

SPHERE GEOMETRY AND INVARIANTS

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ABSTRACT. A finite abstract simplicial complex G defines two finite simple graphs: the Barycentric refinement G_1 , connecting two simplices if one is a subset of the other and the connection graph G' , connecting two simplices if they intersect. We prove that the Poincaré-Hopf value $i(x) = 1 - \chi(S(x))$, where $\chi(S(x))$ is the Euler characteristic of the unit sphere $S(x)$ of a vertex x in G_1 , agrees with the Green function $g(x, x) = (1 + A')_{xx}^{-1}$, where A' is the adjacency matrix of the connection graph G' of the complex G . By unimodularity $\psi(G) = \det(1 + A') = \prod_x (-1)^{\dim(x)} = \phi(G)$, the Fredholm matrix $1 + A'$ is in $\text{GL}(n, \mathbb{Z})$, where n is the number of simplices in G . We show that the set of possible unit sphere topologies in G_1 are combinatorial invariants of the complex G , and establish so that also the Green function range of G is a combinatorial invariant. The unit sphere character formula $g(x, x) = i(x)$ applies especially for the prime graph $G(n)$ and prime connection graph $H(n)$ on square free integers in $\{2, \dots, n\}$ playing the role of simplices. In $G(n)$, integers a, b are connected if $a|b$ or $b|a$ and where in $H(n)$ two a, b are connected if $\gcd(a, b) > 1$. The Green function $g(x, x)$ in $H(n)$ relate there to the index values $i(x)$ in $G(n)$. To prove the invariance of the unit sphere topology we use that all unit spheres in G_1 decompose as $S^-(x) + S^+(x)$, where $+$ is the join and S^- is a sphere. The join renders the category X of simplicial complexes into a monoid, where the empty complex is the 0 element and the cone construction adds 1. The augmented Grothendieck group $(X, +, 0)$ contains the graph and sphere monoids $(\text{Graphs}, +, 0)$ and $(\text{Spheres}, +, 0)$. The Poincaré-Hopf functionals $G \rightarrow i(G) = 1 - \chi(G)$ or $G \rightarrow i_G(x) = 1 - \chi(S_G(x))$ as well as the volume are multiplicative functions on $(X, +)$. For the sphere group, both $i(G)$ as well as $\psi(G)$ are characters. The join $+$ can be augmented by a product \cdot so that we have a commutative ring $(X, +, 0, \cdot, 1)$ in which there are both additive and multiplicative primes and which contains as a subring of signed complete complexes $\pm K_i$ isomorphic to the integers $(\mathbb{Z}, +, 0, \cdot, 1)$. Both for addition $+$ and multiplication \cdot , the question of unique prime factorization appears open.

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1. PREFACE

A finite abstract simplicial complex G has a Barycentric refinement G_1 which is the Whitney complex of a graph. If χ denotes the Euler characteristic functional, we identify the values $i(x) = 1 - \chi(S(x))$ of unit spheres $S(x)$ in G_1 as Green function values $g(x, x) = (1 + A(G'))_{xx}^{-1}$, where $A(G')$ is the adjacency matrix of the connection graph G' of G . The graph G' has like G_1 the set of simplices in G as vertex set but connects two if they intersect; in G_1 , two simplices are connected if one is contained in the other. Having established that $1 + A(G')$ is unimodular [24], the matrix entries $g(x, y)$ became interesting. We can now look at the connection graph G'_1 of G_1 . Its Green function values are again related to unit spheres in the Barycentric refinement G_2 of G_1 . We compare the unit sphere topologies in G_1 with the unit sphere topologies of G_2 . We show here that in G_2 , no new topologies appear. It follows that the sphere index spectrum, the set of Green function values is a combinatorial invariant of a simplicial complex

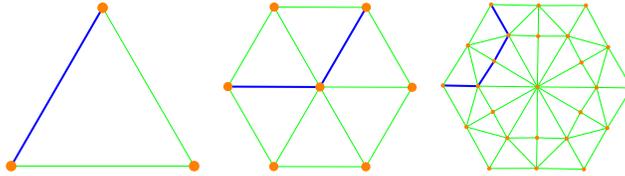


FIGURE 1. We see the triangle G and two of its Barycentric refinements G_1 and G_2 . A choice of unit spheres in G_1 and G_2 are marked.

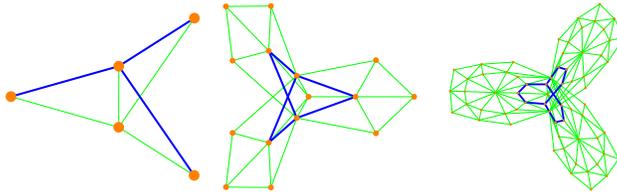


FIGURE 2. For the windmill graph G , unit spheres in G_1 can have topologies different from unit sphere topologies in G . While all unit spheres in G are contractible, there is a unit sphere $S(x)$ in G_1 which as a suspension $P_2 + P_3$ of the 3-point graph P_3 has $\chi(S(x)) = \chi(P_2) + \chi(P_3) - \chi(P_2)\chi(P_3) = -1$. But in G_2 , no new topologies appear in unit spheres.

2. INTRODUCTION

The quest to find invariants of topological spaces is particularly concrete when searching for combinatorial invariants in abstract finite simplicial complexes. These are quantities which do not change when applying a Barycentric refinement [2]. Abstract simplicial complexes have a surprisingly rich geometry despite the fact that they are one of the simplest objects mathematics knows. Every partially ordered set for example defines a complex, the order complex. Besides matroids also graphs are a source for complexes, like subcomplexes of the clique complex or then graphic matroids. Other type of simplicial complexes were introduced by Jonsson in [10]. Graph theory enters naturally as any Barycentric refinement G_1 of an arbitrary abstract simplicial complex is already is the Whitney clique complex of a graph. The faces of G are the vertices of G_1 and two faces connected if one is contained in the other. The complex G also defines the connection graph G' on the same vertex set, where two simplices connect if they intersect. Both G_1 and G' produce aspects of the geometry on G which are well accessible as graphs are not only intuitive, they also serve well as data structures.

Examples of combinatorial invariants of simplicial complexes are simplicial cohomology, Euler characteristic, homotopy groups (discrete notions of spheres and discrete notions of homotopy allow them to be defined combinatorially), the Bott invariants [2], the clique number (which is 1 plus the maximal dimension), minimal entries of a f -vector [26, 27], Wu characteristic, as well as connection cohomology attached to Wu characteristic [35, 23] or minimal possible entries in the f -matrix. Examples of quantities which are not combinatorial invariants are the f -vector itself (as it gets multiplied by a fixed upper triangular matrix when applying a refinement), the f -matrix, telling about the cardinalities pair intersections, the chromatic number (it can decrease under refinements but will stabilize already after one step to the clique number) or dimension (it can increase under refinements and will converge to the maximal dimension when iterating the Barycentric refinement process), neither is the Fredholm characteristic ψ (as it stabilizes to 1 already after one Barycentric refinement).

Focusing on finite simplicial complexes G allows to do geometry on finite sets but still be close to classical topology: the topology of a smooth compact manifold M for example is completely determined by the topology of the simplicial complex G defined by a triangulation

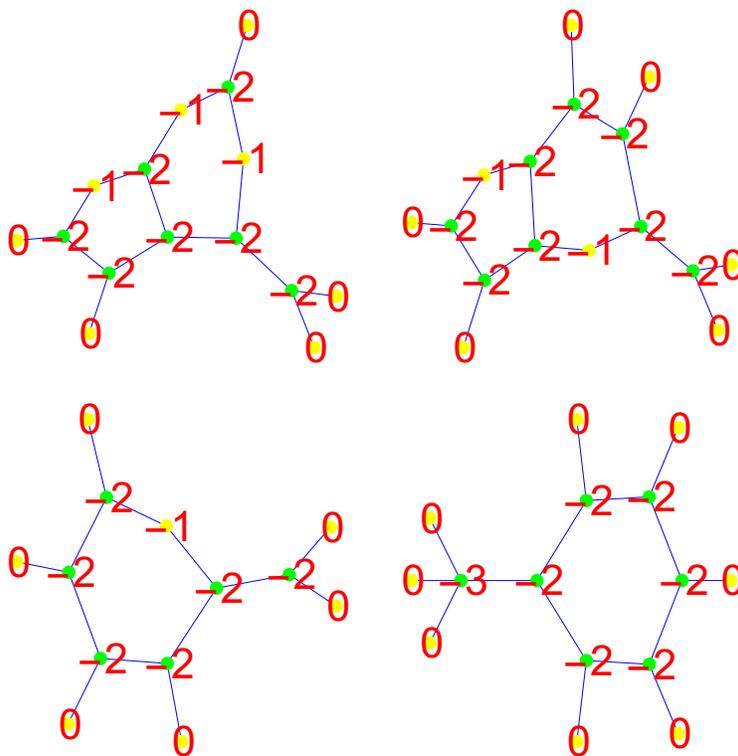


FIGURE 3. Adenine, Guanine, Cytosine and Thymine graphs with sphere spectrum. These are all 1-dimensional graphs. The Fredholm Characteristic ψ is 1 for Adenine and -1 for the others. The Euler characteristic χ is -1 for Adenine and Guanine and 0 for Cytosine and Thymine. The sphere spectral values σ depend only on vertex degrees as for all 1-dimensional graphs. It is $\sigma = \{-2, -1, 0\}$ for the first three. Thymine alone has a different sphere spectrum $\sigma = \{-3, -2, 0\}$. The values ψ, ϕ, σ are different for all. Also the f -vectors $(15, 16), (16, 17), (13, 13), (15, 15)$ differ. All four graphs are prime in the Zykov monoid of graphs.

of M in the form of a d -graph and the abstract Whitney complex describing it contains all topological information about M . While it is possible to look at topological invariants by embedding the complex into a continuum or by realizing it as a polytop in an Euclidean space, we prefer here to look at invariants defined and computed in a finite combinatorial manner. This is pragmatic computer science point of

view.

Both the set of abstract finite simplicial complexes as well as the set of finite simple graphs form a Boolean lattice: one can intersect and take unions. They both do not form a Boolean ring however as the Boolean addition, the symmetric difference operation, $A\Delta B = A \cup B \setminus A \cap B$ is no more in the same category. A valuation is a numerical quantity X satisfying $X(G \cup H) + X(G \cap H) = X(G) + X(H)$. If we had a Boolean ring, a valuation would satisfy the linearity condition $X(G\Delta H) = X(G) + X(H)$. By discrete Hadwiger, any valuation is of the form $X(G) = f(G) \cdot \chi$, where χ is a fixed vector and $f(G)$ is the f -vector of G . Since the only eigenvector of the Barycentric operator $f(G) \rightarrow f(G_1)$ is $(1, -1, 1, -1, \dots)$ Euler characteristic is the only combinatorial invariant which is also a valuation. While Euler characteristic $\chi(G) = \sum_x \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$ is pretty unique among valuations, there are other functionals. One is the Fermi characteristic $\psi(G) = \prod_x \omega(x)$, a multiplicative cousin of Euler characteristic.

Then there are multi-variate versions like the Wu characteristic $\omega(G) = \chi \cdot F(G)\chi$, where $F(G)$ is the f -matrix $F(G)_{ij}$ counting the number of intersections of i and j -dimensional simplices in the complex. Also the Wu characteristic as well as higher order versions are combinatorial invariants. A quadratic valuation $X(G, H)$ is a map for which $G \rightarrow X(G, H)$ and $H \rightarrow X(G, H)$ are both valuations. Related to this had been a question of Gruenbaum [6], whether higher order Dehn-Sommerville relations exist. We answered this affirmatively in [23]. The analysis there shows that Wu characteristic is the only combinatorial invariant among multi-linear valuations. Connection calculus leads to other combinatorial invariants like the Betti numbers of connection cohomology. That this is useful has already been demonstrated in examples. It allows to distinguish the cylinder and the Moebius strip for example.

Quantities satisfying the multiplicative valuation property $X(G \cup H)X(G \cap H) = X(G)X(H)$ must be additive valuations after taking logs and indeed, the Fredholm determinant is an example. It is related to the valuation $f(G)$ counting the number of odd-dimensional simplices in G .

We should point out that cohomology with real-valued cocycles is well defined also for a finitist who does not accept infinity. The reason is that the cohomology groups are determined by Hodge as the nullity

$\dim(\ker(L_k))$ of finite integer-valued matrices L_k , the form Laplacians L_k . The dimension of the kernel as well as the kernel basis itself can be obtained by row reduction, which is a finite process and done on integer matrices. So, cohomology is an acceptable concept for a finitist. For more about the complexity, see [11]. This happens already when following the lead of Kirchhoff, Betti and Poincaré. In topology however still, Euclidean realizations are used even for definitions like Barycentric refinements. But one perfectly can stay within the realm of simplicial complexes or better in the realm of much the more accessible graph category, on which the complex is a subcomplex of the Whitney clique complex.

How do simplicial complexes relate to traditional topology? The usual approach is to realize the complex in some sufficiently high dimensional Euclidean space and call two complexes G, H topologically equivalent if their topological realizations $\overline{G}, \overline{H}$ are homeomorphic in the classical point set topology sense. A topological invariant of the complex is then a quantity produces homeomorphic geometric realizations. For abstract finite simplicial complexes, the notion of topological invariant is equivalent to the notion of combinatorial invariant but the later works in the more restricted axiom system of finite mathematics.

The quest to find a complete finite set of invariants which allow to distinguish an arbitrary pair of simplicial complexes has long shown to be unattainable: it is not possible to build a Turing machine which can distinguish any pair of abstract finite simplicial complexes. Markov used in 1958 the unsolvability of the word problem by Novikov to show that the homeomorphism problem is not solvable. For surveys, see [34, 1, 29].

A constructivist would insist on giving a concrete homeomorphism which establishes the topological equivalence, but this is not so easy, as the Euclidean realizations are piecewise linear pieces of simplices, glued together in some high dimensional space. Also the complexity to decide whether a given pair of finite simplicial complexes are topologically equivalent can be tough. Given two concrete complexes, how do we (by just dealing with finite sets) decide whether they are topologically equivalent? We need other morphism for simplicial complexes in order to mirror topological equivalence. One answer is to use a finite Zariski type topologies which is purely combinatorial. It is conceivable however that we have to go through an exponentially large set of possible admissible finite topologies and check in each case the graph

isomorphism problem.

Let us assume that the complex is the Whitney complex of a graph. As mentioned already, this is not much of a loss of generality as a Barycentric refinement of an abstract simplicial complex is always the Whitney complex of a graph. Since a graph has a natural metric, the geodesic distance, one could see such a complex as an example of a finite metric space. But since every singleton set $\{x\}$ is both open and closed in that metric, the topology is the discrete topology and not that interesting. A weaker topology is Zariski like: it the topology on the vertex set of G_1 defined in such a way that the vertex set of H_1 of any subcomplex H of G is closed. This defines a finite topology on the vertex set of G_1 , (which is in general not Hausdorff similarly as Zariski topology), the homeomorphisms are still the graph isomorphisms. (The analogy is that subgraphs of G play the role of varieties and homeomorphism the role of regular maps. The algebraic nature of the “varieties” comes from the fact that we only allow subgraphs of G and not subgraphs of G_1 . The later would give the discrete topology.)

The Zariski idea is more flexible as we can use the same definition to form much weaker topologies by starting with a smaller set of subgraphs of G defining a sub-base of the topology. For example, we can get topologies on the second refinement G_2 by taking the set of graphs H_2 as closed, where H is a subgraph of H . We have proposed [19] to insist that a finite topology on a graph should be given by a subbase and have the property that the nerve graph of a sub-base should be homotopic to the graph and that a dimension condition is satisfied for intersections. The reasons are that we want any reasonable notion of cohomology to agree with the natural topology of the graph and insist that any notion of homeomorphism should honor the concept of dimension.

Unlike the quest to find a complete set of invariants for a simplicial complex, the task to find concrete combinatorial invariants of an abstract simplicial complex is more accessible. It is part of what we do here. An example of quantities which are interesting are valuations satisfying $X(G \cup H) + X(G \cap H) = X(G) + X(H)$. The discrete Hadwiger theorem [13] has classified all valuations on a complex and identified Euler characteristic as the only combinatorial invariant among them. It states that if the maximal dimension of G is d , the space of valuations is $d + 1$ dimensional. More invariants can be obtained by looking a multi-linear valuations or multiplicative invariants

$X(G \cup H)X(G \cap H) = X(G)X(H)$ like Fredholm characteristic.

Since Fredholm characteristic $\det(1 + A(G'))$ is $\{-1, 1\}$ -valued for simplicial complexes we can look at the Green function values $g(x, y) = (1 + A')_{xy}^{-1}$. Investigating these integers led to the current paper. It is here related to locally defined combinatorial invariants, the sphere Euler characteristic spectrum, which is defined by the collection of indices $i(S(x)) = 1 - \chi(S(x))$ which unit spheres $S(x)$ in the complex can have. One could look at higher order versions like the quadratic sphere spectrum $1 - \chi(S(x) \cap S(y))$, where x, y run both over all the simplices in G . This however does not match the off-diagonal Green function values in general and the off-diagonal values $(1 + A')_{xy}^{-1}$ with $x \neq y$ remain at the moment still unidentified. Analogies from physics suggest that all Green function values should have some natural interpretation and are possibly of a dynamical nature. Work like [3] show how close the discrete case can be to the continuum. In the shifted Fredholm case, it is exciting that the Green function values are integers and quantized. There would be other numbers to consider like $(1 + A')_{xy}^k$ but we seem only get topological invariants for $k = -1$.

In the context of doing arithmetic on graphs, the proof uses join operations. In [7] pp.21 the definition is attributed to A.A. Zykov who introduced it in 1949 [37]. There should be no confusion with the notion of “join” used for partially order sets and especially for simplicial complexes [30]. Topologically, the Zykov join of two graphs has the same properties as in the continuum and which appear in textbooks like [31, 8]. In the discrete, the construction does not need to take any quotient topologies: the vertex set of the join of two graphs is the union of the vertex sets and the edge set is the union of the edge sets of the factors together with all pairs belonging to different graphs. The zero element 0 is the empty graph. As in the continuum, the sum of two spheres is a sphere again. Adding 1 to a graph $G + 1$ is the cone construction.

Unlike for the arithmetic of numbers, where 1 is the only additive prime, there are more additive primes in the join monoid. The 0-dimensional sphere P_2 for example is prime. Adding such a sphere to a graph is the suspension. The maximal dimension of a sum $G + H$ is the sum of the maximal dimensions plus 1. But there are still mysteries about the Zykov monoid. Unlike in the continuum, we suspect (but do not know nor dare to conjecture) that in the discrete, the join operation is a unique factorization monoid, where the empty graph plays the role

of the 0 element and where the 1-point graph K_1 is the smallest example of an additive prime. The complete graph K_n decomposes as $K_1 + \cdots + K_1$ and C_4 can be written as $P_2 + P_2$. We can now for example that a triangulation of a 2d-surface of positive genus is prime, as the only 2d-surfaces which can be factored are of the form $C_n + P_2$ which are all 2-spheres which are prism graphs for $n \geq 2$.

By looking at the numerical quantity of maximal dimension, we can see that every graph G factors additively into primes $G = p_1 + p_2 \cdots + p_k$, but we don't know yet whether the factorization is unique. [I could not get hold yet of the Zykov article [37]. The MathSciNet review of Tutte mentions a unique prime factorization result there.] We have infinitely many prime graphs, like graphs P_n without edges, circular graphs C_n with $n > 4$ or disconnected graphs. A spectral condition is given at the end of this article. In some sense, our understanding of this arithmetic is on the level of Euclid, who did not prove the fundamental theorem of arithmetic yet. But unique prime factorization might not even hold in the sphere sub-monoid within that graph monoid. The sphere monoid might be the easier one to attack. But already there, for each positive dimension, there are infinitely many prime spheres: the number of maximal simplices of $G + H$ is the product of the number of simplices of G and H . A sphere with a prime number of maximal simplices therefore is an additive prime in the sphere monoid. In general, the Euler generating functions $f_G(x) = 1 + \sum_{k=0} v_k x^{k+1}$ for the join satisfies $f_{G+H} = f_G f_H$, which has as a corollary $\chi(G+H) = \chi(G) + \chi(H) - \chi(G)\chi(H)$ which follows from $\chi(G) = 1 - f_G(-1)$. This formula implies that $i(G) = 1 - \chi(G)$ is multiplicative and that graphs with zero Euler characteristic, or graphs with even Euler characteristic and graphs with odd Euler characteristic all form sub-monoids.

At the end we point out that there is a multiplication on simplicial complexes which is compatible with the Zykov join addition in the sense that distributivity holds. If the Zykov join monoid is extended to become a group we get so a commutative ring of simplicial complexes. This ring extends naturally the ring of integers as it contains the ring of integers in the form of complete graphs: $K_n + K_m = K_{n+m}$ and $K_n K_m = K_{nm}$. The empty graph 0 is the zero element and the graph K_1 is the 1-element. We originally looked for a ring structure in order to proof the unimodularity theorem. A purely algebraic proof has not worked yet and might not even exist. Indeed, the Fredholm functional ψ is only multiplicative on the additive subgroup of signed complexes

which have even Euler characteristic. But ψ is a character on the sphere subgroup of the additive group similarly as the Poincaré-Hopf functional $i(G) = 1 - \chi(G)$ which is a character on the sphere group and which plays here an important role.

3. SIMPLICIAL COMPLEXES

A finite abstract simplicial complex is a finite set G of non-empty sets, called simplices, so that G is invariant under the operation of taking non-empty subsets. The dimension of a simplex x in G is defined as the cardinality of x minus 1. In graph settings, the cardinality is called the clique number. The maximal dimension of G is the maximal dimension, which a simplex in G can have. An abstract simplicial complex is sometimes also called hereditary collection [30]. Early references which use finite abstract simplicial complexes are [32, 12]. Before that, textbooks mostly used Euclidean realizations exclusively like [36], a book from 1957.

Having only the subset axiom, simplicial complexes are one of the simplest geometrical structures imaginable, simpler even than algebraic structures, a topology or a measure theoretical structure. Indeed, an abstract simplicial complex is an order structure on a set and every partial ordered set defines a simplicial complex, its order complex. Abstract finite simplicial complexes are therefore closely related to finite posets. In the rest of the paper, we often just say “complex” meaning finite abstract simplicial complex.

The Barycentric refinement G_1 of a complex G is the set of subsets of the simplex sets A which have the property that for any pair (a, b) in A , either a is a subset of b or b is a subset of a . This defines a graph $G_1 = (V_1, E_1)$ where V_1 is the set of simplices in G and E_1 the set of pairs (a, b) such that a is contained in b or b is contained in a . The Whitney complex of this graph is then the complex G_1 . One can also look at the connection complex G' of G which is the Whitney complex of the graph where two simplices (a, b) are connected, if they intersect. It can be easier to work with graphs G_1 and G' however rather than the simplicial complexes.

Remarks.

1) There is a more sophisticated dimension of a complex: the inductive dimension $\dim(x)$ of a simplex x . It is defined as $1 + \dim(S(x))$, where $S(x)$ is the Whitney complex in the unit sphere $S(x)$ of x in G_1 , and

$\dim(G) = (1/|V(G_1)|) \sum_x \dim(x)$. Unlike the maximal dimension, the inductive dimension is a rational number in general. It has some nice properties like satisfying the same dimension inequality than Hausdorff dimension or having computable expectation on Erdős Rényi probability spaces [14].

2) The Barycentric refinement of G can be written as a product $G \times K_1$ [21]: this simplicial product graph $G \times H$ of two complexes is the graph for which the vertices are the ordered pairs (x, y) of simplices x in G and y in H for which two $(a, b), (c, d)$ are connected if either $a \subset c, b \subset d$ or $c \subset a, d \subset b$. Even if G is a one-dimensional graph, it has no relation with the Cartesian product or tensor product for graphs which both are graphs with vertex set $V(G) \times V(G)$. The simplicial product graph $G \times H$ has a vertex set the product of the simplex sets of G and H . The simplicial product graph can be rewritten as the product in the Stanley-Reisner ring. It is a natural product as the Euler characteristic $\chi(G \times H) = \chi(G) \times \chi(H)$ is multiplicative and the Künneth formula for cohomology applies [21].

Given a simplicial complex G , we can look both at the Barycentric refinement G_1 of G as well as the connection graph G' of G . Both graphs will play a role here. The two graphs G' and G_1 have the same vertex set as this is the set of simplices of G . The graph G_1 is a subgraph of G' . In the theory of simplicial complexes one looks also at the Hasse diagram, which is a subgraph of G_1 .

The unit spheres $S(x)$ in G_1 have topologies which are interesting. It turns out that the possible topologies which appear as unit spheres in G_1 form a combinatorial invariant of G . While G_1 can feature new unit sphere topologies, there are no new sphere topologies appearing in G_2 .

Note that we work with general simplicial complexes and do not assume any Euclidean structure. Still, there is intuition which leads to the invariance of the unit sphere topology. A Barycentric refinement can produce new sphere types but after having done one refinement, the local sphere geometry does not change any more in the interior. The inside of refined simplices is now Euclidean. The unit spheres in the interior are discrete spheres and space looks there like Euclidean space. It is getting more interesting at singularities, where unit spheres are no topological spheres any more. An example is the figure 8 graph. Most points in that graph have a zero-dimensional sphere as a unit sphere but there is a point, the singularity, where a sufficiently small

sphere consists of 4 points. The value of the Euler characteristic is 4. It is a value does not change under Barycentric refinement.

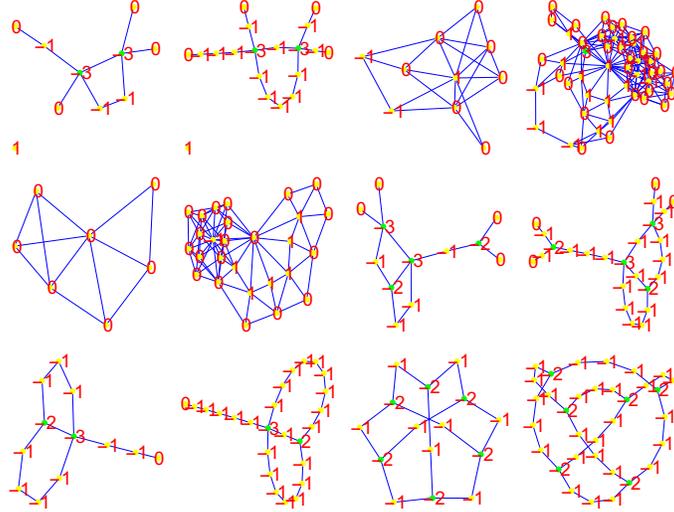


FIGURE 4. Examples of graphs and their Barycentric refinement. In each case, we add the sphere spectral values at each vertex.

Let us call the number

$$i(x) = 1 - \chi(S(x))$$

the Poincaré-Hopf index of the vertex x , where $S(x)$ is the unit sphere in the Barycentric refinement G_1 of the complex G . The name has been chosen because if f is a function taking the maximum at x , then this is the usual Poincaré-Hopf index $i_f(x) = 1 - \chi(S^-(x))$ [15], where $S^-(x)$ is generated by the set of vertices y , where $f(y) < f(x)$. The name character is adequate since on the subgroup of complexes with $\chi(G) \neq 1$, it is a multiplicative character on a group constructed by the join operation on complexes. The set of possible values $\{i(x) \mid x \in V(G)\}$ is the Poincaré-Hopf spectrum of the graph. Since $i_{G+H}(x, y) = i_G(x)i_H(y)$, they form generalized characters. The set of unit sphere topologies is the sphere topology spectrum. We can quantify this numerically for example by attaching to each vertex x the Betti numbers $b_k(S(x))$ of the unit sphere. The set $\{b_k(S(x)) \mid x \in V(G_1)\}$ for example could be called the k 'th Betti spectrum of the complex G .

The invariance of the sphere topology allows to check whether a graph is a Barycentric refinement of a complex. If the unit sphere topology

changes under Barycentric refinement, then G is not the Barycentric refinement of a complex. An example is the windmill graph obtained by gluing three triangles along a common edge. While all unit spheres of G are connected, there is a vertex in G_1 with a disconnected unit sphere. We have seen before that if $\psi(G)$ is not in $\{-1, 1\}$, then G is not the connection graph of a complex. An example is a triangle which has $\psi(G) = 0$.

4. SPHERE TOPOLOGY

In this section we look at discrete spheres and unit spheres $S(x)$ of Barycentric refinements of finite abstract simplicial complexes. Note that since we look at rather general graphs, the unit spheres are rarely spheres but rather general graphs.

Barycentric refined graphs are natural as they are Eulerian [20]. This means that they can be colored with a minimal number of colors, the color being the dimension which the point has when it was a simplex of the complex. This is useful for graph chromatology [18]: one approach to 4-color a planar graph is to embed it first into a 2-sphere, then fill out the interior 3-ball by cut it up using edge refinements until it is Eulerian and so 4-colorable. Having colored the interior and not cutting anything at the boundary, this colors the boundary with 4 colors.

The unit sphere $S(x)$ of a simplicial complex is a sub-complex consisting of all simplices in G which are either contained in x or which contain x . This is better seen in the Barycentric refinement G_1 , for which the simplices of G are now vertices and the unit sphere is the geodesic unit sphere with respect to the graph distance.

Given a simplex x , the set of subsets of x is the complex $S^-(x)$, the stable or negative part of $S(x)$. The set of simplices which contain x define the sub-complex $S^+(x)$, the unstable or positive part of $S(x)$. More intuitive is the graph theoretical reformulation in the graph G_1 : the unit sphere $S(x)$ consists of all vertices in G_1 which are connected to x . The function $f(x) = \dim(x)$ which gives the dimension of x when we look at it as a set in the simplicial complex G . It defines a coloring in G_1 . The dimension functional \dim resembles a Morse function as it allows to partition the unit sphere $S(x)$ into a stable part $S^-(x)$ and unstable part $S^+(x)$. As in the continuum, the join of $S^-(x)$ and $S^+(x)$ is $S(x)$, this relation will also hold here in full generality for any simplicial complex. The property $S^-(X) + S^+(x) = S(x)$ could serve

as a Morse condition without assuming $S^+(x)$ or $S(x)$ are spheres. But here, we don't make any such assumption.

So here is again a formal definition for $S^\pm(x)$ in the case of the function $f(x) = \dim(x)$:

$$S^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$$

and

$$S^+(x) = \{y \in S(x) \mid f(y) > f(x)\}.$$

Despite the fact that the definitions for S^- and S^+ look similar, there is a difference as we do not have any kind of Poincaré duality in the case of a general complex. The stable sphere $S^-(x)$ is always a discrete sphere. It only deals with the inside of a simplex which after Barycentric refinement is of Euclidean structure. The possibly crazy outside world, deals with connections of simplices which are not Euclidean even for very simple examples. For a one-dimensional star graph for example the central vertex x is 0-dimensional and $S^+(x)$ consists of all edges containing x .

The Zykov join of two graphs G, H is is the graph $G + H$ which has as vertex set the union of the vertices of G and H and where two vertices are either connected if they were already connected in G or H or then if they belong to different graphs. We simply call $G + H$ the join of G and H . The join renders the category of graphs into a monoid which can be augmented to an Abelian group on which functionals like $j(G) = -\psi(G)$ or $i(G) = 1 - \chi(G)$ are multiplicative.

A d -sphere or shortly d -sphere is a finite simple graph which has the property that every unit sphere is a $(d - 1)$ -sphere and such that removing a vertex from it renders the remaining graph contractible. This inductive definition gets started with the assumption that the empty graph 0 (the zero element in the monoid) is the only (-1) -sphere. A d -graph is a finite simple graph for which all unit spheres are $(d - 1)$ spheres. A d -complex is a finite abstract simplicial complex for which its Barycentric refinement is a d -graph.

Remark. The notion of Evako d -sphere and d -graph is equivalent to the notion of d -sphere within combinatorial d -manifolds in the theory of simplicial complexes. Notions as such have been put forward in [9, 4]. Related is the notion of a combinatorial d -manifold which is a connected, pure (every facet has dimension d) finite abstract simplicial

complex of maximal dimension d for which each ridge (codimension-1-face) has two facets as a boundary and such that all vertex links are PL-homeomorphic to the boundary of a d -simplex. A combinatorial d -manifold is then a d -sphere if it is PL homeomorphic to the boundary of a d -simplex. These notions rely on Euclidean embeddings. Forman's discrete Morse theory remains in the discrete: a combinatorial d -sphere G is PL-homeomorphic to a standard PL-sphere if and only if it is a Reeb sphere: some Barycentric subdivision G_n admits a discrete Morse function with exactly two critical points [5]. Being a Reeb sphere is equivalent to being an Evako sphere (removing one vertex renders the complex contractible). Starting with the notion of Evako sphere has the advantage of avoiding Morse theory. In any case, detecting spheres can be difficult [28].

We start with a lemma which is motivated by an analogue fact in the continuum, where a Morse function at a critical point x defines two parts $S^\pm(x)$, the stable and unstable part of the sphere. They are obtained by intersecting a small geodesic sphere $S(x)$ with the stable or unstable manifolds $W^\pm(x)$. The sphere $S(x)$ itself is then $S^- + S^+$. This is the picture for the gradient flow of a Morse function f . In the discrete, in full generality, the dimension function $f(x) = \dim(x)$ can serve as a Morse function as we have a stable and unstable part at every point. As pointed [25], the Morse cohomology in G_1 in this setting is directly equivalent to the simplicial cohomology of G . This gives hope for a more general Morse setting for functions with possibly much fewer critical points allowing to compute cohomology more efficiently than with incidence matrices defined by the simplicial complex.

Lemma 1 (Unit sphere decomposition). *If G_1 is the Barycentric refinement of a finite abstract simplicial complex G , then for any vertex x in G_1 , the unit sphere $S(x)$ is the join of $S^-(x)$ and $S^+(x)$.*

Proof. Both $S^\pm(x)$ are subgraphs of the unit sphere $S(x)$. The union of the vertex sets of $S^-(x)$ and $S^+(x)$ is the vertex set of $S(x)$. If $a \in S^-(x)$ and $b \in S^+(x)$, then (a, b) is an edge in $S(x)$ because as simplices, a is a sub simplex of x and x is a sub simplex of b . \square

We can rephrase that also by saying that the unit sphere of a vertex x for which the simplex x is not prime as a complex in G (which is always the case if the dimension is positive) is not prime in G_1 but splits as $S(x) = S^-(x) + S^+(x)$.

Lets now look at the structure of the stable unit sphere:

Lemma 2. *The stable unit sphere $S^-(x)$ of a vertex x in the Barycentric refinement G_1 of an abstract simplicial complex is always a d -sphere for $d = \dim(x) - 1$.*

Proof. $S^-(x)$ is the boundary of the Barycentric refinement of the simplex X in G which belongs to the vertex x in G_1 . We can show by induction in the dimension d that this is a sphere. When looking at the intersection with a sphere $S(y)$ of a vertex y in $S^-(x)$ the dimension is reduced by 1, so that by induction, the unit sphere of a vertex $y \in S^-(x)$ is a $(d - 1)$ -sphere. Also the second condition for a d -sphere, the collapsibility is satisfied by induction. Take away a vertex y of $S^-(x)$. Then every intersection of $S^-(x)$ with an other sphere $S(z)$ which has contained y becomes contractible so that $S^-(x) \setminus y$ is contractible. \square

Examples:

- 1) If x belongs to a 0-dimensional vertex $\{x_0\}$ in G , then $S^-(x)$ is empty so that it is a $d = (-1)$ -dimensional sphere.
- 2) If x belongs to a 1-dimensional simplex (x_0, x_1) in G , then $S^-(x)$ consists of two isolated points $\{x_0, x_1\}$ which is $d = 0$ -dimensional sphere.
- 3) If x belongs to a two-dimensional simplex (x_0, x_1, x_2) in G , then $S^-(x)$ is a cyclic graph C_6 with 6 elements:
 $\{x_0, x_0x_1, x_1, x_1x_2, x_2, x_2x_3, x_3, x_3x_4, x_4, x_4x_0\}$.

We will need a generalization of this which we can see as a higher order version of the just said. Given a vertex x in the second Barycentric refinement G_2 , this corresponds to a simplex (x_0, x_1, \dots, x_d) in G_1 . We can assume without loss of generality that the vertices are ordered so that $x_0 \subset x_1 \subset \dots \subset x_k$, where $x \subset y$ means that as a simplex in G , the simplex x is contained in the simplex y . Now define the second stable sphere

$$S^-(x) = \{z \subset x \mid z \neq x_0, z \neq x_1, \dots, z \neq x_k\}.$$

These second stable spheres are of interest because they appear as unit spheres in the interior of the second Barycentric refinement G_2 of G .

Lemma 3. *The second stable unit sphere $S^-(x)$ of a vertex $x = (x_0, \dots, x_k)$ in the Barycentric refinement G_2 of an abstract simplicial complex G is always a $(l - k)$ -sphere if $\dim(x_k) = l$.*

Proof. We can see this in two different ways: first we can see it as an intersection of k unit spheres $S(x_j)$ in the stable sphere $S^-(x_k)$. Each intersection reduces the dimension by 1.

If we mark the set of dimensions $k_j = \dim(x_j)$ in the integer interval $\{0, \dots, l\}$, then the set of simplices which belong to a gap of dimension values $[a, b]$ which are omitted, then this is a sphere of dimension $b - a - 1$. We call this the gap sphere belonging to $[a, b]$. The second stable unit sphere is now the join of all these gap spheres:

$$S^-(x) = S_{[a_1, b_1]} + \dots + S_{[a_m, b_m]} .$$

□

Examples:

- 1) The set of simplices in the complex of all complete subgraphs of K_4 which are between a vertex p and the tetrahedron x itself can be seen as the virtual stable sphere $S^-(\{x, p\}) = S(x) \cap S(p)$ which is a stable sphere in the Barycentric refinement. These are all the edges and triangles which contain p are contained in x . This is a cyclic graph S_6 . In this case, there is only one gap $[a, b] = [1, 2]$.
- 2) The set of simplices in the complex of all complete subgraphs of K_5 which are different from a fixed vertex p and a fixed triangle t . In this case there are two gaps $[a_1, b_1] = [1, 1]$ and $[a_2, b_2] = [3, 3]$. The two gap spheres are 0-dimensional spheres P_2 and their join is $C_4 = P_2 + P_2$. Again this can be seen as a stable sphere $S^-(\{x, p\})$ of a refinement, where the edge $\{x, p\}$ has become a vertex.

5. THE SPHERE GROUP

In the continuum the monoid of spheres with join operation is not that interesting as two spheres of the same dimension are topologically equivalent. The map \dim from $(\text{Topologicalspheres}, +, 0)$ to \mathbb{N} is an isomorphism of monoids.

This changes completely in a combinatorial setting. The sphere group is now interesting as there is a countable set of non-isomorphic spheres of each positive dimension. First of all we have to establish that the property of being a sphere is invariant under addition given by the Zykov join operation.

The join operation has the same properties as in the continuum: it is associative, commutative and preserves spheres. For associativity, note that the graph $A + B + C$ has the union of the vertex sets as vertices. Two vertices x, y are connected if they were connected in one of the three components A, B, C or then are in different components. The join of two simplices of dimension n and m is a simplex of dimension

$n + m + 1$: shortly $K_n + K_m = K_{n+m}$. The join of two spheres is again a sphere:

Lemma 4 (Joins preserve Spheres). *The set (Spheres, +) is a monoid, a groupoid with a neutral element given by the (-1) -sphere 0 . The join of n -sphere and a m -sphere is a $n + m + 1$ -sphere.*

Proof. Start that the join of two (-1) -spheres is again a -1 sphere. Then use induction with respect to the sum $n + m$ of the dimensions. We have to show that every unit sphere of $G + H$ is a sphere and that removing a vertex of $G + H$ renders the remaining graph contractible. Given a vertex x in $G + H$. Assume it is in G . Its unit sphere is the join of $S_G(x) + H$ which by induction assumption is a sphere. The case when x is in H , is analog. If we remove the vertex x in G , then by definition the graph $G \setminus x$ is contractible because the join of a contractible graph with any other unit sphere graph is contractible. \square

The Grothendieck construction produces then a group from this monoid. Lets call it the sphere group. Elements in this groups are equivalence classes of pairs $A - B$ of spheres, where $A - B \sim C - D$ if there exists a sphere K such that $A + D + K = B + C + K$. The group element $O - C_4$ with octahedron O for example is P_2 . One can see this also because the octahedron is $O = 3P_2$ and the cyclic graph C_4 is $2P_2$ so that $3P_2 - 2P_2 = P_2$.

A d -graph is a graph for which every unit sphere is a $(d - 1)$ -sphere. A d -complex is a finite abstract simplicial complex G for which its Barycentric refinement G_1 is a d -graph.

Corollary 1 (Geometric invariance). *The Barycentric refinement G_1 of a d -complex G is a d -complex.*

Proof. Every unit sphere $S(x)$ is the join of two spheres $S^-(x)$ and $S^+(x)$. $S^-(x)$ is always a sphere. The fact that $S(x)$ is a sphere follows in the case of an original vertex by the assumption and for a simplex (x_0, \dots, x_d) from the fact that $S^+(x)$ is the intersection of the spheres $S(x_k)$ which by definition is a sphere. By the previous corollary, also $S(x)$ is a sphere. \square

While $S^-(x)$ is always a sphere as we have seen before, the unstable sphere $S^+(x)$ is not a sphere in general. For the central vertex p in the star graph G with n spikes for example, the stable sphere $S^+(p)$ is P_n , the graph with n vertices and no edges, which is only a sphere if $n = 2$. In the case of a d -complex G , all unstable spheres $S^+(x)$ are spheres too. The unit sphere decomposition $S(x) = S^-(x) + S^+(x)$ is then the

addition of two spheres.

But in general, the structure of the stable sphere $S^+(x)$ also can be described by unit spheres.

Lemma 5 (The positive sphere lemma). *If G_1 is the Barycentric refinement of a simplicial complex G and if $x = (x_0 \dots x_k)$ with G -vertices x_i is a vertex in G_1 , then $S^+(x) = \bigcap_{j=0}^k S(x_j)$.*

Proof. We only have to show $S^+(x) = \bigcap_j S(x_j)$. Given a vertex $y \in S^+(x)$, then $y = x_0 \dots x_k y_1 \dots y_l$. But this is in the intersection. On other other hand, if we are in the intersection, then we have to be a super simplex of x_0, \dots, x_k . In other words, we have to be in $S^+(x)$. \square

Example: Let G be the octahedron and G_1 its Barycentric refinement. Now, if $x = x_0$ is a vertex in G_1 which was an original vertex, then $S(x) = 0 + S(x)$ where 0 is the empty graph. If $x = (x_0 x_1)$ belonged to an edge, then $S^-(x)$ is a 0-sphere and $S^+(x)$ is a 0-sphere, as an intersection of two spheres. Now $S^+(x) = S(x_0) \cap S(x_1)$.

Given a vertex x in G_2 we define the second positive sphere $S^+(x)$ as part of the unit sphere $S(x)$ consisting of vertices z in G_2 which when written as a simplex Z in G_1 contains the simplex X corresponding to x .

Lemma 6 (Second positive sphere). *Given a vertex $x = (x_0, \dots, x_k)$ in G_2 with $x_0 \leq \dots \leq x_k$. The second positive sphere $S^+(x)$ in G_2 is graph isomorphic to the Barymetric refinement of $S^+(y)$ in G_1 with $y = x_k$.*

Proof. Given $z \in S^+(x)$, then $x_k \leq z$ and $z = (x_0, \dots, x_k, x_{k+1}, \dots, x_l)$. This point z is in $S^+(y) = \bigcap_j S(x_j)$.

On the other hand, given $w \in S^+(y)$, then this is a simplex containing y so that (x_0, \dots, x_k, w) this defines a new vertex z in G_2 which is connected to y . It belongs therefore to $S(x)$. \square

Let x be a vertex in G_2 with $x = (x_0, \dots, x_k)$ in G_1 with $\dim(x) = k$ so that $S(x)$ is the join of a $(k - 1)$ -sphere $S^-(x)$ and $S^+(x)$. We have defined $S^+(x)$ and $S^-(x)$ for vertices x . Assume the vertex x in G_2 belongs to the simplex $x = (x_0 \dots x_k)$ in G_1 . Define $S_1^-(x)$ as the simplex $(x_0 \dots x_k)$ equipped with the $(k - 1)$ -skeleton complex structure, so that it is a sphere. Define also the virtual positive sphere $S^+(x) = \bigcap_j S(\{x_j\})$ in G_1 . The graph $S_1^-(\bar{x}) + S_1^+(\bar{x})$ is the virtual unit sphere of the simplex x in G_1 . It is topologically equivalent to $S(x)$ in G_2 . We will see in a moment that it is equivalent to an actual

unit sphere $S(y)$, where $y = x_k$.

First a simple property of Barycentric refinement:

Lemma 7 (Intersection complexes). *Let H, K be subcomplexes of G so that H_1, K_1 are subgraphs of G_1 . Then $(H \cap K)_1 = H_1 \cap K_1$.*

Proof. Given a simplex x in $H \cap K$. This means $x = \{x_0, \dots, x_k\}$ is a set which is both in H and K and all are connected. Now, also (x_0, \dots, x_k) is a simplex in H_1 as well as in K_1 . The simplex is in the intersection. on the other hand, given a vertex x in $H_1 \cap K_1$. Then x is both in H_1 and K_1 , meaning that $x = \{x_0, \dots, x_k\}$ is a face in H and in K . \square

For every vertex x in G_2 corresponding to a vertex $(x_1, \dots, x_k) \in G_1$, each of the elements x_j are vertices in G_1 . The following lemma makes a step to relate (x) with a new unit sphere $S(y)$.

Lemma 8. $S^+(x)$ with G_2 is the Barycentric refinement of a virtual positive sphere in G_1 which agrees with the positive sphere $S^+(y)$ in G_1 .

Proof. $(x_0, x_1, \dots, x_k) \subset G_1$. The positive sphere $S_2^+(x)$ is the Barycentric refinement of $\bigcap S(x_k)$ in G_1 . Each x_k corresponds to a simplex X_k in G . We have now to find a vertex $y \in G_1$ such that $\bigcap S(x_j)$ is equivalent to $S^+(y)$. Such an y corresponds to a simplex (y_0, y_1, \dots, y_k) in G . We can construct that k -simplex from the simplices X_0, \dots, X_k inside G . The simplices X_k are all contained in each other since the x_j are all connected. Assume without loss of generality that things are numbered so that $X_0 \subset X_1 \cdots \subset X_k$. We can now pick $y_0 \in X_0, y_1 \in X_1 \setminus X_0, y_2 \in X_2 \setminus X_1, \dots, y_k \in X_k \setminus X_{k-1}$. Now $S^+(y)$ is a subgraph of G_1 containing all vertices which belong to simplices in G containing all the y_j . $S^+(y)$ is $\bigcap S(\{y_j\})$ which is the Barycentric refinement of $\bigcap_j S(y_j)$. $S^+(x)$ is $\bigcap S(x_j)$ whose Barycentric refinement is $S^+(x)$ in G_2 . \square

Examples.

1) If $x = (x_0)$ is 0-dimensional and (x_0) is the vertex in G_1 which belongs to x , then we can take $y = x_0$. The sphere $S(y) = S^+(y)$ has as a Barycentric refinement the sphere $S(x) = S^+(x)$.

2) If $\dim(x) = 1$ and $x = (x_0, x_1)$ then we get the virtual unit sphere $S((x_0, x_1)) = P_2 + S^+(x)$ in G_1 . The graph $S^+(x)$ in G_2 is isomorphic to the refinement of $S(x_0) \cap S(x_1)$.

6. THE TWIN PROPOSITIONS

Given a connection graph G' of a complex G , we can either remove a vertex x in $V(G') = V(G_1)$ or we can add a vertex x . We have seen

already that adding a vertex changes the Fredholm determinant by a factor $i(x)$. It is a bit surprising but in both cases, the Fredholm characteristic $\psi(G) = \det(1 + A'(G))$ gets multiplied by an integer.

Here is again the case when adding a cell. It is a bit more general than before as we now do not insist that the new cell has a unit sphere smaller dimensional cells only. The proof is the same as before and uses the concept of a valuation on simplicial complexes. This is a functional X on subcomplexes, which satisfies the linearity condition $X(H \cup K) + X(H \cap K) = X(K) + X(H)$ (we can not write $X(H \Delta K) = X(K) + X(H)$ as $H \Delta K = H + K$ is not a complex any more but it explains the name linearity as the symmetric difference Δ is the addition in a larger Boolean ring of structures). The prototype of a valuation is the Euler characteristic. As in the continuum, where valuations are defined on convex subsets of Euclidean space (which naturally come with a simplicial complex structure). It is important to see that a valuation is not a functional on subsets like in measure theory and indeed valuations do not extend to the measure theory level except in geometric measure theory, but where we deal with a Fermionic analogue of valuations.

In the discrete also, it is not just a functional on subsets. The Euler characteristic for example depends on the choice of simplicial complex structure. The Euler characteristic is the prototype of a valuation. Actually there is a $(\dim(G) + 1)$ -dimensional space of valuations by discrete Hadwiger and the Euler characteristic is up to a scaling factor the only which is a combinatorial invariant: the Barycentric operator on f -vectors has only one eigenvalue 1 and eigenvector $(1, -1, 1, \dots)$. Repeating the Barycentric refinement leads to a central theorem [22]. In the proof, we again make use of the fact that the only valuation which assumes the value 1 on all simplices must be the Euler characteristic.

Proposition 1 (Attaching cells). *Let G be a simplicial complex and G' its connection graph. If we chose a sub-complex H of G leading to a subgraph H' of G' and attach a new cell x to it, leading to a new complex $G' \cup_H x$, then the Fredholm characteristic ψ of $G + x$ gets multiplied by a factor $i(x) = 1 - \chi(H)$:*

$$\psi(G \cup_H x) = (1 - \chi(H))\psi(G) .$$

Proof. (i) The map $H \rightarrow X(H) = \psi(G \cup_H x) - \psi(G)$ is a valuation. The reason is geometric: as this is a super count of the number of new paths which are allowed when adding the vertex x . Since x is a single

vertex and only one path can occupy it, the linearity condition is satisfied.

(ii) Also the map $H \rightarrow Y(H) = X(H)/(-\psi(G))$ is an integer valued valuation. The reason is that by unimodularity (or induction assumption if we want to use this proposition as a proof), $\psi(G)$ is $\{-1, 1\}$ valued and because when looking at a function of H , the complex G is fixed.

(iii) The map $H \rightarrow Y(H)$ takes the value 1 on simplices H . The reason is that $H \cup \{x\}$ is again a simplex and that we look now at the collection of all Leibniz paths in $G' \cup_H \{x\}$ which go through x . Now $Y(H) = 1$ is equivalent to $X(H) = -\psi(G)$ and so $\psi(G \cup_H x) = 0$. The reason why this is zero is because $\psi(K_n)$ is zero for every complete graph K_n with $n > 1$. every path in $G \cup_H x$ can now be paired with an other path of different parity.

(iv) Because all valuations which take the value 1 on simplices must be the Euler characteristic, we have $Y(H) = \chi(H)$. \square

In the new CW complex, $S(x)$ is the unit sphere in the Barycentric refinement graph of $G + x$.

Now lets start with a complex and write $G - x$ for the complex, where the cell x has been removed. Now what matters is the sphere $S(x)$ in G_1 and not the sphere in G' . The reason is that if we remove the cell x , we remove all links to sub-simplices and super simplices. The proof is very similar. We write it again down so that we see the difference.

Proposition 2 (Removing cells). *If G is a simplicial complex and G' its connection graph. If we chose a vertex x in G' and remove x , leading to the new complex $G' - x = (G - x)'$, then the Fredholm characteristic of $G - x$ is the Fredholm characteristic of G multiplied by a factor $i(x) = 1 - \chi(S(x))$.*

$$\psi(G - x) = (1 - \chi(S(x)))\psi(G) .$$

Proof. (i) The map $H \rightarrow X(H) = \psi(G) - \psi(G - x)$ is a valuation. (ii) Scale it to $Y(H) = X(H)/\psi(G)$. Again this is integer valued complete subgraphs.

(iii) The valuation Y takes the value 1 on complete subgraphs.

(iv) As any valuation taking the value 1 on complete subgraphs is Euler characteristic, we get $Y = \chi(H)$. This implies $\chi(S(x)) = X(S(x))/\psi(G) = (\psi(G) - \psi(G - x))/\psi(G) = 1 - \psi(G - x)/\psi(G)$ and so $\chi(S(x)) - 1 = -\psi(G - x)/\psi(G)$ which is equivalent to the claim. \square

7. UNIT SPHERE TOPOLOGY

We establish now that the set of unit sphere topologies is stable from G_1 on. Further Barycentric refinements do not produce new sphere topologies:

Theorem 1. *a) Every unit sphere $S(y)$ in G_1 is topologically equivalent to a sphere $S(x)$ in G_2 .*

b) Every unit sphere $S(x)$ in G_2 is topologically equivalent to a sphere $S(y)$ in G_1 .

Proof. a) The Barycentric refinement $S(x)_1$ is isomorphic to $S((x))$ in G_2 , if (x) is the vertex in G_2 which corresponds to the singleton vertex (x) .

b) Given now a unit sphere $S(x)$ in G_2 . It can be decomposed as $S^-(x) + S^+(x)$. If $x = (x_0, \dots, x_k)$ is the simplex in G_1 which belongs to the vertex x in G_2 , take $y = x_k$. We have seen that $S^-(y)$ is a sphere of the same dimension than $S^-(x)$. It is therefore topologically equivalent. Furthermore, we have seen that $S^+(x)$ is the Barycentric refinement of $S^-(x)$. It is also topologically equivalent. \square

Examples:

1) If G is a one dimensional graph, then the set of unit spheres of G and the set of unit spheres of G_1 agree as we have $S(x) = P_{d(x)}$ with $d(x)$ meaning the vertex degree. But this is special as the Barycentric refinement of a graph without edges is the graph itself.

2) If G is a graph with only one dimensional spheres, then this is true too for G_1 . For an octahedron or icosahedron from example, the unit spheres are cyclic graphs. After parametric refinement, this remains so.

The next lemma assures that the join operation honors topological equivalence. Now this is obvious if we look at topological realizations in some Euclidean space. Since we don't want to use the functor of topological realizations of abstract simplicial complexes to actual simplicial complexes, and since we don't need topological equivalence in the sense of topology [19], lets for the following just take a weaker notion of equivalence. We only look at the Euler characteristic because this is the only thing we really need for establishing the Barycentric invariance of the Green function diagonal values.

So, we say that two finite abstract simplicial complexes are topologically equivalent if their Euler characteristic agree:

Lemma 9. *Given two pairs of simplicial complexes A, B and C, D . If $\chi(A) = \chi(C)$ and $\chi(B) = \chi(D)$, then $\chi(A + C) = \chi(B + D)$.*

Proof. This follows from fact that $G \rightarrow 1 - \chi(G)$ is a group homomorphism. $1 - \chi(A + C) = (1 - \chi(A))(1 - \chi(C))$. \square

Note however that the join does not commute with the Barycentric refinement:

Examples.

1) The Barycentric refinement $(C_4 + C_4)_1$ of the join $C_4 + C_4$ is a graph with 80 vertices. It is larger than the join of the refinements $C_8 + C_8 = (C_4)_1 + (C_4)_1$ which is a graph with 16 vertices. Still, both are 3-spheres.

2) The Barycentric refinement $(P_3 + P_3)_1$ of the utility graph $P_3 + P_3$ is a graph with 15 vertices and 18 edges. The join of the Barycentric refinements is the utility graph itself: $(P_3)_1 + (P_3)_1 = P_3 + P_3$.

Remarks:

1) More generally if $H \sim H'$ and $K \sim K'$, then $H \sim K$ is equivalent to $H' \sim K'$. This could be shown most conveniently using an embedding.

2) It is not true that $H_1 + K_1 = (H + K)_1$ as the example of $H_1 = C_4, K_1 = P_2$ shows. In this case both are equivalent 2-spheres.

8. PRIME AND PRIME CONNECTION GRAPHS

Given an integer $n \geq 2$, we can look at two graphs G_n and H_n . Both have the vertex set $\{2, \dots, n\}$. In the **prime graph** G_n case, vertices are connected, if one divides the other. In the prime connection graph H_n , two vertices are connected, if they have a common divisor larger than 1. The graph G_n is part of the Barycentric refinement of the complete graph on the set of primes and the graph H_n is part of the connection graph of the complete graph on the set of primes.

We can now express the sphere spectrum of G_n in terms of the adjacency matrix A' of H_n . It is an application of the connection we have built between the Green function values, the diagonal values of the inverse of the Fredholm matrix, and the Euler characteristic of the corresponding sphere.

Corollary 2. $1 - \chi_{G_n}(S(x)) = (1 + A_{H_n})_{xx}^{-1}$.

Proof. Given n , let $P = \{p_1, \dots, p_k\}$ be the primes which appear as factors of square free integers in $V_n = \{2, \dots, n\}$. This defines now a abstract finite simplicial complex G . The sets are the subsets of G_n such that their product is in V_n . Because for any given A and any non-empty subset B of A also the product of the elements in B is in V_n ,

this is indeed an abstract finite simplicial complex. The Barycentric refinement G_1 of G is now the graph G_n (there should be no confusion since $n > 1$ prevents the Barycentric refinement G_1 to be mixed up with G_n). The connection graph G' is the graph H_n . \square

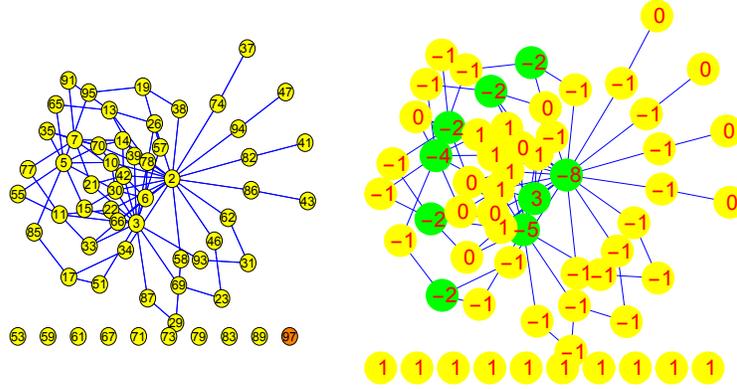


FIGURE 5. We see the prime graph G_{100} , first with the vertex labels given by square free integers. In the picture to the right we see the sphere spectrum $i(x) = 1 - \chi(S(x))$. For primes x , we have $i(x) = 1$. The unit sphere of $x = 2$ is a graph with vertex set $\{6, 10, 14, 22, 26, 30, 34, 38, 42, 46, 58, 62, 66, 70, 74, 78, 82, 86, 94\}$, which has $v_0 = 10$ connectivity components. There is furthermore a nontrivial loop $6, 42, 14, 70, 10, 30, 6$ in the unit sphere which gives a non-trivial fundamental group and $b_1 = 1$. There are no triangles in $S(x)$ so that $\chi(S(x)) = 1 - (10 - 1) = -8$.

We can wonder now how to estimate the Green function values from above or from below for the graphs G_n .

9. GRAPH ARITHMETIC

The multiplicative monoid $(\mathbb{N}, \cdot) = (\{1, 2, 3, \dots\}, \cdot)$ has the unique factorization property, telling that every non-unit can be factored uniquely into a product of prime numbers. It is a unique factorization monoid. More generally, the multiplicative group of any unique factorization domain is a unique factorization monoid. The additive monoid of $\{0, 1, 2, 3, \dots\}$ also has unique prime factorization but it is trivial since only 1 is an additive prime and any number n naturally can be decomposed as $n = 1 + 1 + 1 + \dots + 1$. The question of prime factorization

can be extended to networks. It turns out that there is both an addition and multiplication which are compatible by distributivity. We work here with graphs even so one can take more general simplicial complexes.

The set \mathbb{G} of finite simple graphs $G = (V, E)$ becomes an additive monoid with the join operation $G + H$. The definition is as follows: we have $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{(a, b) \mid a \in V(G), b \in V(H)\}$. The 0 element is the empty graph (\emptyset, \emptyset) . Adding one to G is the cone over G and adding the 0-sphere P_2 to G is the suspension of G .

The maximal dimension $\overline{\dim}(G)$ of a graph G is the dimension of the largest complete subgraph K_{d+1} which appears in G . The number $\overline{\dim}(G) + 1$ is also called the clique number. The relation $\overline{\dim}(H + K) = \overline{\dim}(H) + \overline{\dim}(K) + 1$ follows from $K_n + K_m = K_{n+m}$. (Less obvious is the same inequality for the inductive dimension $\dim(G) = \sum_{v \in V(G)} [1 + \dim(S(x))]/|V|$):

$$\dim(G + H) \geq \dim(G) + \dim(H) + 1 .$$

Lets call a graph G an additive prime graph if it can not be written as $G = H + K$ with H, K different from the 0 element. An example of an additive prime graph is K_1 . All the K_n can be factored as $K_n = K_1 + K_1 + \dots + K_1 = K_1^n$ and are therefore not prime. A star graph S_n can be factored as $P_n + K_1$ where P_n is the graph with n vertices and no edges.

Lemma 10. *Every finite simple graph G can be decomposed into additive prime factors $G = p_1 + \dots + p_n$, where p_i are prime graphs.*

Proof. If there is a factor H , then its maximal dimension has to be smaller. By induction, we can then break up every factor into smaller factors or leave it if it is an additive prime. \square

Lemma 11. *There are many additive prime graphs: every disconnected graph is an additive prime, every non-contractible graph with a prime volume is an additive prime.*

Proof. The join of two graphs is connected so that a disconnected graph must be prime. Every circular graph C_n with $n > 4$ or every point graph P_k is prime. For the circular graph, one can see that if $C_n = K + H$, then both K, H have to have maximal dimension 0. But we can list $P_1 + P_1 = K_2$, $P_1 + P_2 = L_3$, $P_2 + P_2 = C_4$. Any other $P_k + P_l$

contains a star graph S_k or S_l . If both $k, l > 2$, then this is incompatible with C_n . \square

As we can list all one-dimensional graphs which are not prime we see that most one dimensional graphs are prime. For example, every tree which is not a star graph is prime.

As in the case \mathbb{N} , the question about unique prime factorization is more difficult than the question of the existence of a factorization. Experiments indicate that graphs form a unique factorization monoid: still the question remains: is it true that every graph has a unique factorization $G = p_1 + \dots + p_n$ into prime graphs p_k .

Note that this is not true for the join in the standard topology case as if we take the join of a k -sphere and a m -sphere, we get a $k+m+1$ sphere. So, we can for example take a 3-sphere and add a 4-sphere, which gives topologically the same than adding a 2-sphere to a 5 sphere. In each case, we get a 8-sphere. In the discrete $C_4 = P_2 + P_2$ but C_5 can not be written as a sum.

Here is a formula which gives the f -vector for the join of two complexes. Given a simplicial complex with v_k simplices of dimension k . Define the extended f -vector generating function = extended Euler polynomial of G as

$$f_G(x) = 1 + \sum_{k=0} v_k x^{k+1} .$$

The empty graph has the generating function $f_G(x) = 1$. (Originally, I had defined $f_G(x) = 1/x + \sum v_k x^k$ and defining the multiplication as $xf_G(x)g_G(x)$ but June-Hou Fung pointed out to me that a multiplication with x simplifies the generating function.)

Proposition 3. *The join $G + H$ of two complexes has the f -generating function of the form*

$$f_{G+H} = f_G f_H .$$

Proof. Given a simplex x in G and a simplex y in H , this produces a simplex $x + y$ of dimension $\dim(x) + \dim(y) + 1$. \square

This formula can shed light on the unique prime factorization problem. If $G + H$ can be factored, then its rational function can be factored in the above sense. We have then for example

$$f_{p_1+p_2+p_3} = f_{p_1} f_{p_2} f_{p_3} .$$

But even if we have a unique factorization algebraically in the polynomials with multiplication $fg(x) = f(x)g(x)$, we still have the problem to see whether the factors can be realized as simplicial complexes and furthermore require that the factors have positive entries.

Since $\chi(G) = 1 + f(-1)$. We have $\chi(G+H) = 1 + (-1)f_G(-1)f_H(-1) = 1 - (\chi(G) - 1)(\chi(H) - 1) = \chi(G) + \chi(H) - \chi(G)\chi(H)$:

Corollary 3. *The Euler characteristic of the join of two complexes satisfies*

$$\chi(G + H) = \chi(G) + \chi(H) - \chi(G)\chi(H) .$$

Of especial interest is the Euler-Poincaré functional $i(G) = -f(-1) = 1 - \chi(G)$. (Again, it was June-Hou Fung who pointed out to me that the above corollary implies that $1 - \chi(G)$ is multiplicative). From the above formula we see that it behaves like a multiplicative character on the group of graphs. However, since we have to define $i(-G) = 1/i(G)$ to extend it to the entire group, we can not define $i(-G)$ for graphs G which have $i(G) = 0$. We see especially that i is not defined on the “integers” \mathbb{Z} given by the graphs K_n and their negatives $-K_n$ as $i(K_n) = 0$.

Corollary 4. *For any two simplicial complexes G, H , the functional $i(G) = 1 - \chi(G)$ satisfies*

$$i(G + H) = i(G)i(H) .$$

While $i(G)$ can be zero, we can look at the subgroup of the join group for which $i(G) \neq 0$. This is still a group.

Corollary 5. *The monoid of graphs with join $+$ has the following submonoids:*

- a) *the submonoid of graphs with even $\chi(G)$*
- b) *the submonoid of graphs with odd $\chi(G)$*
- c) *the submonoid of graphs with zero $\chi(G)$*
- d) *the submonoid of graphs with $\chi(G) \in \{0, 2\}$*
- e) *the submonoid of sphere graphs*
- f) *the submonoid of graphs with $i(G) \neq 0$.*

Proof. This all follows from the sum formulas for χ and i as well as the fact that the join leaves spheres invariant. \square

Corollary 6. *$i(G)$ can be extended to a multiplicative character on the additive group of spheres.*

Proof. It takes there the values ± 1 on the monoid. Define $i(-G) = i(G)$. Now $i(G - H) = i(G)/i(H)$. The functional i takes values in $\{-1, 1\}$. \square

Examples.

1) If $F = P_n, G = P_m$ are both 0-dimensional point graphs, then $f(P_n)f(P_m) = (1 + nx)(1 + mx) = 1 + nx + mx + nmx^2$ showing that the graph $F + G$ has nm edges and $n + m$ vertices and Euler characteristic $n + m - nm$. Looking at the number of edges, we see immediately that a 1-dimensional graph with a prime number of edges must be prime. More generally, any graph with prime volume must be prime.

2) If $G + H$ is 2-dimensional, then one of the graphs, say G has to be 0-dimensional and so $f_G(x) = 1 + nx$. If $f_H(x) = 1 + mx + kx^2$ is the extended Euler polynomial of a 1-dimensional graph. There are $f = kn$ triangles and $e = k + mn$ edges and $v = m + n$ vertices. We see also the Euler formula confirmed as $\chi(G + H) = \chi(G) + \chi(H) - \chi(G)\chi(H) = n + (m - k) - n(m - k)$ which agrees with $v - e + f$.

3) If G, H are two spheres, where at least one has Euler characteristic 2, then their product has Euler characteristic 2. Only if both have Euler characteristic 0, it is possible that the product has Euler characteristic 2. We can define the monoid of all graphs with Euler characteristic in $\{0, 2\}$. We could also look at the sub-monoid of all graphs with even Euler characteristic, or the monoid with odd Euler characteristic, or the set of graphs with 0 Euler characteristic. These graphs do not change the Euler characteristic of the graph it is multiplied with.

4) Lets look at two 1-spheres C_n and C_m . Their product is a 3-sphere with $f_{C_n+C_m}(x) = (1 + nx + nx^2)(1 + mx + mx^2)$. The new f -vector therefore is $(m + n, m + n + mn, 2mn, mn)$ which has zero Euler characteristic as $\sum_k (-1)^k v_k = 0$.

A special case of the unique prime factorization problem in the sphere monoid is the question whether any 3-sphere is either prime or then a unique sphere of two 1-spheres. In this simple case, we can answer unique factorization: it boils down to the question whether $a + b = m + n, ab = mn$ implies $m = a, n = b$ or $m = b, n = a$ which is true. But this could be a case for the Guy law of small numbers. We could imagine for example that there would be a 7-sphere which can be written as $p_1 + p_2 + p_3$ with prime 2-spheres p_i but also be able to write it as $q_1 + q_2$ as a sum of two prime 3-spheres. We have just not seen such an example yet and it probably does not exist.

As class field theory in rings of integers like $\mathbb{Z}[\sqrt{-5}]$ shows, many monoid belonging to rings of integers in function fields do not have a unique prime factorization and this could be the case here too.

Remarks.

1) The join $G + H$ can be obtained within an extended Stanley-Reisner ring as follows. If $V = \{x_1, \dots, x_n\}$ and $W = \{y_1, \dots, y_n\}$ represent the vertices of the graph G and H and $f = 1 + \sum_I a_I x^I, g = 1 + \sum_J b_J x^J$ represent the complex for G and H , then fg represents the complex for $G + H$. For example, if $f = 1 + x + y + z + xy + yz + xz + xyz$ represents the Whitney complex of $G = K_3$ and $g = 1 + a + b + ab$ represents the complex for $H = K_2$, then $fg = 1 + x + y + z + a + b + xa + xb + ya + yb + za + zb + xya + xyb + xza + xzb + yza + yzb + xyza$ represents the complex $G + H$.

2) Trying to generalize a modular arithmetic, one could try to modify the join operation to subgraphs of a given background graph P and define the graph with $V(G + H) = V(G) \Delta V(H)$ and $E(G + H)$ containing all edges contained in $V(G) \Delta V(H)$ or then connecting two different vertices $a \in V(G) \cap V(G) \Delta V(H)$ with $b \in V(H) \cap V(G) \Delta V(H)$. Mathematically we would look at a Stanley-Reisner ring with \mathbb{Z}_2 valued coefficients over a fixed number of variables. We have not succeeded to get finite rings like that.

3) Any bipartite graph is a subgraph of $P_n + P_m$ as by definition, we can split the vertices into two disjoint subsets. The graph $P_n + P_m$ is the most extreme case. The utility graph $P_3 + P_3$ is the most famous example.

Finally, lets look at the relation between the Fredholm characteristic:

Corollary 7. *On the sub monoid of graphs with even Euler characteristic, the Fredholm characteristic is multiplicative: Then $\psi(G + H) = \psi(G)\psi(H)$. This happens especially on spheres so that on that group ψ becomes a character when extended to negative graphs with $\psi(-G) = 1/\psi(G)$.*

Proof. Check cases:

If $\psi(G) = \psi(H) = 1$, then $\psi(G * H) = 1$.

If $\psi(G) = \psi(H) = -1$, then $\psi(G * H) = 1$.

If $\psi(G) = -\psi(H) = 1$, then $\psi(G * H) = -1$. □

Corollary 8. *In the sub-monoid of graphs with odd Euler characteristic, then $\psi(G + H) = -1$. This implies that if $\psi(G) = 1$, then G is prime in that monoid.*

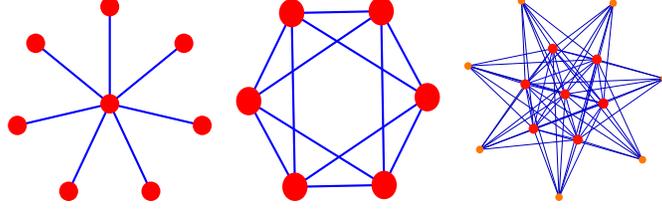


FIGURE 6. We see the addition of a star graph S_7 and an octahedron graph. Since the star graph is a cone $S_7 = P_7 + K_1 = P_7 + 1$ and the Octahedron graph factors into $\text{Oct} = K_2 + C_4 = K_2 + K_2 + K_2 = K_3 \cdot K_2$, the graph to the right has the additive prime decomposition $P_7 + K_1 + K_2 + K_2 + K_2$. In this case, the additive prime decomposition is unique. In all cases we have seen so far, the composition is unique, indicating that $(\mathbb{G}, +)$ could be a unique factorization monoid.

Proof. Check cases:

If $\psi(G) = \psi(H) = 1$, then $\psi(G + H) = -1$.

If $\psi(G) = \psi(H) = -1$, then $\psi(G + H) = -1$.

If $\psi(G) = -\psi(H) = 1$, then $\psi(G + H) = -1$. □

Examples.

- 1) Among complete graphs only K_1 has Fredholm characteristic 1. All others have Fredholm characteristic -1 and can therefore not be prime.
- 2) There is a two dimensional graph G with the topology of the projective plane. Its f -vector is $(15, 42, 28)$ so that $\psi(G) = 1$. Its Euler characteristic is 1 too. Therefore, G must be prime.
- 3) A tree with an even number of edges must be prime in the monoid of odd Euler characteristic graphs.

When mentioning the arithmetic of graphs to An Huang, he asked me about relations of the spectrum of the join and the individual components. Here are three remarks:

Lemma 12. *If $G = H + K$ then there is an eigenvalue $|V(H)| + |V(K)|$ of the scalar Laplacian L . Consequently, any graph G for which there is no eigenvalue $v_0(G)$ for L is prime.*

Proof. If $m = |V(H)|$ and $n = |V(K)|$, then the vector which is constant n on H and constant m on K is an eigenvector to the Laplacian L with eigenvalue $n + m$. The eigenvector is perpendicular to the constant. □

Corollary 9. *The graph $nG = G + G + \dots + G$ has the eigenvalue $n|V(G)|$ with multiplicity at least $n - 1$.*

Proof. The multiplicity follows because we can write the sum in $n - 1$ different ways as a sum of two graphs $A+B$. The proof above determine then the eigenvector. \square

The example $K_n = K_1 + K_1 + \dots + K_1$, where n appears with multiplicity $n - 1$ is an extreme case. An other case is the $(n - 1)$ -dimensional cross polytop $nP_2 = P_2 + \dots + P_2$ which has the eigenvalue $2n$ with multiplicity $n - 1$.

Of interest is also the smallest non-zero eigenvalue λ_2 of the Laplacian.

Lemma 13. $\lambda_2(G + H) = \min(|V(H)|, |V(K)|) + \min(\lambda_2(G), \lambda_2(H))$.

Proof. The Courant-Fischer formula tells $\lambda_2 = \inf_{v \cdot 1=0} (v, Lv)/(v, v)$. Let $v_2(H)$ be the eigenvector of $L(H)$ with eigenvalue $\lambda_2(H)$ and $v_2(G)$ the eigenvector of $L(H)$ with eigenvalue $\lambda_2(G)$. Assume they are normalized. Extend $v_2(G)$ and $v_2(H)$ onto $G+H$ by putting 0 on the other part. They still have norm 1 on the product space. They are also perpendicular to the constant vector. Now $(v_2(H), Lv_2(H)) = \lambda_2(H) + m$ and $(v_2(G), Lv_2(G)) = \lambda_2(G) + n$. This shows

$$\lambda_2(G + H) \leq \min(|V(H)|, |V(K)|) + \min(\lambda_2(G), \lambda_2(H)) .$$

On the other hand, since $G + H$ can be obtained from the disjoint union of G and P_n (the graph without edges), by adding edges, we have $\lambda_i(G) + n \leq \lambda_i(G + H)$. Similarly $\lambda_i(H) + m \leq \lambda_i(G + H)$. See Corollary 4.4.2. in [33]. \square

Here is an other spectral result. It deals with the highest form Laplacian. Let $D = d + d^*$ be the Dirac operator of a simplicial complex. The form Laplacian $L = (d + d^*)^2$ splits into blocks called form Laplacians L_k , the restriction of L on discrete k -forms. The nullity of L_k is the k 'th Betti number b_k [17, 16]. We have already seen that if we have two graphs G, H , then the volume of $G + H$ is the product of the volumes of G and volumes of H . Lets write $L_v(G)$ for $L_{\dim(G)}(G)$ for the volume Laplacian, the Laplacian belonging to the largest dimension. Now if G has volume m and H has volume n then we can look at the volume eigenvalues $\lambda_1, \dots, \lambda_n$ of $L_v(G)$ and the volume eigenvalues μ_1, \dots, μ_m of $L_v(H)$. What are the volume eigenvalues of $H + G$?

Lemma 14. *For two arbitrary simplicial complexes G, H , the volume eigenvalues of $G + H$ are given by $\lambda + \mu$, where λ runs over the volume eigenvalues of G and μ runs over the volume eigenvalues of H .*

Proof. If U is the volume Laplacian of G and K is the volume Laplacian of H . Given $Uv = \lambda v$ and $Vw = \mu w$. The volume Laplacian W of $G + H$ is a $(nm) \times (nm)$ matrix because the volumes of G and K have multiplied. Define on the facet xy of $G + H$ joining the facets x and y of G and H the value $f_{xy} = v_x w_y$. Now $Wf = \lambda f + \mu f$. \square

10. A RING OF NETWORKS

The group $(X, +, 0)$ with Zykov addition $+$ (graph join) has a compatible multiplication. We formulate it only in the graph case, meaning the case of simplicial complexes which are Whitney complexes of graphs, even so it would also work for general simplicial complexes.

Given two graphs $G = (V, E), H = (W, F)$, define the graph $G \cdot H = (V \times W, Q)$, where the vertex set $V \times W$ is the Cartesian product of the two vertex sets and where the edge set Q consists of all pairs

$$\{((a, b), (c, d) \mid (a, c) \in E \text{ or } (b, d) \in F\} .$$

This means that we connect (a, b) with any (c, x) if $(a, c) \in E$ and with any (a, b) with (x, d) if $(b, d) \in F$. This product is obviously associative and commutative.

Lemma 15. *This product is compatible with the join addition 0 in the sense that the distributivity law $G \cdot (H + K) = G \cdot H + G \cdot K$ holds.*

Proof. Like in the case of the ring of integers, spacial geometric insight can help by placing the graphs H, K and $H+K$ into one coordinate axes and G onto the other. Now both $G \cdot H + G \cdot K$ and $G \cdot (H + K)$ have as the vertex set the product sets of the vertices. As connections in a sum $H+K$ consist of three types, connections within H , connections within K and any possible connection between H and K , two points $(a, b), (c, d)$ are connected if either (a, c) is an edge in H , or (b, d) is an edge in K or then if either a, c or b, d are in different graphs. \square

Examples:

All equality signs mean here “graph isomorphic”.

- a) $P_n \cdot P_m = P_{nm}$
- b) $K_n + K_m = K_{n+m}$
- c) $K_n \cdot K_m = K_{nm}$
- d) $1 + G$ is cone over G
- e) $1 + C_n = W_n$ wheel graph
- f) $P_2 + G$ suspension of G
- g) $P_2 + C_4 = P_2 + P_2 + P_2 = K_3 \cdot P_2$ octahedron graph: Oct
- h) $K_{3,3} = P_3 + P_3 = K_2 \cdot P_3$ utility graph

- i) $K_{n,m} = P_n + P_m$ complete bipartite graph
- j) $1 + P_n = S_n$ star graph
- k) $1 + S_3 = 1 + 1 + P_n = K_2 + P_n$ windmill graph
- i) $K_2 \cdot C_4 = C_4 + C_4$ three sphere
- j) $K_3 \cdot \text{Oct} = \text{Oct} + \text{Oct} + \text{Oct}$ is an 7-dimensional sphere
- k) $K_{n,m} + K_{n,m} = K_2 \cdot K_{n,m} = K_2(P_n + P_m) = K_{n,n} + K_{m,m}$
- l) $W_4 + P_2 = 1 + C_4 + 1 + C_4 = K_2 + C_4 + C_4$.

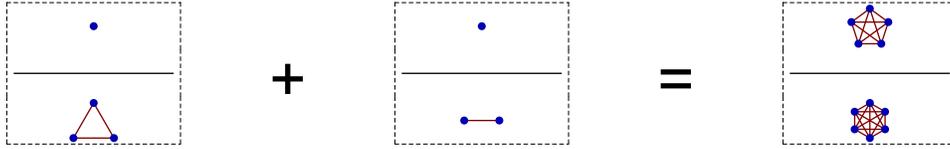


FIGURE 7. Working in the field generated by the ring of networks is analogues to work with fractions. In school arithmetic, we start with the monoid $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, then introduce the integers \mathbb{Z} , then build multiplication leading to the ring of integers and finally produce the field of fractions. Here we see the identity $1/3 + 1/2 = 5/6$ in network arithmetic. All graphs involved are here complete graphs where arithmetic is equivalent to the arithmetic in the rationals \mathbb{Q} .

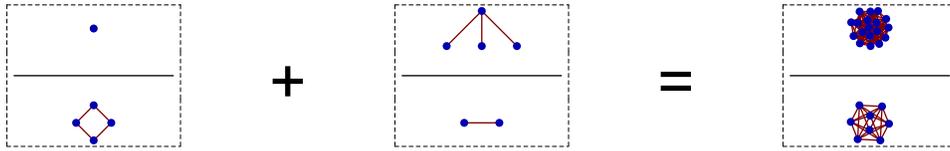


FIGURE 8. While the field of networks contains the field \mathbb{Q} as $K_n + K_m = K_{n+m}$ and $K_n \cdot K_m = K_{nm}$, we can work with more general graphs. Here we see the result of the addition of two fractions: $1/C_4 + S_3/K_2$, where C_4 is the circular graph and S_3 is the star graph. The addition is done as in school arithmetic. We get $K_1 \cdot K_2 + S_3 \cdot C_4 = K_2 + S_3 \cdot C_4$ in the nominator and $C_4 \cdot K_2$ in the denominator.

One of the consequences of the fact that we know the maximal eigenvalue of $G + H$ is that we can get the maximal eigenvalue of some graphs immediately.

Corollary 10. *For $n > 1$, the maximal eigenvalue of $K_n \cdot G$ is $n|V(G)|$.*

This implies for example that the maximal eigenvalue of K_n is n for $n > 1$ and the maximal eigenvalue for $K_{n,m}$ is $n + m$ for $n, m \geq 1$. This is extremal as $\lambda_k \leq |V(G)|$ for all eigenvalue. It also shows that because $K_n P_2$ is the $n - 1$ dimensional sphere (cross polytop), that the maximal eigenvalue is $2n$. Indeed, the octahedron $K_3 \cdot P_2$ for example has maximal eigenvalue 6 and the 16-cell $K_4 \cdot P_2$ has maximal eigenvalue 8 with multiplicity at least 3.

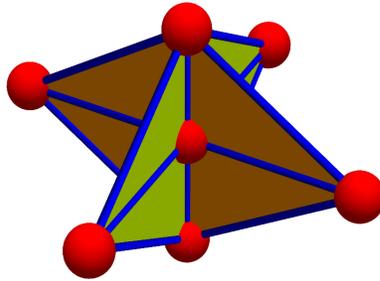


FIGURE 9. The graph $1 + P_4 + P_2 = S_4 + P_2$ is the suspension of the star graph $S_4 = 1 + P_4$. As a non-prime graph, its maximal eigenvalue is the number of vertices, which is 7. Since P_4 has only 0 eigenvalues, the graph $S_4 = 1 + P_4$ has $\lambda_2 = 1$. According to the spectral lemma for λ_2 above, we know that $P_2 + S_4$ has eigenvalue $2 + 0 + 1 = 3$.

11. QUESTIONS

Lets finish with some questions.

A) We still have not identified the off diagonal terms $(1 + A(G'))_{xy}^{-1}$ in topological terms, where G' is the connection graph of a complex. Having the diagonal terms related to natural sphere index functionals, which are very closely related to Morse theory, it is likely that also the

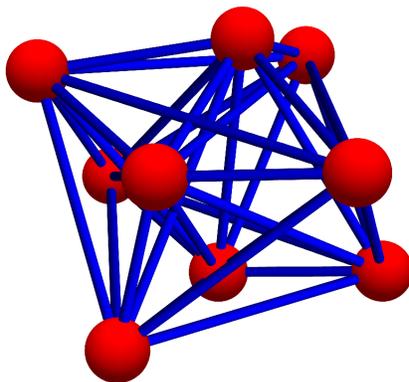


FIGURE 10. The graph $K_3 \cdot L_2 = K_3 \cdot (1 + P_2) = K_3 + K_3 \cdot P_2 = K_3 + \text{Oct} = 1 + 1 + 1 + \text{Oct}$ is the sum of the triangle and the octahedron but it is also 3 times the linear graph L_2 of length 2. It is also the suspension of the cone of the octahedron or the cone of a 3 sphere. As the graph is not an additive prime, its maximal eigenvalue is the number of vertices, which is 9. The multiplicity result above only assures the maximal eigenvalue to appear with multiplicity 2. The eigenvalue 9 actually appears with multiplicity 5.

off diagonal terms refer to some interesting topological or dynamical notion.

B) Since we have identified two characters $i(G) = 1 - \chi(G)$ and $\psi(G)$ on the sphere group (a homomorphism from the sphere group to the group $\{|z| = 1\} | z \in \mathbb{C}\}$), it is natural to ask whether there is more representation theory of the sphere group for which the character are classical characters, represented by a trace of a matrix associated to a sphere. We especially would like to know the collection of all characters. Since the sphere group is discrete, one could take any distance (like the minimal number of vertices or edges to be modified to get from one to an isomorphic image of the other). This is then a topological

group which has a Haar measure as discreteness assures that it is locally compact and such that the Haar measure, the counting measure. There are many characters on the sphere group like assigning real numbers $k(x)$ to every vertex $x \in V(G)$ of a graph G and $-k(x)$ to vertices of $-G$. A character is then defined as

$$X(G - H) = \prod_{x \in V(G)} e^{2\pi i k(x)} / \prod_{x \in V(H)} e^{2\pi i k(x)} .$$

Since adding two graphs takes the union of the vertices, this is multiplicative for the additive group of spheres. But as both $i(G)$ and $\psi(G)$ tap into the structure of the simplices in the graph, it is likely that there are more interesting examples of characters.

C) The join and multiplication in which the addition is not extended yet, forms a commutative semiring of type $(2, 2, 0, 0)$: both addition and multiplication are commutative monoids, distributivity holds and $a \cdot 0 = 0$. The question of unique prime decomposition in the additive and multiplicative monoids of abstract finite simplicial complexes is interesting. Maybe the question is more approachable on submonoids like the sphere monoid. On the multiplicative submonoid of complete graphs, the prime factorization is the same than on the multiplicative group of natural numbers without 0.

D) We would also like to be able to compute in the field of fractions defined by the ring and then do a completion solve equations like $G + G = C_5$ leading to $G = C_5/2$ or solving $G \cdot G = K_5$ with solution $\sqrt{K_5}$. Both equations can not be solved in the ring because C_5 is an additive prime (every cyclic graph C_n with $n > 4$ is an additive prime) and K_5 is a multiplicative prime because every graph K_p with p vertices is a multiplicative prime if p is a rational prime.

E) There are interesting questions related to the number theoretically defined prime graphs G_n and prime connection graphs H_n . One can for example try to estimate the Green function values from above. The topology of G_n is linked the Riemann hypothesis as it is directly linked to the growth of the Euler characteristic of G_n by the Mertens connection [25]. The Fredholm characteristic of truncations of truncated prime graphs on the other hand gives a signature, comparing the parity of square free integers $\leq n$ which have an even number of prime factors. Such integers correspond to odd-dimensional simplices in the complex. The Green function values for prime graphs, the matrix entries of $(1 + A(H_n))_{xy}^{-1}$ could contain some interesting number theory.

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