Basics on Dirichlet Series 08/2008, Oliver Knill

The finite sum $S(x,y) = \sum_{k=x}^{y} a_k$ is a discrete version of a **definite integral**, $\Delta a_n = a_{n+1} - a_n$ the discrete version of **the derivative**. A discrete version of the **partial integration** formula $\int_x^y f(x)g(x) = F(x,y)g(y) - \int_x^y F(x,t)g'(t)\,dt$ with $F(x,t) = \int_x^t f(s)\,ds$ is:

Lemma: (Abel's summation formula)

$$\sum_{k=x}^{y} a_k b_k = S(x,y)b(y) - \sum_{k=x}^{y-1} S(x,k)\Delta b(k) .$$

Proof. The statement is true for $x = y$. Induction with respect to y allows to compare the $L(y)$ with the right hand side $R(y)$: the equation $L(y+1) = R(y+1)$ reads

$$L(y)+a(y+1)b(y+1) = R(y)-S(x,y)b(y)+S(x,y+1)b(y+1)-S(x,y)\Delta b(y) = R(y)+a(y+1)b(y+1) .$$

□

Lemma: $|\Delta e^{-\lambda_n s}| \leq \frac{|s|}{\sigma}\Delta e^{-\lambda_n \sigma}$, if $s = \sigma + i\tau$.

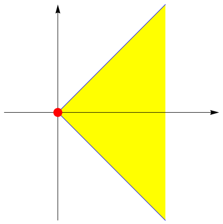
Proof.

$$|\Delta e^{-\lambda_n s}| \leq \int_{\lambda_n}^{\lambda_{n+1}} s e^{-us}\,du \leq |s| \int_{\lambda_n}^{\lambda_{n+1}} e^{-us}\,du = \frac{|\sigma|}{\sigma}\Delta e^{-\lambda_n \sigma} .$$

□

Theorem: (Jensen-Cahen) If the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ is convergent for s_0, then it is convergent for s in any cone $|\arg(s - s_0)| \leq \alpha < \pi/2$.

Proof.



We can assume $s_0 = 0$ without loss of generality. For s is in the cone, we have $|s|/\sigma \leq 1/\sin(\alpha)$. Abel's summation formula allows to write the sum as

$$\sum_{k=m}^{n} S(m,k)\Delta e^{-\lambda_k s} - S(m,n)e^{-\lambda_n s} .$$

By assumption, given $\epsilon > 0$, there exists m_0 so that for $k \geq m \geq m_0$ $|S(m,k)| < \epsilon/\sin(\alpha)$. The sum is therefore smaller or equal to

$$\epsilon(\sum_{k=m}^{n-1} \Delta e^{-\lambda_k \sigma} + e^{-\lambda_n \sigma}) = \epsilon e^{-\lambda_m \sigma} < \epsilon .$$

□

It follows that for every region G in the cone, the series is uniformly convergent as well as any of its derivatives $f^{(n)}(s) = (-1)^n \sum_{k=1}^{\infty} a_k \lambda_k^n e^{-\lambda_k s}$.

Theorem: In every cone intersected with $\mathrm{Re}(s) \geq \sigma > \sigma_0$, there are only finitely many roots of the Dirichlet series, unless the function is identically zero.

Proof. We show that there can not be any accumulation points of roots in any such intersection E. Assume there were roots $s_n = \sigma_n + i\tau_n$ with $\sigma_n \to \infty$. Then the function $g(s) = e^{\lambda_1 s} f(s) = a_1 + \sum_2^{\infty} a_n e^{-(\lambda_n - \lambda_1)s}$ is uniformly convergent in E and $g(s) \to a_1$ for $s \to \infty$ uniformly along any path. Because $g(s_n) = 0$, we must have $a_1 = 0$. We can now continue in the same way to get $a_k = 0$ for any integer $k > 1$. □

Assume a Dirichlet series is not convergent for $s = 0$. In other words, the series $S(k) = \sum_{n=1}^{k} a_n$ does not converge. The following formula generalizes the formula for the radius of convergence $\limsup_{k\to\infty} |S(k)|^{1/k} = 1/r_0$ for Taylor series, where $\lambda_k = k$ and where the radius of convergence r_0 is related with the abscissa of convergence σ_0 by $e^{-\sigma_0} = r_0$.

Theorem: (Cahen's formula) Assume the series $S(k)$ does not converge, then the abscissa of convergence of the Dirichlet series is

$$\sigma_0 = \limsup_{k\to\infty} \frac{\log|S(k)|}{\lambda_k} .$$

Proof. Because the sequence $S(k)$ does not converge and especially not converge to 0, there is a constant C and infinitely many k for which $\log|S(k)| > -C$. Therefore, $\sigma_0 \geq 0$.
(i) Given $s = \sigma_0 + \delta > \sigma_0$ show that the series converges. Given $0 < \epsilon < \delta$ $\log|S(k)| \leq (\sigma + \delta - \epsilon)\lambda_k$ or $|S(k)| \leq e^{(\sigma+\delta-\epsilon)\lambda_k}$ for large enough k. Now use Abel's formula to show that the sum converges.
(ii) Assume $\sum_k a_k e^{-\lambda_k s} = \sum_k b_k$ converges. Now write with Abel

$$S_k = \sum_{n=1}^{k} b_n e^{\lambda_n s} = \sum_{n=1}^{k} B(n)\Delta e^{\lambda_n s} - B(k)e^{\lambda_k s}$$

showing that there exists C for which $|S(k)| \leq Ce^{\lambda_k s}$. □

Cahen's formula links the growth of the random walk $S(k) = \sum_{n=1}^{k} a_k$ with the convergence properties of the zeta function $\zeta(s)$.

Source: [1].

References

[1] G.H. Hardy and M. Riesz. *The general theory of Dirichlet's series.* Hafner Publishing Company, 1972.