

THE STRANGE ATTRACTOR THEORY OF TURBULENCE

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INTRODUCTION

It is a fact of experience almost too familiar to notice that dissipative physical systems subject to weak steady driving approach states of dynamic equilibrium that are independent of initial condition. As the strength of the driving is increased, these systems typically undergo a sequence of transitions—the details depending on the system—and arrive eventually at behavior that may be described as chaotic or turbulent. The turbulent motion is not entirely without regularity, but the regularity is statistical in character and appears only when long-term time averages are examined.

Ideally, the mechanisms producing the transition from steady to chaotic behavior, and the detailed nature of the motion in the chaotic regime, should be deducible directly from the equations of motion for the system in question, i.e. the Navier-Stokes or Boussinesq equations in the case of classical kinds of fluid systems. Direct attacks on these equations, however, meet with overwhelming difficulties. On the one hand, control over the analytic properties of the equations is not yet good enough either to prove or to disprove the existence of regular solutions for all times and arbitrary regular initial data. On the other hand, it seems quite hopeless to try to compute explicit analytic solutions with chaotic behavior, to say nothing of computing, from first principles, the statistical distribution describing the behavior of typical solutions. To circumvent the difficulties of a direct approach, a number of oblique lines of attack have been developed. One of these approaches, known as the *strange attractor* theory of turbulence, is the subject of this review.

This approach focuses on the time dependence of turbulent motion; the fundamental idea on which it is based is:

Turbulent time dependence is not an exceptional feature of particular equations of motion but a property shared by a broad class of typical differential equations.

Adopting this point of view changes the perspective from one of studying particular—and intractable—equations to trying to answer the question:

How does a typical solution of a typical differential equation behave over the long run?

A substantial body of deep mathematical theory is available to be applied to this question, and mathematical work in this area in recent years has been both invigorated and focused by interaction with the physical study of the chaotic behavior of dissipative systems.

The approach has at least two obvious drawbacks. One is that there is no guarantee that the Navier-Stokes equation will indeed turn out to be typical. This objection is not as serious as it might appear. The Navier-Stokes equation is after all only an approximation, albeit a very good one for most purposes. Even if it were to turn out to have nontypical properties, the very notion of “typical” means that most small perturbations on it would produce an equation with typical behavior. Furthermore, the discovery of a nontrivial exceptional qualitative property of the Navier-Stokes equation would be a great step towards understanding that equation, so the program can further our understanding even if its fundamental pre-supposition ultimately turns out to be wrong.

The second drawback is that, at best, the investigation of typical behavior can furnish only a list of alternatives. Which of the alternatives actually occurs for a particular equation can only be determined by a detailed study of that equation (or by performing an experiment, either a computer experiment on the equation or an actual experiment on the physical system it describes.)

Up to now, at least, this approach has not contributed very much to the solution of the traditional questions about turbulence or to the practical computation of critical parameter values, phenomenological parameters like effective turbulent viscosity, etc. Its successes have come more in suggesting new questions to be investigated experimentally than in explaining the results of prior experiments. Although the mathematical theory has developed some very powerful methods of analysis, it has generally not been possible to sum up the principal insights in a few concise theorems that can be applied without regard to the reasoning behind them. In short, it is a better source of tools than of recipes.

Terminology

We will be discussing differential equations. By a *state* for a differential equation, we mean a complete specification of initial condition; the space of all states will be called the *state space*. Thus, for a Hamiltonian system, the state space means the phase space rather than the configuration space. For an incompressible fluid system, a point of the state space is a velocity

field, defined on the physical region occupied by the fluid, with vanishing divergence and satisfying appropriate boundary conditions. We will use the term *orbit* to refer to a solution to the differential equation regarded as a curve in the state space, and call *solution flow* the motion on the state space that advances each point along its respective orbit. We will say that a stationary or periodic orbit is *stable* or *attracting* if all orbits starting sufficiently near to it converge to it; this property is frequently called *asymptotic stability in the sense of Lyapunov*.

The terms *turbulent*, *chaotic*, and *stochastic* (applied to describe time dependence of solutions to a differential equation) will be used interchangeably. Note, however, a slight and potentially question-begging difference in connotation; *stochastic*, as normally used, implies the existence of a well-defined average behavior.

Some General References

The idea that chaotic time dependence of turbulent fluid flows might be understood as a property of fairly general differential equations was first advanced in a way that attracted widespread attention in Ruelle & Takens (1971a,b). [A very suggestive example had been pointed out earlier by Lorenz (1963), but Ruelle & Takens were not aware of Lorenz's work.] The paper of McLaughlin & Martin (1975) was very influential in popularizing these ideas. Recent general surveys include Ruelle (1978a,b, 1980a,b), Lanford (1981), and Eckmann (1981).

CHAOTIC BEHAVIOR

It is often felt that there is something paradoxical about having solutions to a deterministic equation behave in a chaotic or stochastic fashion. There is, however, no real paradox; the solutions are, in fact, uniquely determined by the initial conditions, but the effects of small changes in the initial conditions are so amplified by the equations of motion that any *finite-precision* information about the initial conditions provides no *finite-precision* information about the state of the system at much later times. In other words:

An important element in the explanation of the chaotic behavior of solutions of deterministic equations of motion is the sensitive dependence of solutions on initial conditions.

The use of probabilistic concepts in the analysis of deterministic motion should, in any case, be familiar from classical equilibrium statistical mechanics. In fact, sensitive dependence on initial condition, in the form of the notion of ergodicity, has long played a central role in one of the standard justifications for the foundations of that subject (see, for example,

Lebowitz & Penrose 1973). It needs to be noted, however, that there are substantial differences between Hamiltonian systems—to which the usual formalism of classical statistical mechanics applies—and the sort of dissipative systems under discussion here. The key distinction lies in the volume-preserving character of the solution flow for Hamiltonian systems (Liouville's Theorem) which has as a consequence the fact that the solution flow is *recurrent* (meaning, roughly, that almost all orbits come back arbitrarily near their initial points infinitely often). For dissipative systems, on the other hand, what usually happens is that most of the points of the instantaneous state space are *transient* in the sense that the orbits that start there eventually go to and stay in another part of the state space. A simple instance is provided by the stable dynamic equilibrium that is usually set up when a dissipative system is driven gently. In this situation, all orbits, no matter where in the state space they start, converge eventually to a single stationary solution corresponding to laminar motion. In a certain sense, the system has *no* effective degrees of freedom, although the state space may have large or even infinite dimension. It seems very likely that something similar happens for more strongly driven dissipative systems, even those whose motion is chaotic, viz., that

There are one—or possibly a few—invariant sets of relatively low dimension in the state space to which almost all orbits converge.

These sets are what are called *attractors*. One of the great surprises in this subject is that attractors, except for the very simplest ones, are typically not smooth surfaces in the state space but rather more complicated kinds of sets.

Before taking up the notion of attractor in more detail, we need to elaborate on what is meant in practice by chaotic behavior of a physical system whose equation does not depend explicitly on time. Roughly, the idea is that the system behaves, over a long period of time, in a repetitive but not strictly periodic fashion. It should be emphasized that, in this article (and in the field it surveys), the analysis is focused on understanding temporal—not spatial—chaos. There is a tendency to identify chaotic motion with spatially complicated flow patterns. This identification is not entirely mistaken, since laminar flow generally has simple geometry, but it may be misleading. It is possible to have either

a fluid flow with extremely complicated intrinsic spatial structure and no time dependence at all (in, for example, large-aspect-ratio convection)

or

chaotic motion with relatively simple spatial structure (small-aspect-ratio convection).

It is not even true that *apparent* temporal complexity necessarily indicates chaotic behavior. Convective systems, for example, can undergo quite complicated periodic motion before they become aperiodic. When they do become stochastic, the motion can often be decomposed, at least roughly, into a small stochastic component superimposed on a much larger periodic component. It is often not easy to distinguish between such weakly stochastic motion and complicated but purely periodic motion simply by watching the system. The standard way to detect a stochastic component in the motion is to measure the *power spectrum* of some dynamical variable. Stripped of technicalities, this means the following: Measure, at equally spaced times t_i some numerical quantity such as one component of the velocity at a particular point of the fluid. Call the measured quantities X_1, \dots, X_N . Subtract the mean and take the discrete Fourier transform, i.e. form

$$\tilde{X}(\omega) = \frac{1}{\sqrt{N}} \sum_{j=1}^N (X_j - \bar{X}) e^{i\omega j}, \quad (1)$$

where ω is of the form $2\pi k/N$ and where

$$\bar{X} = \frac{1}{N} \sum_{j=1}^N X_j. \quad (2)$$

Then see whether $|\tilde{X}(\omega)|^2$ is concentrated in a series of sharp peaks. The appearance of a “broad-band” component in $|\tilde{X}(\omega)|^2$ (beyond that due to the finite precision of the measurements) is generally taken as the operational definition of stochastic behavior of the system in question.

ATTRACTORS

One of the most fruitful ways of studying the mathematical structures underlying observed stochastic behavior of physical systems has been the careful study, mostly with the aid of computers, of simplified models, and one model that has been particularly informative is the *Lorenz system*, a set of three coupled differential equations:

$$\begin{aligned} \frac{dx}{dt} &= -\sigma x + \sigma y; & \frac{dy}{dt} &= rx - y - xz; \\ \frac{dz}{dt} &= -bz + xy, \end{aligned} \quad (3)$$

where b, σ , and r are constants. It is hard to imagine a much simpler system that is neither linear nor two-dimensional, but the solutions to these equations nevertheless do very complicated things. E. N. Lorenz (1963) discovered numerically a striking mathematical structure which has

come to be known as the *Lorenz attractor* and which occurs for these equations with

$$b = 8/3 ; \quad \sigma = 10 ; \quad r = 28 . \quad (4)$$

The exact parameter values are not crucial, but the behavior of typical solutions definitely does depend on the parameters and is quite different in other regions of parameter space. It should also be noted that, in spite of overwhelming numerical evidence, there is to my knowledge no complete proof that the structure about to be described actually does occur for these specific equations. It is not hard to see that it does occur for *some* equations.

The phenomenology is as follows: The equations admit three stationary solutions, one at the origin and the other two (which we will denote by C_{\pm} and refer to as *centers*) at $x=y=\pm\sqrt{b(r-1)}$; $z=r-1$. All three stationary solutions are unstable. Orbits that start near the origin escape monotonically; those that start near the centers escape through growing oscillations. If a solution is computed starting from some more or less randomly chosen initial point, what is found without exception is that the orbit will, after an initial transient regime of variable length, settle down to a motion in which, most of the time, it can be thought of as performing oscillations about one of the centers. The oscillation grows in amplitude; when it reaches a critical size, the orbit abruptly makes a transition to oscillation about the other center. This oscillation again grows and the orbit eventually makes a transition back to oscillating about the first center, and so on. A representative orbit is shown in Figure 1.

The amplitude of oscillation immediately after transition varies from transition to transition, and it in turn determines the number of oscillations before the next transition. The sequence of numbers of oscillations between transitions appears random, and the power spectra of the coordinates are continuous (see Figure 2). Thus, the motion both appears chaotic and satisfies the standard operational test for chaotic behavior.

The mathematical object responsible for this behavior is sketched schematically in Figure 3. This sketch represents a family of orbits in the three-dimensional state space for the Lorenz system. It is not even approximately to scale; proportions have been distorted in the hope of making the mathematical structure more transparent. To a first approximation, the structure looks like two reasonably flat loops of ribbon, one lying above the other along a central band, and the two glued together at the bottom of that band. The motion flows around the loops, clockwise on the left and counter-clockwise on the right. Going once around the right-hand loop constitutes a single oscillation around C_{+} . Orbits beginning either above or below the ribbon are attracted quickly down to its immediate vicinity and then follow the flow on it. The double-loop structure is strictly invar-

invariant under the solution flow; any point on it has an orbit that can be traced both forward and backward for all time without leaving it.

The central band is divided in half by orbits that flow essentially straight down to the stationary solution at the origin; these orbits are exceptions to the pattern of growing oscillations followed by transitions displayed by typical orbits. Orbits to the left of this boundary will make their next oscillation around C_- ; those to the right will go next around C_+ . The fact that oscillations around the centers are growing in amplitude means that, for example, a loop around C_+ brings the orbit back to the left of where it started out. A transition from oscillation around C_+ to oscillation around C_- occurs when an orbit making a loop around C_+ comes back to the left of the dividing boundary.

The central band divides in half laterally at the bottom of this boundary, and each half, after having made a loop around the appropriate center, has become wide enough to cover almost the entire top of the band. Thus,

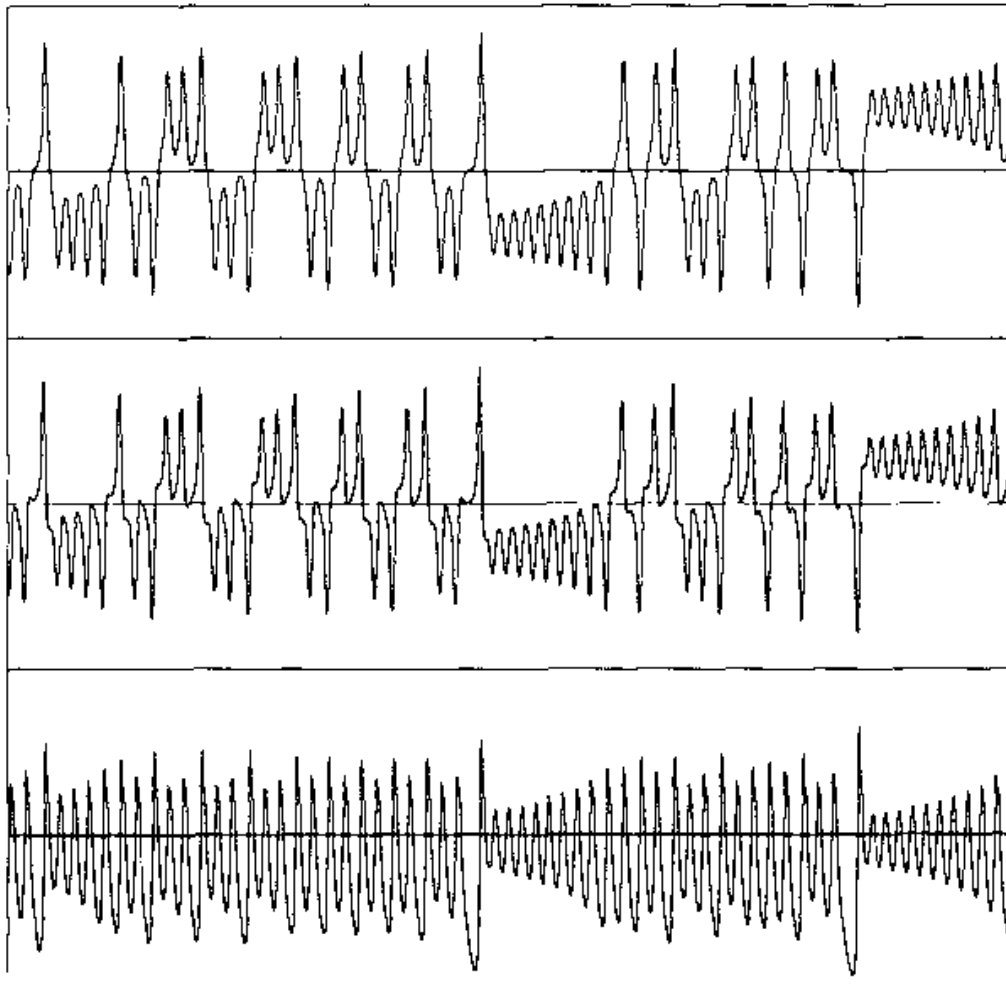


Figure 1 A representative orbit for the Lorenz system. From top to bottom: x , y , and z plotted against t .

orbits are pulled apart laterally as they flow around the loops, and this accounts for the observed sensitive dependence on initial conditions.

A typical orbit on this structure wanders over the surface, coming arbitrarily close to each point infinitely often. There are, however, a great many nontypical orbits. We have already mentioned the orbits in the middle of the central band which simply converge down toward the origin. There are also many periodic orbits, all unstable, as well as orbits with more subtle kinds of atypical behavior.

We next take a closer look at the ribbons and argue that they cannot be simple surfaces but must rather have infinitely many layers. Start at the top of the central band where there are two approximately parallel ribbons, one on top of the other. As we have drawn the picture, the upper ribbon is made up of orbits returning to the central band after a loop around C_+ ; the lower, around C_- . As the orbits flow down the central band, the two ribbons are drawn together. At the bottom, they form a two-sheeted surface which proceeds to split laterally in two with half going left around C_- and half right around C_+ . Thus, the ribbon of orbits going around C_+ or C_- has at least two sheets, the upper one made up of orbits whose previous circuit was around C_+ , the lower of orbits whose previous circuit was around C_- . These sheets are carried closer together by the flow but the separation remains nonzero, so the upper ribbon at the top of the central band actually has two layers rather than just one. The same is true for the lower ribbon, and therefore the structure at the bottom of

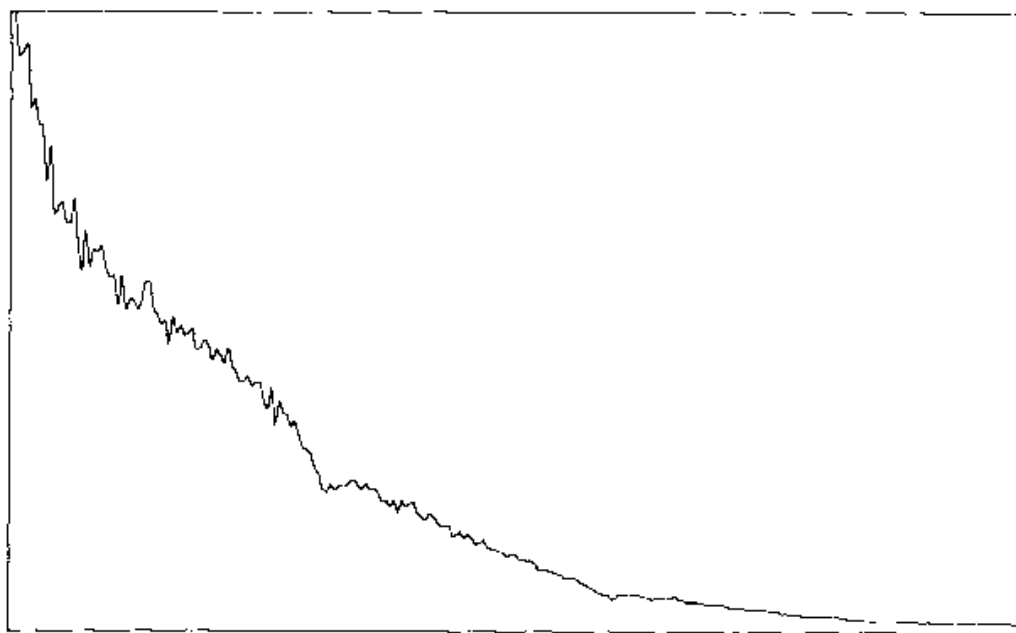


Figure 2 The power spectrum of the x coordinate of the Lorenz system (on a linear scale). The frequency ranges from 0 to 5; the oscillations apparent in Figure 1 have a frequency of about 1.5.

the central band actually has four layers rather than just two. Thus, the ribbons going around C_+ and C_- are actually four-sheeted, and so on. Continuing to argue in this way we see that all the ribbons must have infinitely many sheets.

This object is an instance of what has come to be called a *strange attractor*. The formulation of the definitive definition of the term *attractor* must await a more complete understanding of what the possible phenomena are. The general idea is clear, however: an attractor is a set that attracts nearby orbits (i.e. an orbit that starts near the attractor stays near it and converges to it as time goes to infinity). It should also be required that the set be closed and invariant under the solution flow (i.e. be made up of complete orbits defined for all time) and that the solution flow on the set be *recurrent*, i.e. that most orbits return infinitely often to the vicinity of their starting points. A situation which occurs frequently and which suffices to guarantee that the motion is adequately recurrent is that there is a single orbit in the attractor passing arbitrarily near to every point.

Stable stationary solution and limit cycles are elementary and trivial examples of attractors. It has turned out that the other kinds of attractors that occur most frequently are structures with infinitely many layers like the Lorenz attractor described above. Hence the epithet “strange.” The general investigation of attractors has barely begun. Obtaining a complete classification looks, at present, extremely remote. On a less ambitious and more pragmatic level, there does not exist a convincing list of the “simplest” possibilities. Even worse, there are very simple examples—notably, the *Hénon* attractor (Hénon 1976, Hénon & Pomeau 1976, Curry 1979)—whose properties really aren’t understood at all. There does exist, however,

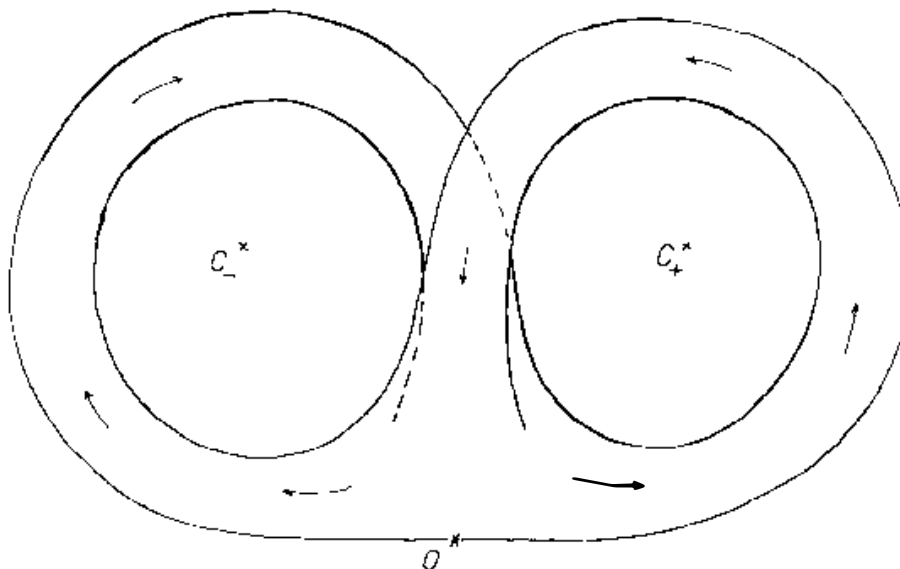


Figure 3 A schematic view of the Lorenz attractor.

one class of attractors for which there is a detailed and deep mathematical theory. These are the attractors satisfying a condition introduced by S. Smale (1967) and known as *Axiom A*. Roughly, Axiom A requires that orbits on or very near to the attractor have a strong and technically convenient form of sensitive dependence on initial condition, for both positive and negative time; it also requires that arbitrarily close to each point of the attractor there passes a periodic orbit. We will return in the final section of this review to one of the most important properties of attractors satisfying Axiom A.

For a more detailed discussion of the Lorenz attractor see Lorenz (1963), Guckenheimer (1976), Ruelle (1976a), Lanford (1978), Guckenheimer & Williams (1979), and Williams (1979). The literature on attractors satisfying Smale's Axiom A is extensive. For an introductory survey, see Bowen (1978).

SCENARIOS

In addition to asking how typical solutions of a typical differential equation behave, it is also possible to ask how this behavior varies as a parameter in the differential equation changes. Of particular interest is the study of the transition process from an equation whose solutions are asymptotically regular (i.e. stationary or periodic) to one with some kind of strange attractor. One can hope that there will turn out to be comparatively few such transition processes, each with distinctive characteristics permitting it to be identified without a detailed analysis of the underlying equations. J.-P. Eckmann has introduced the term *scenario* for such typical transition processes.

The Hopf bifurcation is an excellent classical example of a successful application of this approach. Let us review briefly how it works. The idea is to see what happens when, as a parameter is changed, a stationary solution to a differential equation loses stability. We consider, then, a differential equation depending on a parameter μ and a stationary solution for that equation which may also depend on μ . The particular situation we want to examine is that

for $\mu < \mu_c$, all the eigenvalues of the linearization of the equation at the stationary solution have strictly negative parts,

but

at $\mu = \mu_c$, a single complex-conjugate pair of simple nonreal eigenvalues crosses into the right half-plane with nonzero speed.

The Hopf Bifurcation Theorem says that the qualitative nature of the motion near the stationary solution is determined by the sign of a certain nonlinear function d of the first, second, and third derivatives of the dif-

ferential equation with respect to the state variables at the stationary solution for μ_c . (The explicit formula for d is extremely complicated but need not concern us here.) For $d > 0$, what happens is that, for μ slightly larger than μ_c , the equation has an attracting periodic orbit—i.e. a stable nonlinear oscillation—in the neighborhood of the now unstable stationary solution. As μ decreases to μ_c , the oscillation collapses down to the stationary solution; its amplitude is asymptotically proportional to $\sqrt{\mu - \mu_c}$. For $d < 0$, something equally specific (if less interesting) happens: for μ slightly less than μ_c , there is a nonattracting periodic orbit which, as μ increases to μ_c , collapses down to the stationary solution with amplitude asymptotically proportional to $\sqrt{\mu_c - \mu}$. One of the great strengths of the theorem is that it assures us that, under our assumptions about the eigenvalues of the linearization, these are the *only* two possibilities except in the degenerate case $d = 0$. (For a detailed discussion of the Hopf bifurcation, and proofs of the results cited above, see Marsden & McCracken 1976.)

The most interesting applications of the Hopf Bifurcation Theorem to physical systems do *not* proceed by verifying that the equations of motion satisfy the hypotheses. What is done, rather, is to observe experimentally the way the system in question goes from a stationary to an oscillatory regime as the parameter passes through a critical value. If the amplitude of the oscillations grows like $\sqrt{\mu - \mu_c}$ as μ passes μ_c , it is fairly safe to conclude that the transition process is a Hopf bifurcation, the square-root behavior serving as an experimentally verifiable signature. In this way, it is possible to arrive at a fairly precise picture of the transition from steady to oscillatory behavior even in situations where it is impossible to compute the stationary solution accurately, either analytically or numerically.

The Hopf bifurcation is thus an extremely successful scenario for the comparatively elementary transition from steady to periodically oscillatory motion. A few scenarios for the more interesting transition from periodic to aperiodic motion have been analyzed; a useful practical summary of this area has been given recently by Eckmann (1981). We will concentrate here on just one scenario, known as the *Feigenbaum transition*, which has been identified unequivocally in numerical studies of a number of simple models and perhaps observed in convection experiments as well.

The transition process is not a single bifurcation but an infinite sequence of them; it may be described as follows: The starting point is an attracting periodic orbit which loses stability at a first critical parameter value μ_0 . In losing stability, it produces a new attracting periodic orbit in much the same way as an attracting periodic orbit is produced in the Hopf bifurcation. The new orbit tracks the old one closely but goes around it twice before closing (see Figure 4).

Thus, the period of the stable oscillation for μ just above μ_0 is twice that for μ just below μ_0 . We will therefore refer to this bifurcation as the *period-*

doubling bifurcation; it is also frequently called the *pitchfork bifurcation*. As the parameter is further increased, a second critical value μ_1 is reached at which the doubled orbit itself loses stability through a period-doubling bifurcation. Thus, for μ just above μ_1 , the system has an attracting periodic orbit that follows four times around the fundamental orbit before closing, i.e. a stable oscillation with period about four times the base period. This orbit in turn undergoes a period doubling bifurcation (at $\mu = \mu_2$) producing an oscillation with period about eight times the base period, and in fact the cascade continues ad infinitum in a finite parameter interval. The parameter value μ_∞ at which the sequence of doublings accumulates represents the onset of chaotic behavior.

A qualification is necessary here. The fact that such infinite sequences of period doublings do occur and cannot be eliminated by small changes in the differential equation is well established, as are a number of striking features of the accumulation process. What is not well understood is the character of the motion for μ slightly above μ_∞ . Computer experiments strongly suggest that for most parameter values just above μ_∞ there is a strange attractor or perhaps several strange attractors, similar in character to the Hénon attractor. Unfortunately, nothing like this has been proved as yet; it simply isn't known for certain that a strange attractor *ever* appears at the accumulation of period doublings for a differential equation depending on a parameter. One thing that is known is that, if such an attractor exists, it must be very unstable with respect to changes in μ ; there are values of μ above but arbitrarily close to μ_∞ for which there is an attracting periodic orbit.

The accumulation of period doublings has some characteristic features, discovered by M. Feigenbaum (1978, 1979b, 1980), which are consider-

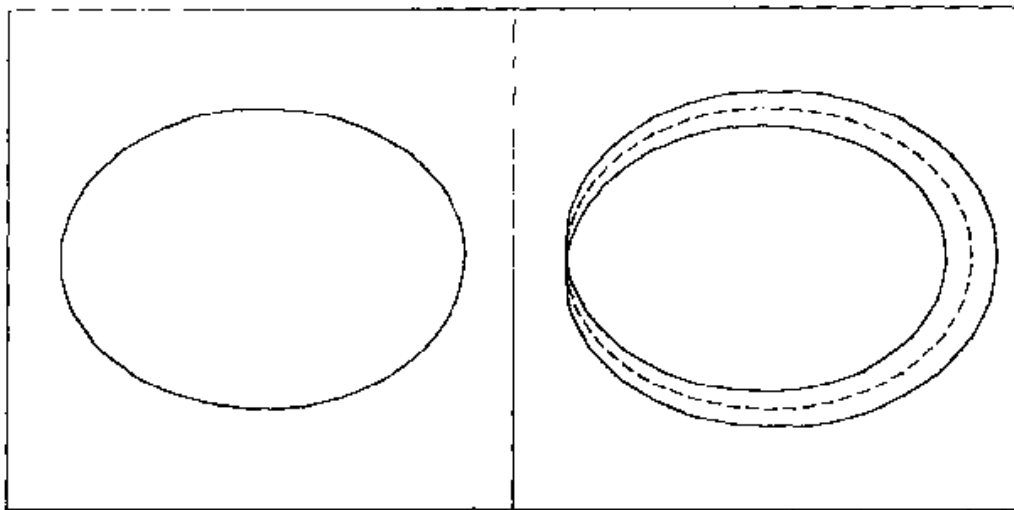


Figure 4 The period doubling bifurcation. On the left: the stable orbit for $\mu < \mu_0$. On the right: the solid curve represents the doubled stable orbit for $\mu_0 < \mu < \mu_1$; the dotted curve the now-unstable doubled orbit.

ably more distinctive than the square-root behavior of the amplitude in the Hopf bifurcation. The first, and easiest to state precisely, concerns the rate of convergence of the successive critical parameter values (μ_n) to their limit and says that, excluding degenerate cases, the convergence is geometric with a universal ratio:

$$\lim_{n \rightarrow \infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = 4.6692 \dots \quad (5)$$

This ratio has been observed in a number of model systems, and its origin and universality are well understood theoretically. Because of the relatively large value of the limiting ratio, all but the first few μ_n 's can be expected to be very close together, and this makes the observation of successive period doublings and physical measurement of the ratio extremely difficult. Unless the parameter can be controlled extraordinarily well, what will be observed is one or two period doublings and then chaotic behavior. In fact, up to now, the limiting ratio has not been measured, even roughly, in any physical system.

Feigenbaum has also argued that the power spectrum for essentially any dynamical variable for μ near μ_∞ should display a universal ratio between the strengths of lines corresponding to periods of about $2^n \tau_0$ and those of nearby lines corresponding to periods of about $2^{n+1} \tau_0$. (Here, τ_0 denotes a base period). This part of Feigenbaum's analysis has not yet been put on a firm mathematical footing, but it does appear that something similar to what Feigenbaum predicts has been observed in convection experiments performed by Libchaber & Maurer (1980). This area is currently under very active investigation, and the situation should be clarified soon.

To close this section, we should point out that, although the notion of scenario is an appealing and powerful one, it does not apply to all transitions from regular to chaotic behavior. Discontinuous transitions, in which the chaotic component of the motion is large for parameter values even slightly above the critical value, can and do occur. The transition from a stable stationary solution to the Lorenz attractor is a mathematical example, and pipe flow appears to be a physical one. (These two situations are, however, not quite parallel. For the Lorenz system, there is a critical parameter value above which the stationary solution is no longer stable. The attractor and the stable stationary solution coexist for parameter values slightly below the critical one. For pipe flow, the stationary solution remains stable for arbitrarily large values of the Reynolds number, but becomes more and more sensitive to perturbations of small but finite amplitude).

For the theory of the Feigenbaum transition see, in addition to the works already cited, Feigenbaum (1979a), Collet et al. (1980, 1981), and Lanford (1980).

STATISTICAL THEORY

The investigation of strange attractors throws some light on the question of what should be meant, in a fundamental sense, by a statistical theory of a dissipative differential equation. The question needs to be turned around from the form usual in the classical statistical mechanics of Hamiltonian systems. Because of Liouville's Theorem, Hamiltonian systems—at least those with bounded and nonsingular energy surfaces—come equipped with natural time-independent probability distributions, the microcanonical ensembles of statistical mechanics. Such an ensemble is essentially just normalized area on an energy surface, and thus is a comparatively elementary and familiar construct; the investigation of whether there are other, radically different, stationary probability distributions can justifiably be dismissed as an empty mathematical exercise. Part of the reason why these ensembles are so natural is that the sets to which they assign probability zero conform to our intuitive notion of negligible sets of initial conditions. That is, if some particular behavior occurs only for a set of initial conditions of microcanonical probability zero, we can reasonably conclude that behavior will never be observed.

There is a general theorem about invariant probability distributions, the Birkhoff Pointwise Ergodic Theorem, which asserts that the long-time limit of the time average of any dynamical variable exists, except perhaps for a set of initial conditions of probability zero. (Here, "dynamical variable" simply means a function on the state space sufficiently well behaved for its integral to be defined and finite). Applied to Hamiltonian systems and the microcanonical ensemble, this theorem says that the time average of any dynamical variable will settle down if watched long enough. The existence of limiting time averages is an extremely familiar fact of experience, both for Hamiltonian and for dissipative systems, and it is commonly assumed that this is a general property of differential equations provided that their solutions remain in a bounded region of the state space. There is, however, no such general theorem for dissipative systems, and it is possible to find differential equations for which time averages do not exist. (An example is given in Ruelle 1980a).

Once the existence of time averages for Hamiltonian systems is established, attention turns to the problem of computing them. The most favorable situation is one where the time average does not depend on the initial condition, but is simply given by the ensemble average of the dynamical variable in question. It is not hard to show that this will be the case for all dynamical variables if and only if the system is *ergodic*, i.e. if and only if there is no way to decompose the energy surface into two parts, each of nonzero microcanonical probability, in a time-invariant way. Determining whether a particular Hamiltonian is ergodic is generally a

delicate mathematical problem, and a great deal of important and deep work has been done on such questions over the past fifty years.

The situation for dissipative systems looks entirely different. Liouville's Theorem does *not* hold; indeed, solution flows generally contract volumes in the state space. As already noted, it is not even automatic that limiting time averages exist. There are, on the other hand, general abstract theorems asserting the existence of time-invariant probability distributions, and even of ergodic probability distributions, provided that solution curves don't run off to infinity. These probabilities are not, as in the Hamiltonian case, spread out over the state space. Consider, for example, what happens in the vicinity of an attractor. Any invariant probability distribution must assign probability zero to the set of all orbits converging to the attractor but not actually lying in it. Thus, whatever invariant probability distribution we might choose to work with, it is no longer justified to interpret a set of initial states of probability zero as physically negligible. Moreover, on a typical complicated attractor like the Lorenz attractor, there are a great many invariant probability distributions and it is not at all apparent how to go about singling out the right one—analogous to the microcanonical ensemble for Hamiltonian systems—to represent the statistics of typical orbits.

Rather than focusing on the choice of an invariant probability distribution, it is more satisfactory to start from limiting time averages of dynamical variables. These are, in any case, the quantities of most direct interest. One might hope that, in good cases, limiting time averages would exist along all orbits and be independent of orbit. This is slightly too optimistic. In the first place, there is no reason why a solution flow should have only one attractor. If there are several, the limiting time average will depend on which attractor the orbit approaches. We will therefore concentrate on a single attractor and study time averages along orbits converging to it. The second qualification is less obvious. It turns out, for the nontrivial attractors that have been studied in detail, that there are always many orbits with exceptional long-term behavior. A simple example is that these attractors generally contain many (unstable) periodic orbits. The best one can hope for, then, is that time averages will be *essentially* independent of orbit, i.e. that among the orbits converging to our attractor there may be a small set of exceptional orbits but that time averages do exist and are independent of orbit as long as the orbit does not belong to the exceptional set. A sensible meaning to give to the word "small" in this case is that the set of exceptional orbits has zero volume in the space of instantaneous states; exactly as for Hamiltonian systems, it is reasonable to suppose that such a set will never be seen in an experiment. If this favorable situation obtains, we will say that the attractor is *ergodic*. It follows from standard theorems that there is then an invariant probability

distribution on the attractor—in physical terms, a stationary ensemble—such that the ensemble average reproduces the time average along nonexceptional orbits for all (continuous) dynamical variables.

Two questions now present themselves:

Are the attractors, which arise in practice, ergodic?

Can the ensemble reproducing the time average along nonexceptional orbits be described directly?

The only answer to the first question available at this time is that the few attractors whose structures are well understood—i.e. the Lorenz attractor and those satisfying Smale's Axiom A—turn out to be ergodic. For Axiom A attractors, this is the important Bowen-Ruelle Ergodic Theorem (Bowen & Ruelle 1975, Bowen 1975, Ruelle 1976b). For this same class a surprising answer to the second question is also available. Omitting numerous technicalities, this answer is roughly as follows: First take a hypersurface slicing through the attractor transversally in such a way that every orbit on the attractor crosses the hypersurface frequently. Introduce "coordinates" on the attractor by describing each point by giving the last place its orbit crossed the hypersurface, and the time since that crossing. Now cut up the intersection of the hypersurface with the attractor into a finite number of sufficiently small pieces, and describe an orbit on the attractor by saying which of these pieces it hits in which order. If the pieces are labeled with, say, $1, 2, \dots, n$, then this procedure associates with the orbit a two-sided infinite sequence of integers in the range from 1 to n . If the cutting-up is done with sufficient care, it can be arranged that

The set of sequences thus obtained from all orbits on the attractor can be described in a simple way; it is the set of all sequences in which certain pairs (i, j) never occur in succession;

The sequences are essentially in one-to-one correspondence with points on the intersection of the attractor with the hypersurface.

The ensemble that reproduces time averages along nonexceptional orbits can be transported to an ensemble on the space of sequences. This transported ensemble turns out to be the thermodynamic equilibrium ensemble for a one-dimensional array of copies of a system with a discrete set of n states, with some nearest-neighbor exclusions and otherwise interacting through a many-body potential, that decreases exponentially as the separation goes to infinity.

The "equilibrium ensemble" for an Axiom A attractor thus looks much more complicated than the microcanonical ensemble for a Hamiltonian system. To construct it, it is necessary both to have a great deal of detailed

information about the solution flow and to find the thermodynamic equilibrium ensemble for an infinite assembly of systems interacting in a non-trivial way. In simple cases at least, the description given might be used as a starting point for the construction of numerical approximations. The main interest of the Bowen-Ruelle Ergodic Theorem is, however, foundational. At least for very well behaved dissipative systems, it answers in a convincing and precise way the question of how, in principle, the equilibrium ensemble is to be defined.

ACKNOWLEDGMENTS

Preparation of this review was begun while the author was a visitor at the IHES in Bures-sur-Yvette, France. Financial support for that visit from the Volkswagen Foundation, and continuing financial support from the National Science Foundation (MCS78-06718), are gratefully acknowledged.

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