqrt{5}\right), \frac{1}{2}\left(3-\sqrt{13}\right) \right\} .$$

The product complex $G \times H$ as a connection Laplacian produces $L(G \times H$ which is the tensor product of $L(G)$ and $L(H)$:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

If we switch the order, we get the tensor product of $L(H)$ with $L(G)$. This is a different matrix, but conjugated to the previous one.

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix} .$$

The eigenvalues $\sigma(L(G) \otimes L(H))$ of the tensor product are the products $\{\lambda_j \mu_k$, where $\lambda_j \in \sigma(L(G))$ and $\mu_j \in \sigma(L(H))\}$.

2.5. **Example 4)** Here is an example of a graphic matroid. If $(V, E) = (\{1, 2, 3, 4\}, \{(14), (12),(13), (23),(34)\})$ is the diamond graph and the edges are relabeeld as $E = \{a, b, c, d, e\}$), we get the list $\{(a, b, c)$, (a, b, d),(a, b, e), (a, c, d),(a, d, e), (b, c, e), (b, d, e), $(c, d, e)\}$ of spanning trees of (V, E) and $G = \{(a), (b), (c), (d), (e), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), (a, b, c), (a, b, d), (a, b, e), (a, c, d), (a, d, e), (b, c, e), (b, d, e), (c, d, e)\}$ is the graphic matroid of the diamond graph. It has the f-vector $(5, 10, 8)$ and Euler characteristic $\chi(G) = 5 - 10 + 8 = 3$. The (23×23)-matrix connection matrix L of G has the inverse whose entries add up to 3.

2.6. **Example 5)** If $G = \{\{1, 2\}, \{1\}, \{2\}\}$ is the set of non-empty subsets of a two point set $V = \{1, 2\}$, then $n = 3, b = 2, f = 1$ and $\chi(G) = 2 - 1 = 1$. The matrices L and g are

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, g = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

We see that $n = 3, b = 2, f = 1$ and $\chi(G) = 1, \phi(G) = -1$. We check $\sum_{x,y} g(x, y) = 1$. There is also the relation $\chi(G) = b - f$. The Fermi characteristic $\phi(G) = (+1)^b(-1)^f = (-1)^f = -1$ agrees with $\det(L) = -1$.

2.7. **Example 6)** If G is the diamond graph complex generated by the set $A = \{(1, 2, 3), (2, 3, 4)\}$, then the connection matrix and its inverse are (we are using $a = -1$ for typographical reasons):

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & a & a & 0 & a & 1 & 1 \\ 0 & 1 & 0 & 0 & a & 0 & a & a & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 & 0 & 1 & 1 & 0 & 0 & a & 0 \\ 0 & a & 0 & 0 & 1 & 0 & 1 & 0 & 0 & a & 0 \\ a & a & a & a & 1 & 1 & 1 & 1 & 1 & a & a \\ 0 & 0 & a & 0 & 0 & 0 & 1 & 0 & 1 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a \\ 1 & 1 & 1 & 0 & a & a & a & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & a & a & a & 0 & 1 \end{bmatrix}$$

The entries of g add up to 1 which is the Euler characteristic $\chi(G) = 4 - 5 + 2 = 1$ of G.

2.8. **Example 7)** The set of sets $G = \{(1,2),(2,3),(1),(3)\}$ is a finite set of non-empty sets but not a simplicial complex. Its Euler characteristic is $\chi(G) = 0$. We can still define the connection Laplacian and have unimodularity $\det(L) = \phi(G) = \prod_{x \in G} \omega(x)$, but the energy theorem $\sum_{x,y \in G} g(x,y) = \chi(G)$ fails as the total energy is 2 which is different from $\chi(G) = 0$. The matrices are

$$L = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, g = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}.$$

3. Poincaré-Hopf

3.1. Every simplicial complex G defines a finite simple graph $G_1 = (V, E)$, where V consists of the sets of G and where two vertices in V are connected by an edge if one is contained in the other as an element in G. The Whitney complex of this graph is called the **Barycentric refinement** G_1 of G. One can define the Barycentric refinement also as the subset G_1 of the power set 2^G, where each simplex $A \in G_1$ has the property that if $x, y \in A$ then either $x \subset y$ or $y \subset x$, where the subset relation refers to the elements x, y of G_1 which are subsets of G.

3.2. A function $f : G \to \mathbb{R}$ defines a function on the vertex set of the Barycentric refinement G_1. Let $S(x)$ denote the **unit sphere** of x: it consists of all simplices y in G which are either strictly contained in x or all simplices y which strictly contain x. The set of sets $S(x)$ is not a simplicial complex in general, but it defines a subgraph of G_1, which carries a Whitney complex which is a simplicial complex. The following Poincaré-Hopf theorem applies for general graphs (V, E) and not only for graphs G_1 defined by simplicial complexes.

3.3. A function $f : G \to \mathbb{R}$ is **locally injective** if $f(x) \neq f(y)$ for any pair $x, y \in G_1$ which are connected in G_1. A locally injective function is also called a **coloring** of G_1, meaning that every set $\{f = c\}$ is independent in the sense of graph theory. Define the **index** $i_f(x) = 1 - \chi(S_f^-(x))$, where $S_f^-(x)$ is the sub-complex of all $y \in S(x) \subset G_1$ with $y \subset x$.

Lemma 1 *(Poincaré-Hopf, [28] 2012).* $\chi(G) = \sum_x i_f(x)$ *for a locally injective* f.

Proof. The formula holds in general for any finite simple graph $G = (V, E)$ with locally injective function f defined on V. To prove it, use induction with respect to the number n of vertices. Take a vertex $v \in G$ for which f is locally maximal. Let $H = G - v$ be the sub-graph of G for which the vertex v and all connections to v were taken away. Since χ is a valuation, it satisfies $\chi(G) = \chi(H) + \chi(B(x)) - \chi(S(x))$. Because $\chi(B(x)) = 1$

and $\chi(S(x)) = \chi(S_f^-(x))$ and by the induction assumption, $\chi(H) = \sum_y i_f(y)$, we have $\chi(G) = \sum_{y \neq x} i_f(y) + i_f(x)$. We have used that for any graph, and for any vertex x, the unit ball $B(x)$ has Euler characteristic 1. □

3.4. If G_1 is the Barycentric refinement of a complex G, one can look at the particular function $f(x) = \dim(x)$. Because connected sets have different dimension, it is a coloring. The minimal number of colors, which is also known under the name **chromatic number**, is equal to the **clique number** which is the maximal cardinality of sets in G. The set $S^-(x)$ is now a sphere complex as it is the boundary of the simplex x. Its Euler characteristic is either 0 or 2 depending on the dimension.

3.5. In the case $f(x) = \dim(x)$, we have $i_f(x) = \omega(x)$ as it is the genus $1 - \chi(S^-(x))$ of the sphere complex $S^-(x)$. Poincaré-Hopf now tells that $\chi(G_1) = \chi(G)$. The fact that Euler characteristic is a combinatorial invariant follows also from the explicit relation between the f-vectors of G and G_1. The f-vector gets multiplied with the Barycentric refinement operator $A = i!S(j, i)$, where $S(j, i)$ are the **Stirling numbers of the second kind**. As the transpose A^T has a unique eigenvector $(1, -1, 1, -1, \dots)$ with eigenvalue 1, the Euler characteristic is the only valuation (linear functional on f-vectors) with the property that it is invariant under Barycentric refinements. Poincaré-Hopf confirms in particular that χ is invariant under refinement.

3.6. For a generalization of Poincaré-Hopf in which the Euler characteristic is replaced by the f-function $f_G(t) = 1 + f_0 t + f_1 t^2 + \cdots f_d t^{d+1}$, see [46]. Given a locally injective function g on the vertex set of a graph, then one has $f_G(t) = 1 + t \sum_{x \in V} f_{S_g(x)}(t)$. There is also a generalization of Poincaré-Hopf from gradient fields to the field type situation given for example in the form of directed graphs (V, E) [47] without triangular cycles. The index is $i(v) = 1 - \chi(S^-(v))$, where $S^-(v)$ is the graph generated by all vertices pointing towards v. If g is a real-valued coloring $g : V \to \mathbb{R}$ then a digraph structure without triangular circular graphs is defined by $v \to w$ if and only if $g(v) < g(w)$. The proof of the more general statement of digraphs which do not necessary have the orientation inherited from a scalar function, one can notice that the absence of triangular cycles produces a total order on each complete sub-graph so that the value $\omega(x) = (-1)^{\dim(x)}$ on each simplex x can be moved to the vertex in x which is maximal with respect to the order.

4. Gauss-Bonnet

4.1. If Ω is the set of locally injective functions on G and $\mathcal{A}$ a σ-algebra on Ω and P a probability measure on the measure space $(\Omega, \mathcal{A})$ we have a probability space $(\Omega, \mathcal{A}, P)$. The **curvature** defined by this probability space is defined as the expectation $K(x) = \mathrm{E}[i_f(x)]$. It obviously satisfies the Gauss-Bonnet formula as it is an average of Poincaré-Hopf theorems:

Corollary 1 *(Gauss-Bonnet).* $\chi(G) = \sum_{x \in G} K(x)$.

Proof. As $\chi(G) = \sum_x i_f(x)$, we have $\chi(f) = \sum_x \mathrm{E}[i_f(x)]$. $\square$

4.2. This is an elegant approach to Gauss-Bonnet and also holds for compact Riemannian manifolds M if one takes a probability space of Morse functions. There are probability spaces which produce the Gauss-Bonnet-Chern integrand for example. One can for example Nash embed M isometrically in an ambient Euclidean space E, then take the probability space of all linear functions in E. Most of them induce Morse functions on M. They produce a probability space of Morse functions on M. Other curvature can be obtained by taking the normalized volume measure on M as P using the heat kernel functions $\Omega = \{f_y(x) = [e^{-\tau L_0}]_{xy}\}$ where L_0 is the Laplacian on scalars. They are Morse for almost all $m \in M$.

4.3. If $\Omega = \prod_{x \in G}[-1,1]$ is the product probability space and if either $P = \prod_{x \in G} dx/2$, where dx is the Lebesgue measure on $[-1,1]$, or P is the uniform counting measure on all c-colorings, where c is the chromatic number of G, we get the **Levitt curvature**

$$K(x) = \sum_{y, x \subset y} \frac{\omega(y)}{|y|} = 1 - \frac{V_0(x)}{2} + \frac{V_1(x)}{3} - \frac{V_2(x)}{4} \dots ,$$

where $V_k(x)$ is the number of k-dimensional simplices in $S(x)$. In the 2-dimensional case, the curvature is $K(x) = 1 - V_0/2 + V_1/3$ which is $K(x) = 1 - \deg(x)/6$ in the case of a triangulation, a case known for a long time [13], the general case appeared in [51]. It was rediscovered in a geometric setting in [26], where it was placed as a Gauss-Bonnet result and studied in the case of low dimensional discrete manifolds. The integral theoretic picture of seeing curvature as an expectation of indices is adapted from integral geometry in the continuum [29,32].

4.4. As for Poincaré-Hopf, also Gauss-Bonnet does not require the graph to be the Barycentric refinement of a complex. It holds for any finite simple graph. Both the index function $i_f(x)$ as well as the curvature function $K(x)$ are defined on the vertices but unlike curvature, the index is always integer-valued. Poincaré-Hopf is even more general: let ϕ be a map from the set V_1 of vertices in the Barycentric refinement G_1 to the set V of vertices in G. Now take the values $\omega(x)$ on V_1 which add up to Euler characteristic and place them to the vertex $\phi(x)$. If $\phi(x) = \{v \in x \mid f(v) < f(w), \forall w \in x\}$ then we get Poincaré-Hopf. Taking a probability space of functions corresponds then to a probability space of transition maps, a Markov process. In the special case, when we distribute the value $\omega(x)$ equally to its 0-dimensional part, the Levitt version of Gauss-Bonnet results.

5. Abstract finite CW-complexes

5.1. A **finite abstract CW-complex** is a geometric, combinatorial object which like Δ-sets or simplicial sets generalizes simplicial complexes. The definition does not tap into Euclidean structures like classical definitions of CW-complexes but closely follows the definition which is used in the continuum. The constructive definition uses a gradual build-up which allows to make proofs easier. Any finite abstract simplicial complex is a CW-complex but CW-complexes are more general.

5.2. The empty complex $0 = \{\}$ is declared to be a CW-complex and assumed to be also a (-1)-sphere. This complex does not contain any cell. To make an extension of a given complex H, choose a sphere S within H and add a **cell** x which has S as the boundary. Given a complex H with n cells and a sub-complex K which is a d-sphere, we add a new $(d+1)$-cell given as the join $x = H + 1$. The integer $d+1$ that is attached to the cell is the **dimension** of the cell. The cells in $S(x)$ are declared to be **a subset** of x or **contained in** x. The maximal dimension of a cell x is the **dimension** of the CW-complex.

5.3. The **unit sphere** $S(x)$ of a cell x is the union of all cells either contained in x or all cells which contain x. A CW-complex G is called **contractible**,[2] if there exists a cell x which has a contractible unit sphere $S(x)$ so that $G \setminus x$ is contractible. To start of the inductive definition, assume that the empty complex $0 = \{\}$ is a (-1)-dimensional sphere which is also declared to be not contractible and that the 1-point complex $1 = K_1$ is declared to be contractible. A CW-complex G of maximal dimension d is called a **d-sphere** if it is a d-complex for which removing any single cell produces a contractible complex. A CW-complex G is a **d-complex**, if all unit spheres $S(x)$ are $(d-1)$-spheres. We do not require however a CW-complex to be a d-complex in general.

5.4. The connection Laplacian L for a CW-complex G is defined in the same way as before if we define two cells to intersect if they have a common sub-cell. It is a $n \times n$ matrix if there are n cells. The construction is inductive too and depends on the build-up of G. Assume the connection Laplacian L for G has been constructed, the connection Laplacian L of the enlarged complex $G+x$ is a $(n+1) \times (n+1)$-matrix, where $L(x,y) = L(y,x) = 1$ if $y \cap (S(x) \cup \{x\}) \neq \emptyset$ and 0 else. In the case of a simplicial complex, it falls back to the already given definition, where $L(x,y) = 1$ if x and y intersect and $L(x,y) = 0$ else.

5.5. The **Barycentric refinement** of a CW-complex G is the Whitney complex of a graph G_1. The cells of G are the vertices of G_1 and two cells are connected, if one is contained in the other. The **connection graph** G' of G is the graph with the same vertices as G_1, but where two cells are connected if they intersect. We can write $L = 1 + A$, where

[2] We identify here contractible and collapsible and would use the terminology "homotopic to 1" for the wider equivalence relation.

A is the adjacency matrix of G' and 1 is the identity matrix, so that the determinant of L is the **Fredholm determinant** of A.

5.6. When adding a cell to a complex, the Euler characteristic changes by $\chi(H) \to \chi(G) = \chi(H)+(1-\chi(S(x)))$. We will show in a later section that the Fermi characteristic changes by $\psi(H) \to \psi(G) = \psi(H)(1-\chi(S(x)))$, when a new cell x is added to a complex H. This illustrates the multiplicative character of ψ in contrast to the additive property of χ.

6. Valuations

6.1. A **valuation** on a simplicial complex G is a function X from the set of sub-complexes of G to $\mathbb{R}$ satisfying the **valuation property** $X(A\cup B)+X(A\cap B) = X(A)+X(B)$ for all sub-complexes A, B of G. A complex is called **a complete complex** if it is of the form $G = 2^A \setminus \{\emptyset\}$ for some finite set A. Every simplex $x \in G$ naturally defines a complete complex $W^-(x) = \{y \neq \emptyset \mid y \subset x\}$. While there is an obvious bijection between complete sub-complexes of G and subsets of G, there is a difference however. Most notably, the empty complex $0 = \{\}$ is a complex but not a simplex in a simplicial complex. Some authors add the empty set $\emptyset$ to any simplicial complex and call it the "void". This leads to the **reduced Euler characteristic** $\tilde{\chi}(G) = \chi(G)-1$, as the **void** $\emptyset$ counts as a (-1)-dimensional simplex which subtracts 1 because the dimension of the void (-1) is odd. We prefer the more topological definition which poetically avoids voids and see the empty-set as a (-1)-dimensional sphere of Euler characteristic 0 and not a simplex. The number $1-\chi(G)$ behaves then like a **genus** (as it is the first Betti number b_1 if G is a 1-dimensional connected complex with Euler characteristic $\chi(G) = b_0 - b_1 = 1 - b_1$ leading to $b_1 = 1-\chi(G)$.)

Lemma 2. *A valuation X which satisfies $X(A) = 1$ for every complete sub-complex A of G must be equal to the Euler characteristic of G.*

Proof. Since the Euler characteristic of a complete complex is 1 we only have to show uniqueness. By the discrete Hadwiger theorem [25], any valuation is of the form $X(A) = X \cdot f(A)$, where $f(A)$ is the f-vector of A and X is a vector and $\cdot$ is the dot product. If G has maximal dimension d, then the space of valuations has dimension $d+1$. If $X(A) = 1$ for every complete graph, this means $X \cdot f(K_k) = 1$, for the f-vectors $f(K_k)$ of K_k. But since these vectors form a basis in the vector space $\mathbb{R}^{d+1}$, we also have uniqueness and $X = \chi$. □

6.2. We have defined a valuation only for simplicial complexes so far. It is possible to extend it to CW-complexes. For CW-complexes, we can define a valuation to be a linear functional $X(G) = \sum_i X_i f_i(G)$ on the f-vector $(f_0(G), f_1(G), \dots, f_d(G))$ of the complex, where $f_k(G)$ counts the number of k-dimensional cells in G. The same definition applies for the Cartesian products of two simplicial complexes or for signed complexes by declaring $X(-G) = -X(G)$ after extending the "disjoint union monoid" of simplicial complexes to a group.

7. Paths

7.1. If A is the adjacency matrix of a graph Γ, the determinant $\det(A)$ is a partition function or a "path integral", in which the underlying paths are fixed-point-free signed permutations of the vertices of the graph. The reason is that the Leibniz definition of the determinant generates **derangements** π for which $\gamma : x \to \pi(x) \to \pi(\pi(x)) \dots$ defines oriented closed paths which depend on the signature of the permutation and are counted positively or negatively. The **Fredholm determinant** $\zeta(\Gamma) = \det(1 + A)$ is a partition function for all oriented cyclic paths π in the graph as the newly added diagonal allows for fixed points $x \to \pi(x)$. In short,

$$\det(1 + A) = \sum_{\gamma} (-1)^{|\gamma|}$$

summing over all 1-dimensional oriented cyclic path sets γ of Γ and where $(-1)^{|\gamma|} = \text{sign}(\gamma) = \phi(\gamma)$ is the signature of the corresponding permutation. If for a cyclic part C we have $\phi(C) = (-1)^{|C|}$ as $|C|$ counts the number of odd-dimensional simplices in C and $\phi(\gamma) = \prod_{C \in \gamma} \phi(\gamma)$. In this sense, the Fredholm determinant is more natural as it sums over all dynamical systems on the network and not only over fixed-point-free ones as the determinant does.

7.2. The unimodularity theorem tells that if the graph Γ is the connection graph G' of a simplicial complex G and L is the connection Laplacian of G, then

$$\det(L) = \sum_{\gamma \subset G} (-1)^{|\gamma|} = \prod_{x \in G} (-1)^{|x|-1} = \phi(G) .$$

As $L = 1 + A$ where A is the adjacency matrix of Γ, the above path picture in the connection graph applies. The energy theorem tells that the Euler characteristic of G is

$$\sum_{x,y} L^{-1}_{x,y} = \sum_{x \in G} (-1)^{|x|-1} = \chi(G) .$$

Cramer's determinant formula shows that the left hand side has a path interpretation too. In other words, both the determinant of L as well as the Euler characteristic of G have a path integral representation summing over one-dimensional closed oriented loops of the complex.

7.3. If A is a $n \times n$ matrix and if $\text{per}(A)$ denotes the **permanent** of A, then $\text{per}(A)$ is the number **derangements** of the vertex set of the graph while $\text{per}(1 + A)$ is the number of all **permutations** of the vertex set. For complete graphs $G = K_n$ for example, $\mathrm{per}(1 + A(K_n))$ generates the **permutation sequence** $1, 2, 6, 24, 120, 720, \dots$ while the permanent of the adjacency matrix $\mathrm{per}(A(K_n))$ generates the **derangement sequence** $0, 1, 2, 9, 44, 265, \dots$

7.4. To every path belongs a sign, the signature of its permutation. The determinant of L can now be expressed combinatorially. The unimodularity theorem assures that the number of even paths and the number of odd paths in a connection graph G' differ by 1. The following lemma will be used later. It is a special case of the multiplicative Poincaré-Hopf lemma. If G is a CW-complex with complete subcomplex H and $G_x = G +_H x$ is the extended complex, then, we can look at the new connection graph $G'_x = G' +_{H'} x$.

Lemma 3 *(Fredholm extension lemma). If H is a complete subcomplex of G then $\psi(G' +_{H'} x) = 0$.*

Proof. If H is a complete subcomplex of G, then it defines a vertex h in G' (as G_1 also G' has the sets in G as vertices). If a path does not hit H, then it can be paired with a path adding the hx path of length 2. If a path does hit H, then it has to hit a neighbor k which is also connected to x. We can now pair such a piece hk with the extended path hxk and again get a cancellation. How look at all paths which do hit h and do not hit hk etc. We see that the paths can be partitioned into a finite set of paths, where each can be paired with an extended path with opposite sign. □

7.5. Note that H is a subcomplex of G, and not of G'. We build then the complex $G_x = G +_H x$ which produces the connection graph $G'_x = G' +_{H'} x$. The above Fredholm extension Lemma 3 tells that the Euler characteristic of H and not of H' matters. This is important because $\chi(H)$ and $\chi(H')$ differ in general. Comparing the Fredholm determinant of G' and $(G +_H x)'$ is not the right thing. The added vertex x over H attaches a new cell to the CW-complex. Also on the level of graphs, the extension lemma would fail as the join over H adds a lot more simplices on the connection level. The Fredholm determinant of the expanded cell complex G_x which on the connection graph level makes a cone extension over H'.

8. Proof of unimodularity

8.1. The key of the proof is a **multiplicative Poincaré-Hopf result** for CW-complexes which immediately proves the unimodularity theorem. If a newly added cell x is odd-dimensional, then $S(x)$ is an even-dimensional sphere with Euler characteristic 2 and $(1-\chi(S(x))) = -1$ switches the sign. If the newly added cell x is even-dimensional, then $S(x)$ is odd dimensional. In this case, $1 - \chi(S(x)) = 1$ and the sign of the product stays the same. In the following proposition, it is not assumed that $A = S(x)$ is a sphere. It applies therefore for more general complexes.

Proposition 1 *(Multiplicative Poincaré Hopf). If x is a new cell attached to a sub-complex $A = S(x)$ of G, then $\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$.*

Proof. (i) The map

$$Y : A \to (\psi(G \cup_A x) - \psi(G))$$

is a valuation.
(ii) This valuation satisfies $\psi(G \cup_A x) = 0$ if A is a complete sub-graph. This follows from the Fredholm extension Lemma 3.
(iii) It follows from (ii) that $Y(A) = -\psi(G)$ if A is a complete sub-graph.
(iv) By Lemma 2, because the valuation $-Y(A)/\psi(G)$ takes the value 1 for complete sub-graphs, we have $-Y(A)/\psi(G) = \chi(A)$ which means $Y(A) = -\chi(A)\psi(G)$ But $(\psi(G \cup_A x) - \psi(G)) = -\chi(A)\psi(G)$ is equivalent to the claim $\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$. □

8.2. An algebraic argument proving this would use the Laplace expansion of the connection matrix of $G \cup_A \{x\}$ with respect to the newly added column. We will see that in the proof of Theorem 2. However, the analysis of the minors requires a similar insight into the path expansion as in Lemma 3.

8.3. If more general objects H than spheres would be used in the extension construction $G \to G \cup_H \{x\}$, where the interface H is a general sub-complex of G, then the unimodularity theorem would fail in general as the Fermi characteristic could become zero. For a CW-complex, the sub-complexes H are assumed to consist of spheres $S(x)$. Because they have Euler characteristic 0 or 2, this assures that $1 - \chi(S(x))$ is always 1 or -1.

8.4. Every simplicial complex is a CW-complex: first start with 0-dimensional cells, the vertices, then add 1-dimensional simplices x. Every sphere $S(x)$ of such a simplex x is a 0-dimensional sphere. After having added all 1-dimensional simplex, each triangle is still only a 1-dimensional complex C_3, the 1-skeleton of K_3 and a sphere. Now add the 2-dimensional simplices. This converts a 1-dimensional cyclic cell complex C_3 into a 2-dimensional complex K_3. We can build up any simplicial complex recursively by starting with sets of cardinality 1, then adding sets of cardinality 2, then 3 etc. It then follows from the Euler characteristic or Fermi characteristic changes that $\chi(G) = \sum_x \omega(x)$ and $\psi(G) = \prod_x \omega(x)$, where $\omega(x) = (-1)^{\dim(x)} = 1 - \chi(S^-(x))$ and $S^-(x)$ is the unit sphere of the cell x at the moment it has been added to the CW-complex.

8.5. Discrete CW-complexes are strictly more general than simplicial complexes: we can for example add a cell x to the cyclic graph C_4 of length 4. The Barycentric refinement of the resulting CW complex is then the Whitney complex of a wheel graph. It has only $9 = 8 + 1$ cells, where the 8 cells came from the circular graph C_4 generated by $\{(1,2),(2,3),(3,4),(4,1)\}$. The wheel graph W_4 obtained as $W_4 = C_4 + 1$ (the cone extension) on the other hand has more cells: as a Whitney complex, the wheel graph W_4 has 5 vertices, 8 edges and 4 triangles leading to a finite simplicial complex with 17 cells. Topologically the two CW-complexes are equivalent, but the first has 9 cells, the second one has 17 cells. As in the continuum, discrete CW-complexes have both practical advantages as well as proof theoretical advantages.

8.6. Here is an other example. The 2-dimensional cube $G = K_2 \times K_2 = \{(12),(1),(2)\} \times \{(12),(1),(2)\} = \{\ (12)\times(12),\ (12)\times(1),\ (12)\times(2),\ (1)\times(12),\ (1)\times(1),\ (1)\times(2)\ (2)\times(12),\ (2)\times(1),\ (2)\times(2)\ \}$ is not the Whitney complex of a graph. It can also not be written as a simplicial complex. The Cartesian product of two simplicial complexes is not a simplicial complex if both factors are positive dimensional. We can see $K_2 \times K_2$ however as a CW-complex with $3 \cdot 3 = 9$ cells. The Barycentric refinement of this element is defined as the Whitney complex of a graph $G_1 = (V, E)$, where V is the set of 9 cells and where $(a, b) \in E$, if either $a \subset b$ or $b \subset a$. Completely analog is the boundary of the solid cube $K_2 \times K_2 \times K_2$. This boundary is what we usually call the "cube" or hexahedron. It can be realized by starting with the 1-dimensional graph representing the cube and adding 6 cells. This gives a CW-complex with one 3-dimensional cell, six 2-dimensional cells, twelve 1-dimensional faces and eight 0-dimensional cells. In total, there are $3^3 = 27$ cells. It is only as a discrete CW-complex that we regain the familiar picture of the cube as a 2-dimensional sphere.

9. Joins

9.1. Let us first define the join for simplicial complexes. Given two sets $x, y \in G$, let $x + y$ be the disjoint union and $x \cup y$ the union. Given two simplicial complexes G and H, then $G + H = G \cup H \cup \{x \cup y \mid x \in G, y \in H\}$ is a simplicial complex called the **Zykov join**. It is the discrete analogue of the join in topology.

9.2. With the notation $1 = K_1$, the complex $G + 1$ is a **cone extension** of G. If $P_2 = S_0 = \{\{1\},\{2\}\}$ denotes the 0-dimensional sphere, the complex $G + P_2$ is the **suspension**. The Zykov monoid has the class of spheres as a sub-monoid. The complex $P_2 + P_2$ is the circle C_4, the complex $P_2 + C_4 = O$ is the octahedron. The complex $nP_2 = P_2 + P_2 + \cdots + P_2$ is a $(n-1)$-dimensional sphere. A complex is an **additive Zykov prime**, if it can not be written as a sum $H + K$, where H, K are complexes.

9.3. The following lemma holds for general CW-complexes. But we need it only for complexes.

Lemma 4. *If G is contractible and H is arbitrary, then $G + H$ is contractible.*

Proof. Use induction with respect to the number of cells. Assume $G = K +_A x$, where A is contractible, then $G + H = K + H +_{A+K} x$. By induction, $A + K$ is contractible and $K + H$ is contractible. Therefore $G + H$ is contractible. □

9.4. For example, the cone extension $G + 1$ of any complex is contractible. As an other example, any unit ball $B(x)$ as it is the cone extension of the sphere $S(x)$. The just observed statement implies:

Corollary 2. *Contractible complexes form a sub-monoid of all simplicial complexes. An additive prime factor of a non-contractible complex is not-contractible.*

9.5. For example, $C_4 = S_0 + S_0$, where S_0 is the zero dimensional 2-point complex which is the 0-dimensional sphere. C_4 is not contractible so that also S_0 is not contractible.

9.6. A **d-ball** is a punctured sphere $S - x$, a d-sphere S for which one vertex x has been taken away. By definition, a d-ball is contractible. By definition a d-ball is a d-complex with boundary which has a $(d - 1)$-sphere as a boundary. For example, the cone extension $G + x$ of a sphere G is a ball because making an other cone extension $G + x + y$ is a suspension $G + S_0$ which is a sphere so that $G + x = G + S_0 - y$ is a ball. Unlike spheres, balls do not form a sub-monoid of the join complex. The ball $1 = K_1$ for example gives $1 + 1 = K_2$ which is the 1-simplex and not a 1-ball.

9.7. One of the nice things about the Zykov monoid is that it contains spheres as a sub-monoid.

Lemma 5. *If G and H are both spheres, then $G + H$ is a sphere. If G is a sphere and H is a ball, then $G + H$ is a ball.*

Proof. The proof is inductive. Note that for $x \in G$, the sphere $S_{G+H}(x) = S_G(x) + H$ and for $y \in H$ we have $S_{G+H}(y) = G + S_H(y)$. This shows that the unit spheres in $G + H$ are all spheres if the unit spheres in G and H are and that the unit spheres in $G + H$ are balls if one of the G and H is a ball. As for the induction assumption, for $G = H = 0$, we have $G + H = 0$ and for $G = 0, H$ we have $G + H = H$.

Write $G = K +_A x$, where A is a smaller dimensional sphere and K is a smaller dimensional ball. Now $G + H = K + H +_{A+H} x$. We know $K + H$ is a ball by induction assumption and that $A + H$ is a sphere, again by induction assumption. □

10. Hyperbolicity

10.1. Given a simplicial complex A, its **genus** is defined as $\gamma(A) = 1 - \chi(A)$. The reason for the name is that if A is 1-dimensional connected complex then $\chi(A) = b_0 - b_1 =$

$1-b_1$ is the Euler-Poincaré formula relating the combinatorial and cohomological Euler characteristic then $b_1 = \gamma(A)$ is the genus of the curve, the number of "holes". The analogy is less established in higher dimensions: for surfaces already, where $\chi(A) = b_0 - b_1 + b_2 = 2 - b_1$, it is custom to define the genus as $1 - \chi(A)/2$. For a torus for example, $\gamma(A) = 1 - \chi(A)/2 = 1$ and for a sphere, the genus $\gamma(A) = 1 - 2/2 = 0$ is zero. The following product formula shows that the genus γ is natural:

Lemma 6 *(Product formula).* $\gamma(A+B) = \gamma(A)\gamma(B)$.

Proof. If $f(t) = 1 + \sum_{k=1}^{d} f_k t^{k+1} = 1 + f_0 t + f_1 t^2 + \cdots + f_d t^{d+1}$ is the **generating function** for the f-vector $(f_0, f_1, \ldots, f_d)$ of G, then the Euler characteristic is expressible as $\chi(G) = -f(-1)$. It satisfies therefore $f_{A+B} = f_A f_B$. This implies $\chi(A+B) = \chi(A) + \chi(B) - \chi(A)\chi(B)$ because $\chi(G) = 1 - f_G(-1)$. We see that the genus $\gamma(G) = 1 - \chi(G)$ is multiplicative. □

10.2. This formula implies immediately for the complete graph $K_n = n = 1 + 1 + \cdots + 1$ that $\gamma(n) = 0$ and that in general for a contractible graph G, the genus is zero. In particular, the cone extension $G+1$ always has genus 0 because $\gamma(G+1) = \gamma(G)\gamma(1) = \gamma(G) \cdot 0 = 0$. The **suspension**, the join with a 0-sphere, $G \to G + S_0$ changes the sign of the genus $\gamma(G+S_0) = -\gamma(G)$. For example, for the n-sphere $S_{n-1} = nS_0 = S_0 + \cdots + S_0$ we have $\gamma(S_{n-1}) = (-1)^n$ like for the circle $S_1 = C_4 = S_0 + S_0$ where it is $\gamma(C_4) = 1$ or the octahedron $O = S_0 + S_0 + S_0$ where it is $\gamma(O) = -1$ as $\chi(O) = 2$ by Euler's formula $\chi(G) = f_0 - f_1 + f_2 = 6 - 12 + 8 = 2$.

10.3. If $G = (V, E)$ is a graph, let f be a locally injective function[3] and $S(x)$ the unit sphere of x in the Barycentric refined graph G_1. The later is the simplicial complex in which the vertices are the simplices which are either contained in x or which are simplices containing x. Let $S_f^-(x) = \{y \in S(x) | f(y) < f(x)\}$ and $S_f^+(x) = \{y \in S(x) | f(y) > f(x)\}$. All graphs $S(x), S_f^-(x)$ and $S_f^+(x)$ stand for their Whitney complexes.

Lemma 7 *(Hyperbolic structure). For the function* $f = \dim$ *on a simplicial complex* G*, we have* $S(x) = S_f^-(x) + S_f^+(x)$.

Proof. This is stated as Lemma (1) in [39]. Every simplex y in $S(x)$ is either a simplex in $S_f^-(x)$ or a simplex in $S_f^+(x)$. The function f satisfies $f(y^-) < f(y)$ if $y^- \subset y$ and $f(y^+) > f(y)$ if $y^+ \supset y$. □

10.4. For a general locally injective function f we don't have $S(x) = S_f^-(x) + S_f^+(x)$ as not all elements in the stable part $S_f^-(x)$ are necessarily connected to the unstable part.

[3] There should be no confusion with the f-vector notation.

10.5. **Example.** If $G = W_4$ is the wheel graph with 4 spikes and $f(x)$ is alternating 1 or -1 on the cyclic boundary S_4 and takes the value 0 on the central point c, then $S(c) = C_4$ and $S^+(c)$ is a 0-dimensional sphere and $S^+(c)$ an other 0-dimensional sphere. The sphere $S(c)$ is the sum of the two spheres.

10.6. **Example.** If $G = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,5\},\{3,6\},\{4,5\},\{1,3,6\},\{1,4,5\}\}$ and $x = \{1,3\}$. The set of vertices connected to x in G_1 are $\{\{1\},\{3\},\{1,3,6\}\}$. This is not a simplicial complex. We can however form the graph with these vertices and connecting two if one is contained in the other. This is what is called the sphere $S(x)$ of x in G_1. We have $S_f^-(x) = \{\{1\},\{3\}\}$ and $S_f^+(x) = \{\{1,3,6\}\}$. Now $1-\chi(S(x)) = 1-1 = 0$ and $1-\chi(S^-(x)) = 1-2 = -1$ and $1-\chi(S^+(x)) = 1-1 = 0$.

10.7. For any $x \in G$, define the **index** $i(x) = 1 - \chi(S(x))$. It is the genus of the unit sphere of x. For any locally injective function f, the index $i_f(x)$ of f and the index $i_{-f}(x)$ of $-f$ are linked by the following relation:

Corollary 3 *(Dual index).* $i_f(x)i_{-f}(x) = i(x)$.

Proof. We have $i_{-f}(x) = 1 - \chi(S_f^+(x))$ and $i_f(x) = 1 - \chi(S_f^-(x))$. Now use Lemma 7 and Lemma 6. □

10.8. It follows in the special case $f = \dim$ that

$$k(x) = \omega(x)(1 - \chi(S(x)))$$

is also a curvature satisfying a Gauss-Bonnet formula.

Corollary 4 *(Sphere Gauss-Bonnet).* $\sum_x k(x) = \chi(G)$.

Proof. The sum $\sum_x k(x)$ is equal to the sum $\sum_x \omega(x)(1 - \chi(S(x)))$ which is $\sum_x i_f(x)$, where $f(x) = -\dim(x)$. The result follows now from the discrete Poincaré-Hopf theorem. □

11. McKean-Singer formula

11.1. The **trace** of a matrix L on the finite dimensional Hilbert space of all functions $G \to \mathbb{R}$ is defined as $\mathrm{tr}(L) = \sum_x L(x,x)$. The **super trace** of L is defined as

$$\mathrm{str}(L) = \sum_x \omega(x)L(x,x)\ .$$

11.2. If g is the inverse of L, then its diagonal entry $g(x,x)$ is the genus $\gamma(S(x))$ of $S(x)$:

Lemma 8 *(Green's function formula).* $g(x,x) = \gamma(S(x)) = 1 - \chi(S(x))$.

Proof. By the Cramer formula, $g(x,x)$ is $\psi(G \setminus x)/\psi(G)$ but by the multiplicative Poincaré-Hopf theorem, this is $1 - \chi(S(x))$, where $S(x)$ is the unit sphere in G_1. □

11.3. The sum

$$V(x) = \sum_{y \in G} g(x,y)$$

is the **total potential energy** of x (note that the sum includes also $y = x$). We have now:

Corollary 5 *(Mc-Kean Singer).* $\mathrm{str}(L^{-1}) = \chi(G)$.

Proof. This is a reformulation of the sphere Gauss-Bonnet formula (4) which told

$$\chi(G) = \sum_x \omega(x)(1 - \chi(S(x))) = \sum_x k(x) ,$$

which used that $k(x) = i_f(x)$ for $f = -\mathrm{dim}$. □

11.4. This is the analog of the McKean-Singer formula [53]

$$\mathrm{str}(e^{-tH}) = \chi(G)$$

which holds for the Hodge Laplacian $H = (d + d^*)^2$ of G [30]. The proof of the discrete case of the Hodge Laplacian $H = (d + d^*)^2$ follows closely the continuum proof given in [11]. Here, for the connection Laplacian, we have $\mathrm{str}(L^k) = \chi(G)$ for $k = -1, 0, 1$, where the cases $k = 0, 1$ are both the definition of Euler characteristic. For $|k| \geq 2$, there is no such identity any more in general. This is related to the fact that L has a short range Green functions while the Green functions of H are singular as a non-invertible operator.

Corollary 6. *For any simplicial complex* $\sum_x \omega(x)\chi(S(x)) = 0$.

Proof. This follows directly from $\chi(G) = \sum_x \omega(x)(1 - \chi(S(x))) = \sum_x \omega(x)$. The entry $\omega(x)$ was the Poincaré-Hopf index of the function $f(x) = \mathrm{dim}(x)$ and $\omega(x)(1 - \chi(S(x)))$ was the Poincaré-Hopf index of the function $f(x) = -\mathrm{dim}(x)$. □

12. Proof of the energy theorem

12.1. Denote by $B(x) = B_{G'}(x)$ the **unit ball** of x in the connection graph G'. We can write the **connection vertex degree** $d(x) = d_{G'}(x)$ of a vertex x in terms of stable spheres $S^+(y) = \{z \in S(y) \mid z \subset y\}$ in the Barycentric refinement graph G_1 which is a sub-graph of G'.

Lemma 9 *(Unit Ball lemma).* $d(x) = \sum_{y \in B(x)} \chi(S^+(y))$.

Proof. From the multiplicative property of the genus under the joint operation $S^+(x) + S^-(x) = S(x)$, we have

$$\sum_{y \in B(x)} (1 - \chi(S^+(y))) = \sum_{y \in B(x)} \omega(y)(1 - \chi(S(y))) \; .$$

Use apply the Gauss-Bonnet theorem on the right hand side to see that the value is equal to $\chi(B(x)) = 1$. The left hand side is $\sum_{y \in B(x)} 1$ - $\sum_{y \in B(x)} \chi(S^+(y))$ which is

$$1 + d(x) - \sum_{y \in B(x)} \chi(S^+(y)) = 1 \; . \quad \square$$

12.2. **Example:** if $G = K_3$ is the triangle graph then G_1 is a wheel graph with 6 boundary points. The graph G' additionally connects all three edges and has three more vertices than G_1. If x is the central vertex of G_1, then $d(x) = 6$ and every $\chi(S^+(y))$ in $B(x)$ except x itself has $\chi(S^+(y)) = 1$. In a general complex G, if x is a facet (an element in G of maximal dimension) in G, then $\chi(S^+(x)) = 0, \chi(S^+(y)) = 1$ for all neighbors.

12.3. Theorem 2 suggests to lump all the potential energies $g(x, y)$ together and consider it to be a curvature. This indeed works. The **potential** $V(x) = \sum_y g(x, y)$ of the simplex x is the same as the sphere curvature:

Lemma 10 *(Potential is curvature).* $V(x) = \sum_y g(x, y) = \omega(x) g(x, x) = k(x)$

Proof. The claim

$$\sum_y g(x, y) = (-1)^{\dim(x)} g(x, x) = k(x)$$

can be restated in vector form as

$$g1 = k \; ,$$

where 1 is the constant vector $1(x) = 1$. Write $L = 1 + A$, where A is the adjacency matrix of the connection graph. As $g = L^{-1} = (1 + A)^{-1}$, this is equivalent to

$$Lk = (1 + A)k = 1 \; .$$

We prove this now. Because $k(x) = 1 - \chi(S^+(y))$ and A is the adjacency matrix of G', this means

$$Lk = (1 + A)(1 - \chi(S^+(y))) = 1 + d_{G'}(x) - \sum_{y \in B_{G'(x)}(x)} \chi(S^+(y)) \; .$$

Using Lemma 9, this is equal to 1. □

13. Proof of Theorem 2

13.1. The proof is inductive with respect to the number n of cells in G. It is again easier to prove the result in the more general class of CW-complexes. Let L be the connection matrix of G and K the connection matrix of $G+x$. Define $K(t)(y,z) = K(y,z)$ if z is different from x and $K(t)(y,x) = tK(y,x)$ if $y \neq x$ and similarly $K(t)(x,y) = tK(x,y)$ if $y \neq x$:

$$K(t) = \begin{bmatrix} L_{11} & L_{12} & . & . & . & L_{1n} & tL_{1x} \\ L_{21} & L_{22} & . & . & . & . & tL_{2x} \\ . & . & . & . & . & . & tL_{3x} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ L_{n1} & . & . & . & . & L_{nn} & tL_{nx} \\ tL_{x1} & tL_{x2} & \dots & \dots & \dots & tL_{xn} & 1 \end{bmatrix} .$$

Then $K(0) = L \oplus 1$ and $K(1) = K$. The matrix $K(0)$ has the eigenvalues of L and an additional eigenvalue 0.

13.2. Let $L[y,x]$ denote the sub-matrix of L in which row y and column x of L are deleted.

Lemma 11. *The minor* $\det(L[y,x])$ *is* $\omega(y)$ *if* y *is a subset of* x *and* x *is facet, a maximal element in* G.

Proof. The Green-Star formula (proven below) gives

$$g(x,y) = \omega(x)\omega(y)\chi(W^+(x) \cap W^+(y)) .$$

Now $\chi(W^+(x) \cap W^+(y)) = \chi(W^+(x))$ because of inclusion and this is equal to $\omega(x)$ due to maximality. So, $g(x,y) = \omega(x)\omega(y)\omega(x) = \omega(y)$. □

This immediately implies $\sum_{y \subset x} \det(L[y,x]) = \chi(S(x))$ if x is a facet, a set in G which is not contained in a larger subset.

13.3. The following proposition is a version of the multiplicative Poincaré-Hopf result Lemma 3. It will relate $\psi(G +_A x)$ with $\psi(G)$, where A is the sphere to which x has been attached.

Proposition 2. $\det(K(t)) = (1 - t^2\chi(A))\det(L)$.

Proof. A Laplace expansion with respect to the last column (which by definition belongs to a maximal cell) gives

$$\det(K(t)) = \det(L) - t^2 \sum_{y \subset x} \det(L)\omega(y) .$$

Now use the previous lemma $\det(L[y,x]) = \omega(y)$ to get

$$\sum_{y \subset x} \det(L)\omega(y) = \det(L)\chi(A) . \quad \square$$

13.4. Because we are in a CW-complex situation, in which A is a sphere with $\chi(A)$ being either 0 or 2, the lemma implies that in the later case, at $t = 1/\sqrt{2}$, the determinant is zero, with a single root meaning that a single eigenvalue crosses 0. In the former case $\chi(A) = 0$, the determinant $K(t)$ stays constant meaning that no eigenvalue can cross 0.

14. The strong ring

14.1. Theorem 1 and Theorem 2 can be generalized to a ring generated by simplicial complexes in which the disjoint union is the addition and where the product is the Cartesian product defined as a CW-complex. The notion of "Cartesian product" is pivotal for any geometry. The fact that the Cartesian product of simplicial complexes (as sets) is not a simplicial complex has as a consequence that the connected simplicial complexes are the **multiplicative primes** in a ring in which additive primes - the connected elements - have a unique prime factorization. (General elements can have different multiplicative prime factorizations as pointed out in [21,18]. It is related to the non-unique polynomial prime factorization: $(1+x+x^2)(1+x^3) = (1+x^2+x^4)(1+x)$ in $\mathbb{N}[x]$.) The product enjoys both spectral compatibility for the Hodge Laplacian H as well as connection Laplacian L. Both for the Hodge Laplacian as well as the connection Laplacian the spectrum adds. For the connection Laplacian, the spectrum multiplies.

14.2. With the disjoint union of simplicial complexes as addition, the set of simplicial complexes is an additive monoid for which the empty complex $O = \{\}$ is the zero element. One can extend it to a group which is then the additive group in a ring, the **Zykov ring**. The set theoretical **Cartesian product** $G = A \times B$ of two simplicial complexes is not a simplicial complex any more in general. It can however be given the structure of a discrete CW-complex and still defines two graphs G_1, the Barycentric refinement of G, as well as G', the connection graph of G. Both graphs have the sets in $A \times B$ as vertices. In $G_1 = A \times B$, two sets x, y are connected, if $x \subset y$ or $y \subset x$. In G', two different sets x, y are connected if $x \cap y$ is not-empty. The **refined Cartesian product** $(A \times_s B)_1$ is now a simplicial complex which satisfies all the properties of the continuum like the Künneth formula [33]. The refined Cartesian product multiplication is not associative however as

$(A \times K_1) \times K_1$ is the second Barycentric refinement of A while $A \times (K_1 \times K_1) = A \times K_1$ is the first Barycentric refinement.

14.3. The disjoint union and the Cartesian product are operations which define the **ring** $\mathcal{G}$ generated by simplicial complexes. The multiplicative unit is $1 = K_1$ and the empty complex $\emptyset$ is the 0**-element**. We call it the **strong ring** because the corresponding connection graphs get multiplied with the **strong Sabidussi multiplication** for graphs which is dual to the multiplication in the Zykov-Sabidussi ring. See [40,45]. The ring elements can also be represented by graphs, the Barycentric refinement graph G_1 in which two elements are connected if one is contained in the other or the **connection graph** G' in which two elements are connected if they intersect.

14.4. Every element $G = G_1 \times \cdots \times G_n$ of the ring has a connection Laplacian $L(G)$. Just define $L(G) = L(G_1 \times \cdots \times G_n)$ as a tensor product of $(L(G_1) \otimes L(G_2) \cdots \otimes L(G_n)$. This matrix can be seen as a connection matrix of the strong product of $G'_1 \boxtimes \cdots \boxtimes G'_n)$ of connection graphs of G_i. The **strong product** of two finite simple graphs $G = (V, E)$ and $H = (W, F)$ is the graph $G \boxtimes H = (V \times W, \{((a, b), (c, d)) \mid a = c, (b, d) \in F\} \cup \{((a, b), (c, d))\ b = d, (a, c) \in E\} \cup \{((a, b), (c, d))\ (a, c) \in E \text{and} (b, d) \in F\})$. It is an associative product introduced by Sabidussi [58]. See [21,18]. Added in proof: this is explored a bit more in [45]: the arithmetic ring of graphs with **Zykov join** $(V, E) + (W, F) = (V \cup W, E \cup F \cup \{(a, b), a \in V, b \in W\}$ as addition and **Sabidussi multiplication** $(V, E) \star (W, F) = (V \times W, \{(a \times b, c \times d),\ (a, c) \in E$ or $(b, d) \in F\}$ defines an associative ring going back to Zykov [69] and Sabidussi [58]. Under the dual operation of **graph complement**, the join becomes disjoint sum and $\boxtimes$ becomes $\star$.

Lemma 12. *$(G \times H)' = G' \boxtimes H'$.*

Proof. In both cases, we deal with a graph with vertex set $V \times W$, where V is the set of simplices in G and W the set of simplices in H. Two elements in $G \times H$ intersect if they either agree on one side or if they are connected on both sides. This is exactly what the Sabidussi multiplication does. □

14.5. The energy theorem extends from abstract simplicial complexes to elements in the strong ring $\mathcal{G}$. First of all, we have to see that for any $G \in \mathcal{G}$, the Laplacian L is unimodular $g = L^{-1}(x, y)$ and satisfies $\chi(G) = \sum_{x,y} g(x, y)$. Because the proof of the unimodularity theorem worked more generally for discrete CW-complexes, the proof goes over. Also for CW-complexes, every unit sphere can be decomposed into a stable and unstable part as a join. The Gauss-Bonnet (or Poincaré-Hopf) formula for the curvature $1-\chi(S^-(x))$ is the definition of Euler characteristic, for $K^+(x) = 1-\chi(S^+(x))$. The formula $g(x, x) = 1 - \chi(S(x))$ extends in the same way. Also the formula $V(x) = \sum_y g(x, y) = g(x, x)$ interpreting the diagonal entries of g as a potential energy of the simplex x generalizes and the energy theorem follows:

Corollary 7. *The energy functional is a ring homomorphism from $\mathcal{G}$ to $\mathbb{Z}$: $\chi(G+H) = \chi(G)+\chi(H)$ and $\chi(G \times H) = \chi(G)\chi(H)$.*

14.6. The Cartesian product of cell complexes produces a strong product for the connection graphs and a tensor product for the connection matrices:

Proposition 3. *If G and H are two simplicial complexes with connection matrix $L(G)$ and $L(H)$. Then $L(G \times H) = L(G) \otimes L(H)$.*

Proof. If $x_1, \dots x_n$ are the cells in G and $y_1, \dots, y_m$ are the cells in H, we have basis elements $e_1, \dots, e_n$ in the Hilbert space of G and $f_1, \dots f_m$ the basis elements in the Hilbert space of H. Now build the basis $e_1 \otimes f_1, \dots, e_1 \otimes f_m, e_2 \otimes f_1, \dots, e_2 \otimes f_m, f_m \otimes f_1, \dots, e_n \otimes f_m$ for the tensor product of the two Hilbert spaces. In that basis, the connection matrix is the tensor product of the $L(G), L(H)$. In the first row, we have the blocks $L(G)_{11}[L(H)], \dots, L(G)_{1n}[L(H)]$. □

14.7. As stated in Theorem 5, the strong ring so has a **representation** in a tensor algebra of matrices. Every element G in the ring is given a connection matrix L. The addition in the ring is the direct sum $\oplus$ of matrices. The multiplication in the ring produces the tensor product of matrices. (While the tensor product of matrices is not commutative, it is modulo coordinate changes where $L \otimes K$ is identified with $K \otimes L$.)

14.8. The upshot is that the Zykov-Sabidussi ring (with Zykov join as addition and Sabidussi multiplication) in its dual form relates to a ring of simplicial complexes (with disjoint union as addition and Cartesian product as multplication) and becomes on the connection graph level a ring of connection graphs (with disjoint union as addition and strong product as multiplication) and then is algebraically represented as a tensor ring of connection matrices (with direct sum of matrices as addition and tensor product as multiplication).[4] It is one ring which binds them all. The story indicates that the geometry of simplicial complexes is part of an extended arithmetic in which signed graphs play the role of the integers.

15. Green function entries

15.1. The **star** $W^+(x)$ of a simplex $x \in G$ is defined as $W^+(x) = \{y \in G \mid x \subset y\}$. It contains also x. We think of it as the **unstable manifold** passing through x. Unlike the **core** simplicial complex $W^-(x) = \{y \in G \mid y \subset x\}$, the star is not a simplicial complex in general. The **core** of x is the symplicial complex generated by the simplex x. We can write $L(x, y) = \chi(W^-(x) \cap W^-(y))$ and $\omega(x) = \chi(W^-(x) \cap W^+(x))$ as $W^- \cap W^+ = \{x\}$ is a set of sets with only one element x which again is not a simplicial complex in general.

[4] With the usual identification as per literally, both the direct sum of matrices and the tensor product of matrices are non-commutative when done in a particular basis.

15.2. The Green functions, the entries of the inverse matrix $g = L^{-1}$ have a similar formula. It generalizes the formula $g(x,y) = \omega(y)$ if y is a subset of x.

15.3. The proof of the Green Star formula uses a lemma which deals with **complete complexes** G which is the set of all nonempty subsets of a simplex u. We use the notation $y \sim z$ if two simplices y and z intersect.

Lemma 13. *If G is a complete complex with maximal simplex u and $x \in G$, then*

$$\sum_{z \in G, z \sim x} \omega(x)\omega(z)\omega(u) = \delta_{x,u} .$$

Proof. If $x = u$, then the statement re-reads $\sum_{z,z\sim x} \omega(x)\omega(z) = \omega(u)$ which is obvious from $\omega(x) = \omega(u)$ and $\sum_{z \in G} \omega(z) = 1$ for a complete complex.

If $x \neq u$, then the statement reads as $\sum_{z \sim x} \omega(z) = 0$. This follows from $\sum_z \omega(z) = 1$ and $\sum_{z \cap x = \emptyset} \omega(z) = 1$. □

15.4. This implies for example that $\sum_{x,z \in G, x \sim z} \omega(x)\omega(z)\omega(u) = 1$ which is a re-statement that the **Wu characteristic** of G defined by $\omega(G) = \sum_{x \sim y} \omega(x)\omega(y)$ agrees with $\omega(u)$, if u is the maximal simplex in G. For more on Wu characteristic, see [34,41].

Proof. The Green star formula is equivalent to the following statements about stars. From matrix multiplication $Lg = 1$ we have

$$\sum_z L(x,z)\omega(z)\chi(W^+(z) \cap W^+(y)) = 0$$

if $x \neq y$ and

$$\sum_z L(x,z)\omega(z)\chi(W^+(z) \cap W^+(x)) = \omega(x) .$$

We use the notation $z \sim x$ if $x \cap z$ intersect. The first relation means for $x \neq y$

$$\sum_{z \sim x} \omega(z)\chi(W^+(z) \cap W^+(y)) = 0$$

and the second means

$$\sum_{z \sim x} \omega(z)\chi(W^+(z) \cap W^+(x)) = \omega(x) .$$

We can rewrite the second statement as

$$\sum_{u, x \subset u} \sum_{z \subset u, z \sim x} \omega(z)\omega(x)\omega(u) = 1 .$$

By the above lemma, all terms with $x \neq u$ do produce 0 and the identity reduces to

$$\sum_{z \subset x} \omega(z) = 1 \; .$$

Similarly, the first statement is for $x \neq y$

$$\sum_{u, x \subset u} \sum_{z \subset u, z \sim y} \omega(z)\omega(x)\omega(u) = 0 \; .$$

This follows again from the above lemma, as $u = y$ is prevented from the assumption $y \neq x$ and the lemma implies that for $u \neq x$

$$\sum_{z \subset u, z \sim y} \omega(z) = 0 \; . \quad \square$$

15.5. We have made used of this already when showing that the diagonal entries of g are the genus of the unit spheres: $g(x,x) = \chi(W^+(x)) = 1 - \chi(S(x))$.

15.6. If we think of $f(x) = \dim(x)$ as a scalar function on the simplicial complex G, then the discrete Poincaré-Hopf formula shows that every point x is a critical point of f with index $\omega(x)$ and that f naturally can be seen as a **Morse function**. The simplicial complex $W^-(x)$ is in this picture the **stable manifold** of the gradient flow and the star $W^+(x)$ is the **unstable manifold**. The sets $W^+(x) \cap W^+(y)$ and $W^-(x) \cap W^-(y)$ are then the **heteroclinic connection points** and $W^+(x) \cap W^-(x)$ are the **homoclinic connection points**. We see that the matrix entries of L and g are all given by Euler characteristic values of homoclinic or heteroclinic connection sets. The picture indicates that every finite abstract simplicial complex defines a hyperbolic structure whose homoclinic and heteroclinic manifolds build the connections.

15.7. Since $\chi(W^-(x)) = 1$ for all x and $\omega(x) = \chi(W^0(x))$, an elegant symmetric description of the Green-Star formula is

$$L(x,y) = \chi(W^-(x))\chi(W^-(y))\chi(W^-(x) \cap W^-(y)) \; .$$
$$g(x,y) = \chi(W^0(x))\chi(W^0(y))\chi(W^+(x) \cap W^+(y)) \; .$$

16. Footnotes

16.1. The results in this paper have appeared in previous research notes. This document is an attempt for a self-contained publishable write-up. Theorem 1 was announced in the spring of 2016 and appeared first in [35]. An important final step happened in November 2016 with the multiplicative Poincaré-Hopf result. Theorem 2 is in [39], where the set of diagonal elements is outed as a combinatorial invariant as well as [36], where the functional $\mathrm{tr}(L-g)$ is discussed. The arithmetic is discussed in [38,40]. The unimodularity theorem Theorem 1 has been proven differently without the use of CW-complexes in [54].

16.2. Since this paper was submitted, generalizations have appeared. The Poincaré Hopf theorem was formulated for the f-function $f_G(t) = 1 + f_0 t + \cdots + f_d t^{d+1}$ encoding the f-vector of a graph G, then $f_G(t) = 1 + t\sum_{x \in V} f_{S_g(x)}(t)$ [46] where $S_g(x) = \{y \in S(x), g(y) < g(x)\}$ is part of the unit sphere $S(x)$. Gauss-Bonnet then is $f_G(t) = 1 + \sum_x F_{S(x)}(t)$, where F is the anti-derivative of f [36].

16.3. An other generalization appeared in [44]. If G is a simplicial complex and the value $\omega(x) = (-1)^{\dim(x)}$ is replaced by an arbitrary integer-valued function $h(x)$ and $\chi(G) = \sum_x h(x)$ replaces the Euler characteristic, then the connection matrix becomes $L = \chi(W^-(x) \cap W^-(y)$, where $W^-(x) = \{y \in G \mid y \subset x\}$. Define then $g = \omega(x)\omega(y)\chi(W^+(x) \cap W^+(y))$ where $W^+(x) = \{y \in G \mid x \subset y\}$. Then $\det(L) = \det(g) = \prod_{x \in G} h(x)$ and the energy theorem holds $\chi(G) = \sum_{x,y} g(x,y)$. Furthermore, the number of positive eigenvalues of L are the number of x with $h(x) > 0$. If $h(x) \in \{-1, 1\}$ for all $x \in G$, then L is unimoduliar and $L = g^{-1}$. Especially, if $h(x) = 1$ for all x, then L, g are positive definite quadratic forms which are isospectral. If $h(x) = -t^{|x|}$, then the energy theorem becomes $1 - f_G(t) = \sum_{x,y} g(x,y)$. This gives for $t = -1$ the energy theorem proven here.

16.4. Also added in proof we should remark that the energy theorem holds even if we attach to every simplex x a real or complex number $h(x) \in \mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ defining complex-valued energies $\chi(G) = \sum_x h(x)$. The matrices L and g are defined as before and are then complex valued. The unimodularity theorem and the energy theorem generalize to this "electromagnetic" situation: we have $\det(L) = \det(g) = \prod_{x \in G} h(x)$ and $\sum_{x,y} g(x,y) = \chi(G)$ hold for arbitrary $\mathbb{C}$ valued energies. If the values are in $\mathbb{T}$, then $\overline{g}L = 1$. Unexplored is whether this generalizes if $h(x)$ takes values in some non-commutative group like the quaternions. The matrix entries of L and g could be quaternion-valued or take values in the unit sphere $SU(2)$, an "electroweak" situation. There is hope because the Green star formulas for the matrix entries of L and g do not use the multiplicative structure. In a non-commutative case, the order of the cells is important in expressions like $\prod_{x \in G} h(x)$. The multiplicative Poincaré-Hopf still should work however as in the CW complex construction, the order of the build-up matters anyway.

16.5. The name "energy" suggests itself from the fact that for any Laplacian L, the entry $g(x,y) = V_x(y)$ is the **potential energy** at y if a mass has been placed at x so that the sum over all $g(x,y)$ can be seen as the total potential theoretical energy of G. For the Laplacian $L = -\Delta/(4\pi)$ on $\mathbb{R}^3$, the value $V_y(x) = g(x,y) = |x-y|^{-1}$ is the **Newton potential** of a mass point at y and $\iint |x-y|^{-1}\, d\mu(x) d\mu(y)$ is the total energy of a measure

μ. If μ is a **mass density** then this is the classical **total potential theoretic gravitational energy** of the mass configuration. In the case of a simplicial complex, we could look also at the energy $\mathrm{E}(\mu) = \sum_x \sum_y g(x,y) d\mu(x) d\mu(y)$ of a measure μ given by a measure vector $\mu(x)_{x \in G}$ with $\mu(x) \geq 0$. In our case, we look at the uniform measure assigning to every simplex the same weight 1. As mentioned before, [44] provides a generalized set-up.

16.6. The definition of a CW-complex could be generalized by using "homotopic to K_1" instead of contractible. But this would complicate proofs. Examples, where contractibility does not agree with "homotopic to K_1" is the **dunce hat** or the **Bing house**. It has a relation with **shellability**, the property that G is pure of some dimension d and that there is an ordering $x_1, \dots, x_n$ of maximal d-simplices such that the complex generated by $x_1, \dots x_k$ intersected with the complex generated by x_{k+1} is a $(d-1)$-dimensional shellable complex. An elegant generalization which avoids discrete homotopy is to build CW-complexes in which one replaces spheres by **Dehn-Sommerville complexes**, complexes for which the f-function $f_G(t) = 1 + f_0 t + f_1 t^2 + \cdots + f_d t^{d+1}$ has the property that $f(t - 1/2)$ is even for even d and odd for odd d. There is a class of Whitney complexes defined recursively as graph sets $\mathcal{X}_{-1} = \{0\}$, $\mathcal{X}_d = \{G \mid \chi(G) = 1 + (-1)^d$ and $S(x) \in \mathcal{X}_{d-1}\}$ which are all Dehn-Sommerville as a consequence of Gauss-Bonnet $f_G(t) = 1 + \sum_x F_{S(x)}(t)$, where $F_G(t)$ is the anti-derivative of $f_G(t)$. The energy theorem generalizes to this class of CW-complexes because the multiplicative Poincaré-Hopf theorem still works there.

16.7. The development of the notion of a manifold [60] is closely related to combinatorial structures built by Van Kampen, H. Weyl [66] (page 10) or J.H.C. Whitehead, who also happened to introduce CW-complexes. The story of CW-complexes starts with Ehresmann (1933) and is linked to algebraic developments [12] as well as Morse theory. The papers of Whitehead [67] from 1939 to 1941 established that CW-complexes have nice homotopy properties. The definition of discrete CW-complexes was done to prove unimodularity [35]. It depends on the combinatorial definition of what a "sphere" is and uses notions developed in discrete Morse theory [15] or digital topology [22] or [9]. A subclass of CW-complexes is the strong ring generated by simplicial complexes [40]. The strong ring is isomorphic to a subring of the Sabidussi ring [58] and can be seen as a subring of a general algebraic Stanley-Reisner ring construction [64]. The strong ring is a category of objects where Gauss-Bonnet, Brouwer-Lefshetz fixed point theory [31], Euler-Poincaré, arithmetic compatibility with the spectrum and that the energy theorem works and that energy is a ring homomorphism on the strong ring.

16.8. Abstract simplicial complexes and graphs have been close to each other already in the context of graph coloring and topological graph theory as in [17]. Many mathematicians, in particular Poincaré, Birkhoff or Whitney worked both in discrete settings as well as using continuum topology. Graphs are still less used in topology, maybe because *"the origins of graph theory are humble, even frivolous"* to quote [55].

An other reason is that graphs often are treated as **1-dimensional simplicial complexes** and are not equipped with more powerful simplicial complexes like the clique complex. Ivashchenko [22] translated Whitehead's homotopy notion into concrete procedures in graph theory. It has been simplified in [9] which is the version we use and which is crucial for defining "sphere" combinatorially.

16.9. The quest for combinatorial definitions and characterizations of spheres can be traced to [66]. One can define a "sphere" using Morse theory as classically spheres are spaces on which the minimal number of critical points of a Morse function is two. These things are mostly equivalent [48]. The Morse approach is used in Forman's discrete Morse theory [15] and based on the classical Reeb sphere theorem characterizing spheres as manifolds for which the minimal number of critical points is 2. An important property of spheres are Dehn-Sommerville relations which imply that the f-function $f(t) = 1 + f_0 t + \cdots + f_d t^{d+1}$ has the property that $f(x - 1/2)$ being either even or odd allowing to look at geometries where the Dehn-Sommerville spaces replace spheres. It is natural as both Dehn-Sommerville spaces as well as spheres are invariant under the join operation. See [43].

16.10. The first definition of abstract simplicial complexes was given in 1907 by Dehn and Heegaard (see [7]). First attempts to define discrete manifolds go back to Tietze in 1908. More work was done by Brouwer, Steinitz, Veblen, Weyl and Kneser. The first textbook in which abstract simplicial complexes were used heavily and stressed is [1], a book first published in 1947. Alexandroff calls a simplicial complex an "unrestricted skeleton complex" and an arbitrary set of finite sets is called there a **skeleton complex**. The notion of simplicial complex is still mostly used in Euclidean settings like in [19], but abstract simplicial complexes have entered modern topology textbooks like [62,52] of [50]. Its use was amplified with the emergence of **simplicial sets**, which generalize simplicial complexes. In [52], also the join of two simpicial complexes is defined. The analogue definition for graphs origins from Zykov [69]. Many examples of simplicial complexes in graphs are covered in [23]. It uses a notion of abstract simplicial complex in which empty sets (the "void") is present. (This is for example also used in [64]). It leads to **reduced f-vectors** $(f_{-1}, f_0, \dots)$ with $f_{-1} = 1$ and **reduced Euler characteristic** $\chi(G) - 1$ used in enumerative combinatorics [64].

16.11. Notions of **shellability** which are related to homotopy came only later. Bruggesser and Mani [6] in 1971 gave the first recursive combinatorial definition. They cite Rudin's paper [57] who calls a triangulation of a space shellable if there exists an ordering of the maximal simplices x_k such that during the build up, the complexes G_k generated by $\{x_1, \cdots, x_{k-1}\}$ are all homeomorphic to G. Rudin cites D.E. Sanderson [59], written in 1953 and published in 1957 where a d-manifold with boundary is called **shellable** if there is an ordered cellular decomposition into d-cells with disjoint interior such that X_k, the complex generated by x_k, intersected with the boundary G_k is homeomorphic to a $(d-1)$-cell. Also this definition makes uses of homeomorphism and so Euclidean embeddings. Sanderson actually works with what we today call the PL manifolds and shows that for any triangulation, there is as a shellable subdivision. The Sanderson paper cites a paper [5] of the thesis advisor R.H. Bing from 1951 who looks at a "partitioning" into mutually exclusive open sets in Euclidean space whose sum is dense. If G_k is an ordered sequence of these sets, then G_k intersected with X_{k+1} is a connected set. The Bing house can be thickened and triangulated to be unshellable even so the thickened house is a 3-ball.

16.12. Green's functions in general ubiquitous in mathematical physics [8,63,14]. For Green's functions appearing in discrete settings [61]. Discrete calculus on network has been developed and extended in various ways like also for weighted networks [3]. Whenever there is an operator H serving as a Laplacian, then $g(x,y)$ are the matrix entries of the inverse of H. The entries $g(x,x)$ usually do not exist, like for $-\Delta/(4\pi)$ on $\mathbb{R}^3$, where $g(x,y) = |x-y|^{-1}$ is the Newton potential. One looks therefore at the inverse $g_\lambda(x,y)$ of $H - \lambda$, where $\lambda \in \mathbb{C}$ is a parameter. Now, $\lambda \to g_\lambda(x,y)$ is analytic outside the spectrum of H. The diagonal Green's function $\lambda \to g_\lambda(x,x)$ is a Herglotz function if H was self-adjoint. While for real λ, this might not exist, one can look at limits like the **Krein Spectral shift** $\xi(x,\lambda) = \lim_{\epsilon\to 0} \arg(g_{\lambda+i\epsilon}(x,x))$ which exist for almost all λ [16]. If operator-valued random variables are used, the limit of the expectation of $g_{\lambda+i\epsilon}$ often exists almost everywhere and encodes the derivative of a Lyapunov exponent and its conjugate, the density of states. In one-dimensional Schrödinger setups, one has $g_\lambda(x,x) = 1/(m^+(x,\lambda) - m^-(x,\lambda))$, where $m^\pm$ are Oseledec spaces. The unboundedness of g is reflected in **non-uniform hyperbolicity** of the cocycle dynamics, for which stable and unstable directions can get arbitrarily close. In the current discrete setting, where for $\lambda = 0$ the operators stay bounded, we don't have technical difficulties and Green's functions given by topological "curvatures" $g(x,x)$ are bounded. We are in a **uniformly hyperbolic situation** in the terminology of smooth ergodic theory.

16.13. Classically, if we have a **hyperbolic fixed point** x of a transformation or a hyperbolic **equilibrium point** of a flow on a n-dimensional Riemannian manifold M, then a **theorem of Grobman-Hartman-Sternberg** assures the existence of stable and unstable manifolds $W^\pm(x)$ passing through x. If we take a small geodesic sphere $S(x)$ around x, then $S^\pm(x) = S(x) \cap W^\pm(x)$ are $(k-1)-$ and $(d-k-1)$-dimensional spheres and also classically $S(x) = S^+(x) + S^-(x)$, where $+$ is the join. We can see the decomposition of a sphere $S(x)$ into a join of a stable and unstable part as such a hyperbolic structure for the gradient flow of the dimension functional $f(x) = \dim(x)$ on G. Every $x \in G$ is a critical point. If we look at the Morse cohomology on the Barycentric refinement of this complex, we get the original simplicial cohomology. The fact that simplicial cohomology does not change under Barycentric refinement can be seen as a simple example where Morse and simplicial cohomology agree in a discrete setting. We must stress however that this equivalence holds in the discrete for any simplicial complex G. There is no discrete manifold structure required.

16.14. That the structure of hyperbolic dynamics enters in a fundamental level when looking at the dimension functional $f(x) = \dim(x)$ is a bit unique in the discrete and has no direct analog in the continuum as we can not access the lower dimensional parts of space so easily. While we can measure the highest dimensional parts in the form of volume, "counting" lower dimensional parts like "length" of a space needs integral geometric approaches. The theory of hyperbolic dynamics [49,68,24] had its origins also in topology as the work of Morse and Smale shows. Historically, already Poincaré, who saw first the importance of hyperbolic dynamics as an obstacle to integrability, was both a geometer and dynamical systems person, a tradition which was followed by others. In some sense, one can see a homoclinic and heteroclinic tangle in a simplicial complex coming from the gradient flow of the dimension Morse function $f(x) = \dim(x) = |x| - 1$ in a finite set of sets. In some sense, the simplicial complex assumption of being complete under the operating of taking finite non-empty subsets is a Morse-Smale condition. Without this assumption some results fail.

16.15. The total energy $\sum_{x,y} g(x, y)$ is the discrete analogue of classical energy in potential theory. In dimension $d > 2$, the **Newton potential** in $\mathbb{R}^d$ is $g(x, y) = |x - y|^{2-d}$. The prototype is the **scalar Laplacian** $L = -\Delta/(4\pi)$ in $\mathbb{R}^3$. The Gauss law $Lf = \rho$ reproduces via the divergence theorem the Newton law, the inverse square law of gravity or electro statics. Gauss noticed that $g(x, y) = 1/|x-y|$, explaining gravity. But the Gauss discovery allows also to define Newton's law on any space equipped with a Laplacian. The two dimensions, the Laplacian $-\Delta/(2\pi)$ on $\mathbb{R}^2$ gives the Green's function $g(x, y) = -\log|x - y|$ and $-\iint \log|x - y|\, d\mu(x)d\mu(y)$ is the logarithmic energy which can also be finite for singular measures like measures located on Julia sets. For discrete measures μ, one can disregard self-interaction and get the logarithm of a van der Monde determinant. The one-dimensional case $-\Delta/2 = d^2/2dx^2$ with $g(x, y) = |x - y|$ is used in statistics. Its energy $I(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y| d\mu(x)d\mu(y)$ is the **Gini index** or **Gini coefficient**. It plays a role in statistics, where μ is the law of a random variable. Discrete Green functions in graph theory were studied in [10] and appear naturally in any Markov setting. Let us mention that for the Hodge Laplacian $H = (d + d^*)^2$, whose inverse h on the ortho-complement of harmonic functions is defined (which is called the pseudo inverse), we have no interpretation of the total energy $\sum_{x,y} h(x, y)$ yet. There might be no natural one.

16.16. Discrete curvatures have first appeared in graph coloring contexts [4] but only in a two-dimensional setup. See [20]. The discrete Gauss-Bonnet theorem is different from Gauss-Bonnet theorems for polyhedra where excess angles matter (see e.g. [56]). An attempt to produce a discrete second order curvature worked in the simplest Hopf Umlaufsatz situation [27] but it required to formulate carefully what a region is.

After working on this more [28,26] and integral theoretic results [29,32] (where the later are discrete analogues of Banchoff theorems [2]), the situation is transparent: any deterministic or random diffusion process can be used to disperse the original curvatures $\omega(x)$ on simplices to the 0-dimensional part. If done in the most symmetric way to the nearest zero dimensional point, one gets the Levitt curvature $K(x)$ [51]. If done along the gradient of a function f one gets Poincaré-Hopf. An example was $f = -\text{dim}$, where the curvature is $k(x) = \omega(x)(1 - \chi(S(x)))$ we have seen here. If the diffusion is done on the right scale together with adapted Barycentric refinements, one expects to get the Euler curvature in a Riemannian manifold limit.

16.17. The energy theorem is also part of a story in arithmetic: there is a ring $\mathcal{Q}$ of finite simple graphs which with under the graph complement map is isomorphic to a ring $\overline{\mathcal{Q}}$. Now, if we look at the strong ring generated by simplicial complexes, the corresponding connection graphs define a sub-ring $\overline{\mathcal{Q}}$ on which Euler characteristic is by the energy theorem given as a sum of matrix inverse elements. The ring $\mathcal{Q}$ represents in a matrix tensor ring as every element in $\mathcal{Q}$ defines a matrix, the connection Laplacian of its graph complement. We can so attach a spectrum $\sigma(G)$ of a ring element in $\mathcal{Q}$. In addition, the spectra add, under multiplication, the spectra multiply. The same happens with Euler characteristic. The upshot is that every generalized rational number, an element x in $\mathcal{Q}$ defines an object to which we can attach an Euler characteristic $\chi(\overline{x})$ which is a total potential energy of the complex $\overline{x}$ and also attach a matrix $L(x)$ whose spectrum is compatible with the ring operation.

16.18. There are many open questions. We would like to know more about energy moments like $\sum_{x,y} g(x,y)^3$ for example. It appears often to be an integer multiple of the energy $\sum_{x,y} g(x,y)$. While we know $g(x,x) = 1 - \chi(S(x))$ and that the set of these diagonal entries are a combinatorial invariant in the sense that the set is stable already after one Barycentric refinement of a complex we don't have an interpretation for k-moments $\sum_{x,y} g(x,y)^k$ of the energy yet. Especially interesting looks the variance $\sum_{x,y}(g(x,y) - m)^2/n^2$, where $m = \chi(G)/n^2$ is the expectation of the matrix entries of g. It is a measure for the **energy fluctuation** in the complex. Having developed parametrized versions of Gauss-Bonnet [43] and Poincaré-Hopf [46] more recently, there is also a parametrized version of the energy theorem in which the Euler characteristic $f_0 - f_1 + f_2 - \dots$ is replaced with f-functions $f(t) = 1 - f_0 t + f_1 t^2 - \cdots$. The energy theorem then tells that the sum of the matrix entries of the inverse g is the f function of G.

16.19. An inverse spectral problem is to hear the Euler characteristic of G from the spectrum $\sigma(H)$ of the Hodge Laplacian $H = (d+d^*)^2 = dd^* + d^*d$, where $d : \Lambda^k \to \Lambda^{k+1}$ is the exterior derivative, and where Λ^k is the f_k-dimensional space of discrete k-forms, functions on k-dimensional oriented simplices of G. The matrices H and L are both $n \times n$ matrices if G has n elements. If G is one-dimensional, then $L - L^{-1}$ is similar to H [42].

This even holds for products of one-dimensional spaces. One can even hear the Betti numbers of a Barycentric refined G from H or L as b_0 is the number of eigenvalues 1 and b_1 is the number of eigenvalues -1 of L. In dimension 2 or higher, one can not hear the Betti numbers b_k of G from the spectrum $\sigma(L)$ of L in general but it is conceivable that this could be true for Barycentric refinements.

Declaration of competing interest

There is no competing interest.

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