

# SOFT BARYCENTRIC REFINEMENTS

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**ABSTRACT.** The soft Barycentric refinement preserves manifolds with or without boundary. In every dimension larger than one, there is a universal spectral central limiting measure that has affinities with the Barycentric limiting measure one dimension lower. Ricci type quantities like the length of the dual sphere of co-dimension-2 simplex stay invariant under soft refinements. We prove that the dual graphs of any manifold can be colored with 3 colors, which is in the 2-dimensional case a special case of the Grötzsch theorem. It follows that the vertices of a soft Barycentric refined  $q$ -manifold  $G'$  can be colored by  $q + 1$  or  $q + 2$  colors.

## 1. INTRODUCTION

**1.1. A finite simple graph**  $G = (V, E)$  defines a **finite abstract simplicial complex**  $V_1 = \{x = V(H), H \sim K_k \text{ subgraph of } G\}$ , a finite set of non-empty sets closed under the operation of taking finite non-empty subsets. This set of sets  $V_1$  defines the **Barycentric refinement graph**  $\psi(G) = G_1 = (V_1, E_1)$  with vertex set  $V_1$  and edge set  $E_1 = \{(a, b) \in V_1 \times V_1, a \neq b, a \subset b \mid b \subset a\}$ . We define here a **soft Barycentric refinement**. In manifolds without boundary it disregards the co-dimension one simplices  $W$  as vertices and adds edges between maximal simplices which have an intersection in  $W$ . It has similar properties than the Barycentric refinement but it does not increase vertex degrees in 2 dimensions. We focus in this article on two subjects, **1)** the soft Barycentric limit and **2)** the chromatic number of softly refined manifolds. In both of these two stories, the dual  $\hat{G}$  of a manifold  $G$  plays an important role. The skeleton graph  $\hat{G}$  has the facets of  $G$  as points and connects two if they intersect in  $W$ . If the manifold  $G$  is orientable, it can naturally be given a **cell structure** making a dual manifold of  $G$  with reversed f-vector  $(f_d, \dots, f_0)$  of  $G$  and reversed Betti vector  $(b_d, \dots, b_0)$  of  $G$ .

**1.2.** Let us first summarize what we will prove about the **chromatology** of softly refined manifolds  $G$ . The **chromatic number** of  $G$  is defined as the chromatic number of the 1-skeleton complex of  $G$ . It is the minimal number  $c \geq 1$  for which there is a locally injective function  $g : V(G) \rightarrow \{1, 2, \dots, c\}$ , from the vertex list  $V(G)$  of  $G$  ( $V(G)$  is the set of elements in  $G$  with cardinality 1)) to a finite set of colors, where **locally injective** means that  $g(a) \neq g(b)$ , whenever  $(a, b)$  is in the edge set  $E$ , the sets of cardinality 2 in  $G$ . If  $G$  has maximal dimension  $q$ , then  $c \geq q + 1$ . But it can be much larger in general. There are complexes with  $q = 1$  (equivalently triangle-free graphs), for which the chromatic number can be arbitrarily large. We always have  $c(G_1) = q + 1$  for any Barycentric refined graph  $G_1$  because  $g(x) = \dim(x) + 1 \in \{1, 2, \dots, q_1\}$  is an explicit coloring of the vertices  $V_1$  of  $G_1$  ( $V_1$  are the simplices of  $G$ ): given two simplices  $a, b \in V_1$ , the relation  $a \subset b, a \neq b$  forces  $g(a) < g(b)$ . We will see that in the case of a soft refinement, we always have a dichotomy  $c(G') = q + 1$  or  $c(G') = q + 2$ .

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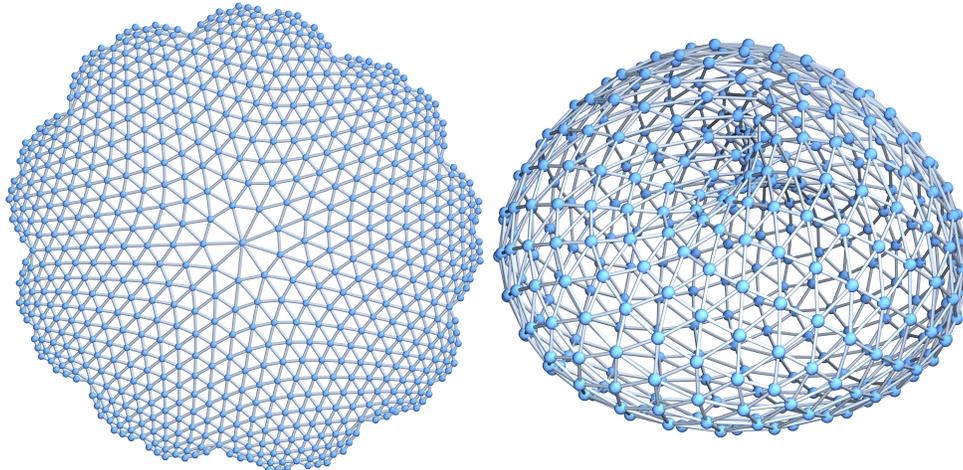


FIGURE 1. We see to the left the 4'th soft Barycentric refinement of a wheel graph with chromatic number  $c(G) = c(\phi^4(G)) = 4$  and the 3'rd softly refined projective plane  $G$  with  $c(\phi^3(G)) = 5$ . No 2-manifold with  $c(\phi(G)) < c(G)$  is known. Giving such an example would settle a conjecture of Albertson and Stromquist [1] which says  $c(G) \leq 5$  for 2-manifolds. The smallest manifold with  $c(\phi(G)) < c(G)$  we know of is the  $q = 5$ -sphere  $G = C_5 \oplus C_5 \oplus C_5$  (the graph join of three 1-spheres) with  $c(G) = 3c(C_5) = 3 * 3 = 9$  for which our result shows  $c(\phi(G)) = q + 2 = 7$ .

**1.3.** Now to **universality**: the spectrum of the Kirchhoff Laplacians  $K(G_n)$  converges in law to a universal measure  $d\mu_q$ , which only depends on the maximal dimension  $q$  of  $G$  [4, 5]. In the case  $q = 1$ , both in the Kirchhoff as well as in the Hodge Laplacian case, the limiting density of states is the **arc-sin** distribution on  $[0, 4]$ , which is the unique potential theoretic equilibrium measure on that set. In the case  $d = 2$ , the limiting central limit measure already showed Cantor-like feature. We so far only know that this measure exists for every  $q$  and that it has finite support if and only if  $q = 1$ . We will note here that also soft Barycentric refinement limits exist. The motivation to look at limiting measures is motivated also when comparing the relation between the number of trees and forests in a graph, as we will see in the next paragraph.

**1.4.** If  $\mu_q$  is either the soft or strong Barycentric limit measure in dimension  $q$ , then the **potential**  $f(z) = - \int_{\mathbb{C}} \log |z - z'| d\mu_q(z')$  is a complex-valued function that is analytic outside the support of the Barycentric limit measure  $\mu_q$  in dimension  $q$ . The potential is of great interest in the Kirchhoff Laplacian case, as it encodes exponential growth rates of trees and forests.

**1.5.** In the case of the Kirchhoff Laplacian, by the **matrix tree theorem**, the **tree index**  $f(0)$  is the exponential growth rate of the number of rooted spanning trees in  $G_n$  as  $n \rightarrow \infty$ . By the **matrix forest theorem**, the forest index  $f(-1)$  measures the exponential growth rate of the number of rooted spanning forests in  $G_n$ . If  $\lambda$  are the eigenvalues of the Kirchhoff Laplacian  $K$ , then the **pseudo determinant**  $\text{Det}(L) = \prod_{\lambda \neq 0} \lambda$  is the number of rooted spanning trees and  $\det(1 + K) = \prod_{\lambda} (1 + \lambda)$  is the number of rooted spanning forests. The tree-forest index  $\tau(G) = \det(1 + K)/\det(K)$  converges in the limit to a number that only depend on the size of the maximal clique in the graph.

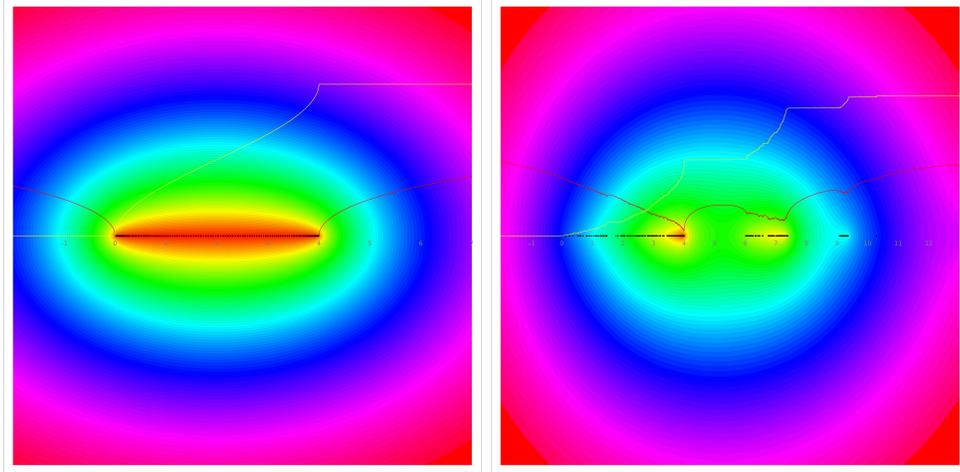


FIGURE 2. We see approximations of the limiting measure for Barycentric refinement in dimension  $q = 1$  and  $q = 2$ . These measures are supported on the real half line  $[0, \infty) \subset \mathbb{R} \subset \mathbb{C}$ . The existence of the limiting measures in any dimension  $q$  is an older story [4, 5].

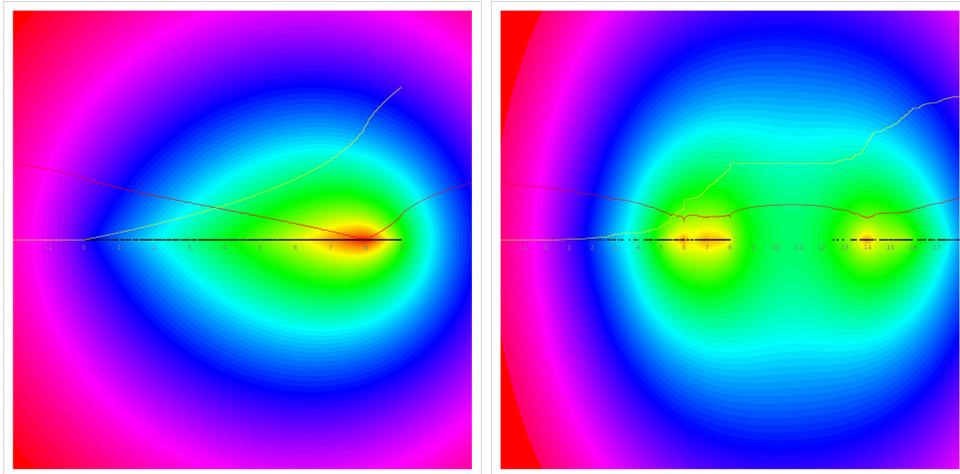


FIGURE 3. We see approximations of the limiting measure for the soft Barycentric refinement limit in dimension  $q = 2$  and  $q = 3$ . This is new. For  $q = 2$ , we have still an absolutely continuous measure of compact support. A Fourier transform allows to express the integrated density of states. It shows a **van Hove singularity** [3], which is related to critical points of the Fourier transform  $\hat{L}(x, y) = 6 - 2 \cos(x) - 2 \cos(y) - 2 \cos(x + y)$  on  $L^2(\mathbb{T}^2)$ , the Laplacian  $L$  of the hex lattice.

**1.6.** The fact that the potential values  $f(0)$  and  $f(1)$  exist, follows from general spectral estimates  $\lambda_k \leq 2d_k$ , where  $\lambda_k \leq \lambda_{k+1}$  are the eigenvalues and  $d_k \leq d_{k+1}$  are the vertex degrees of the graph (see [6]). The quest to estimate the potential  $f(z)$  prompted the research for [7].

**1.7.** The **facets** of  $G$  are the  $q$ -simplices of  $G$ , where  $q$  is the **maximal dimension** of  $G$ . The **boundary faces** of  $G$  are the  $(q - 1)$  simplices of  $G$  that are contained in precisely one facet

of  $G$  The other  $(q - 1)$ -simplices are either **interior faces**, the intersection of two facets, or **singular faces**, the intersection of three or more facets.

**1.8. Definition.** The **soft Whitney complex** of a graph  $G$  is the set of sub-simplices which consist of either  $k$ -simplices in  $G$  for  $k \neq q - 1$ , or then of **boundary facets**, meaning  $(q - 1)$  simplices at the boundary. A boundary facet is a  $(q - 1)$  simplex which is part of exactly one  $q$ -simplex. We can get to the Barycentric refinement from the soft Barycentric refinement by edge refining all the edges connecting facets as such a subdivision.

**1.9.** The **soft Barycentric refinement**  $G_1 = \phi(G)$  of a graph  $G$  has as vertex set the elements in the **soft Whitney complex** and connects two such points if one is contained in the other or if they intersect in an **interior face**, a  $(q - 1)$  simplex which is contained in exactly two  $q$ -simplices.

**1.10.** For  $G = K_{q+1}$  with  $q > 0$  for example, the refinements  $\phi^n(G)$  for  $n > 0$  are balls, meaning  $q$ -manifolds with  $(q - 1)$ -spheres as boundary. For a manifold  $G$ , the growth rate of the  $f$ -vectors of the soft Barycentric refinements  $G'_n$  is slower than the growth rate of the Barycentric refinements  $G_n$ . In dimension 2, a second refinement  $\phi^2$  agrees with the **Loop refinement** [10] defined by Loop in 1978 and the vertex degree stays bounded. For example, if  $G$  is an icosahedron, then  $G' = \phi(G)$  is a stellated dodecahedron. The vertex degree of 2-dimensional manifolds does not grow. We have made the definitions in such a way that singular faces,  $(q - 1)$ -dimensional faces which are the intersection of 3 or more facets, do not enter in the refinement.

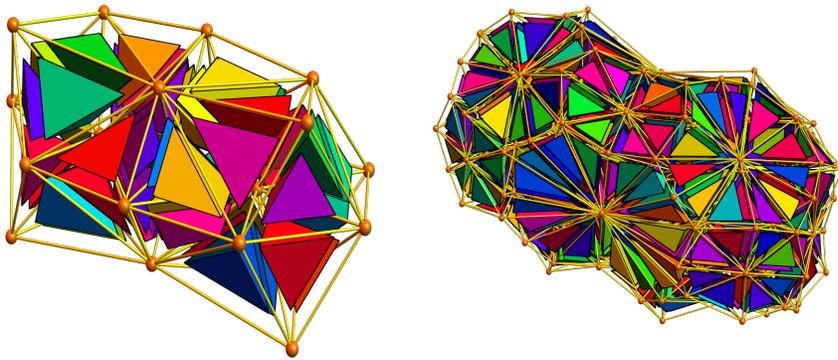


FIGURE 4. The figure shows a 3-manifold with boundary and its first soft Barycentric refinement. Our definition of soft Barycentric refinement features that the boundary undergoes the usual Barycentric refinement.

**1.11.** A finite simple graph is called a  $q$ -**manifold**, if for every vertex  $v \in V$ , the unit sphere  $S(v)$ , the subgraph generated by all neighbors of  $v$ , is a  $(q - 1)$ -sphere. A  $q$ -manifold  $G$  is called a  $q$ -**sphere**, if there is a vertex  $v$  such that  $G \setminus v$  is contractible. A graph  $G$  is called **contractible**, if there exists a vertex  $v$  such that both  $S(v)$  and  $G \setminus v$  are both contractible. These inductive definitions start by declaring  $1 = K_1$  to be contractible and that the empty graph  $0$  to be a  $(-1)$ -sphere. A punctured  $q$ -sphere  $G \setminus v$  if  $G$  is a  $d$ -sphere is called a  $q$ -**ball**. A  $q$ -**manifold with boundary** is a graph such that all unit spheres are either  $(q - 1)$ -spheres

or  $(q - 1)$ -balls. Figure 1 illustrates the following lemma in dimension 2. A wheel graph is a 2-ball and a projective plane is a 2-manifold, meaning a 2-manifold without boundary:

**Lemma 1.** *Soft Barycentric refinement preserves manifolds with and without boundary.*

*Proof.* (i) Let us look first at manifolds without boundary.

A 0-dimensional graph is always a 0-dimensional manifold and a fixed point of  $\phi$ . In one dimensions, a connected manifold is  $C_n$  with  $n \geq 4$ . It is also a fixed point of  $\phi$ . Let now  $G$  be a  $d$ -manifold for  $d \geq 2$ . If in the graph  $G_1$  every edge  $e$  which came from connecting two  $d$ -facets is **edge refined**, (meaning that a new vertex is introduced for  $e = (a, b)$  and connected to all  $S(a) \cap S(b)$ ), we get the Barycentric refinement of  $G$ . In other words, the weakly refined graph  $\phi(G)$  can be obtained from the (strong) Barycentric refinement  $\psi(G)$  by edge collapses of  $(q - 1)$  and  $q$  simplices. (ii) For a manifold  $G$  with boundary  $\delta G$ , we again can just see  $\phi(G)$  can be obtained from  $\psi(G)$  by edge collapses of interior edges connecting a  $q - 1$  simplex with a  $q$  simplex. The proof in (i) shows that the unit spheres in the interior are  $(q - 1)$  spheres. From  $\delta\phi(G) = \psi\delta(G)$  and the fact that the Barycentric refinement operation  $\psi$  preserves the class of manifolds the boundary is a  $(q - 1)$ - manifold.  $\square$

**1.12.** The eigenvalues  $\lambda_j$  (ordered in increasing order  $\lambda_j \leq \lambda_{j+1}$ ) of the Kirchhoff Laplacian give rise to a **spectral function**  $F_G(x) = \lambda_{[nx]}$ , which is encoding the eigenvalues  $0 = \lambda_0 \leq \dots \leq \lambda_n$ . The function  $F$  is piecewise constant, monotone and  $F_G(0) = \lambda_0$  and  $F_G(1) = \lambda_n$ . The **integrated density of states**  $F^{-1}$  is a monotone  $[0, 1]$ -valued function on  $[0, \infty)$ . Its derivative is in general only defined in a distributional sense and defines the **density of states**, a probability measure  $dk = (F^{-1})'$  on  $\mathbb{R}^+$ . For finite  $G$ , the measure  $dk$  is a discrete pure point measure with support on the spectrum of  $L$  (which is a finite set of points). Point-wise convergence of  $F$  implies point-wise convergence of  $F^{-1}$  and so weak-\* convergence of the density of states.

**1.13.** For two graphs on the same vertex set, the **graph distance**  $d(G, H)$  is the minimal number of edges which need to be modified in order to get from  $G$  to  $H$ . In general no host graph is a priori given, in which  $G, H$  can be compared. In that case, we define  $n(G, H)$  to be the minimum number of vertices which a host graph containing both  $G, H$  can have. This means that the distance function  $d(G, H)$  can be extended to graphs which do not have the same vertex set; we just define  $d(G, H)$  as the minimal possible distance we can get by embedding both in a common graph with  $n(G, H)$  elements. It is possible to formulate this differently: the graph distance is the minimal cardinality of the symmetric difference  $E(G)\Delta E(H)$  if  $G, H$  are both seen as a sub-graph in a complete graph  $K_m$ . This distance does not depend on the size  $m$  of the host graph  $K_m$  as long as both  $G, H$  are subgraphs of  $K_m$ .

**1.14.** The proof of the following theorem parallels the proof in the case of the Barycentric refinement  $\psi$ . The proof itself can not be carried over verbatim because the number of simplices of dimension  $q$  and  $q - 1$  essentially grow with the same rate, rendering the multi-scale argument invalid. The argument can be modified easily however. We can split the graph into different identical pieces, evolve each piece separately. The boundary part is renormalized using Barycentric refinement but it grows slower than the interior. If we start with a manifold with boundary, then the boundary can be split.

**Theorem 1.** *In any dimension  $q > 1$ , there is a limiting law for the soft Barycentric refinement. It is universal in the sense, that it only depends on the maximal dimension  $q$  of  $G$ .*

*Proof.* Some of the arguments go over directly from the proof in the Barycentric case. First of all: (i)  $\|F_G - F_H\|_1 \leq 4d(G, H)/n(G, H)$ . Proof: The Kirchhoff Laplacians  $L, K$  of  $G, H$  satisfy  $\sum_{i,j} |L_{ij} - K_{ij}| \leq 4d(G, H)$  because each edge  $(i, j)$  affects only the four matrix entries  $L_{ij}, L_{ji}, L_{ii}, L_{jj}$ . By the **Lidskii-Last inequality**: [12, 9] for any two symmetric  $n \times n$  matrices  $A, B$  with eigenvalues  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$  and  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$ . For two subgraphs  $G, H$  of a common graph with  $n$  vertices, and Laplacians  $L, H$ , the inequality gives  $\|\lambda - \mu\|_1 \leq 4d(G, H)$  so that  $\|F_{G'} - F_{H'}\| \leq 4d(G, H)/n$  if  $G', H'$  are the graphs with edge set of  $G$  and vertex set of the host graph having  $n(G, H)$  vertices. The lemma follows.

(ii) We now check the result for a single  $q$ -simplex  $G = K_{q+1}$ . The soft refined graph  $G_n = \phi^n(G)$  is for  $n > 1$  a union of  $q$  manifolds  $G_{n,k}$  with boundary  $T_{n,k} \cup S_{n,k}$ , where  $S_{n,k}$  is the  $n$ 'th Barycentric refinement of the  $k$ 'th  $(q - 1)$  boundary simplex. While the growth rate of the simplices in  $T_n \cup S_n$  has a similar exponential growth We use that the intersection of two different  $G_{n,k}$  and  $G_{n,l}$  is either empty or then one of the  $T_{n,k}$ , which grows with a different scale. If  $G$  is the disjoint copy of  $(q + 1)$  copies of  $H$ , then  $F(G) = F(H)$ . There exists a constant  $C_q$  such that  $\|F_{G_n} - F_{G_{n+1}}\|_1 \leq C_q \frac{1}{\gamma^n}$  where  $\gamma = 1/(q + 1)$ . So,  $F_{G_n}$  is a Cauchy sequence in  $L^1([0, 1])$  having a limit in this Banach space. The limiting  $F$  produces a limiting measure  $dk$ .

(iii) In general, the boundary of  $G_n$  grows slower than  $G_n$ , even so it is not exponentially slower. Let  $U_n$  denote the interior, the graph generated by interior points. Then  $|F_{U_n} - F_{G_n}|_1 \rightarrow 0$ . A general initial simplicial complex  $G$ , we write it as a union of the open set  $U$  consisting of maximal simplices and the closed set  $K$  which is  $G \setminus U$  and is the **skeleton complex** of dimension  $q - 1$ .

If  $U_n$  is the interior of the refinement  $\phi^n(\bar{U})$ ,  $K_n = \phi^n(K)$  and  $G_n = \phi^n(G)$ , then  $U_n \cup K_n = G_n$ . Again we have  $|F_{U_n} - F_{G_n}|_1 \rightarrow 0$  and since  $U$  is disjoint union of identical parts, the result reduces to the situation in (i).  $\square$

## 2. CHROMATIC NUMBER

**2.1.** We now turn to the second topic of this paper. The **chromatic number**  $c(G)$  of a graph  $G = (V, E)$  is the minimal number of colors  $C = \{1, \dots, c\}$  needed so that there exists a locally injective map  $f : V \rightarrow C$ , meaning  $f(v) \neq f(w)$  if  $(v, w) \in E$ . The **vertex arboricity**  $a(G)$  of  $G$  is the minimal number of forests that are needed to cover  $V$  in such a way that every tree generates itself in  $G$ . (A subgraph  $H$  of  $G$  generates itself if all pairs  $(a, b) \in E(G)$  with  $a, b \in V(H)$  has  $(a, b) \in E(H)$ .) The chromatic number could also be called **vertex seed arboricity** because it asks to cover  $V$  with **seeds** in  $G$ , (a seed collection is forest in which every tree is a single vertex). There is the general relation  $c(G)/2 \leq a(G) \leq c(G)$ : the right inequality holds because every seed is a tree, the left inequality holds because every tree can be colored by 2 colors, leading to  $c(G) \leq 2a(G)$ . Both vertex arboricity and chromatic number are difficult quantities to compute in general. This is in contrast to edge arboricity that can be tackled thanks to the **Nash-Williams** theorem.

**2.2.** Let  $\hat{G}$  denote the **dual manifold** of  $G$ . It has the maximal  $q$ -simplices as vertices and connects two if they intersect in a  $(q - 1)$  simplex. The dual manifold is a graph without triangles. We can attach  $k$ -cells to the duals of  $(q - k)$ -dimensional simplices and so get a natural dual CW complex defined by  $G$ . The  $f$ -vector of  $\hat{G}$  is  $(f_d, f_{d-1}, \dots, f_0)$ . The dual graph has nice features and can be used to define a geodesic flow and sectional curvature as

we will work out elsewhere. The reason to work in the dual is that for paths in  $G$ , the radius of injectivity is always 1. For the dual graph, there is even a chance to have a unique shortest connection between any two points. This happens for example for the octahedron.

**2.3.** Let  $c(\hat{G})$  denote the chromatic number of  $\hat{G}$  and let  $a(\hat{G})$  denote the vertex arboricity of  $\hat{G}$ . The following result can be seen as a version of Grötzsch's theorem, even so it does not prove for  $q = 2$  the more comprehensive Grötzsch's theorem for planar graphs.

**Theorem 2** (Dual 2 forest and 3 color theorem). *For any  $q$ -manifold,  $a(\hat{G}) = 2$  and  $c(\hat{G}) \leq 3$ .*

We will prove this by establishing a more general result. But here is a consequence which had prompted the above theorem.

**Corollary 1.** *If  $G$  is a  $q$ -manifold, then  $c(\psi(G)) = q + 1$  and  $c(\phi(G)) = (q - 1) + c(\hat{G}) \leq q + 2$ .*

*Proof.* The vertices coming from simplices of dimension  $q - 2$  or less can be colored by the dimension  $g(x) = \dim(x)$ . The vertices in  $\hat{G}$  that represent  $q$ -simplices in  $G$  are then colored with a disjoint set of  $c(\hat{G})$  colors. □

**2.4.** This in particular shows that a **4-color theorem for soft Barycentric refined 2-manifolds**  $\phi(G)$  is no problem: first color the vertices that have been vertices in  $G$  and vertices that had been edges in  $G$ . Then use the remaining colors to color the vertices that had been triangles in  $G$ .

**2.5.** A  $q$ -sphere  $G$  is minimally  $(d + 1)$ -colorable if and only if for all  $(d - 2)$  simplices  $x$  of the dual circle  $x'$  have even length. A  $d$ -manifold is  $(d + 1)$ -colorable if and only if every closed circle has odd length. In the planar case, this was an observation of Heawood. In some way, it appears in [8]. Start with coloring one maximal simplex, then the other colors in the neighborhood are determined since  $G$  is simply connected. (Simply connectivity is defined for  $q$ -manifolds by defining two simple paths to be homotopic if they can be transformed into each other by **simple homotopy steps**, which means taking a triangle which contains one or two edges of the graph and replace it with the complement in the triangle, a closed path circling a triangle goes to a point. A graph is simply connected if it is homotopic to a point.) If the graph has the property that every closed loop in the dual graph has even length, then no conflict appears during the coloring.

**2.6.** From the chromatic bound 3, we get two forests covering  $G$  if we can **color acyclically**. The reason is that we can take 2 colors which are not cyclic and form one forest with them. The third color is the third forest (consisting of seeds only). So, we only have to show that we can color  $\hat{G}$  with 3 colors in such a way that there are no **Kempe chains**. Kempe chains are cycles in the graph on which the coloring has 2-colors only.

**2.7.** We prove a stronger statement by induction.

**Theorem 3** (Acyclic 3-color theorem). *For every dual  $\hat{G}$  of a  $q$ -manifold  $G$  with boundary, for which the boundary has been acyclically 3-colored, the acyclic 3-coloring can be extended to the interior, preserving the acyclic property. Also any 2-forest cover on the boundary can be extended to a 2-forest cover of the graph in which the interior is included.*

**2.8.** Using induction with respect to dimension shows that the acyclic color theorem implies the dual 3 color theorem in the case of a manifold  $G$  without boundary. By taking a manifold  $G$  and cutting out a ball, we get two manifolds with boundary. By induction with respect to dimension, the interface  $(q - 1)$ -manifold can be acyclically 3-colored. By the theorem, it can be extended on both sides and so color the manifold  $G$ . Now to the proof of the acyclic 3-color theorem:

*Proof.* The statement holds in dimension  $q = 1$  because every 1-manifold, with or without boundary can be colored with 3 colors and in the boundary case, the color on the boundary (2 isolated points) can be extended to the interior.

Let  $q$  be a minimal dimension with a counter example and let  $n$  be the minimal number of facets of a dual  $q$ -manifold with boundary for which such an extension is no more possible.

Take such a minimal example  $G$ . It must have an interior simplex (and so by the manifold definition has vertex degree  $q + 1$ ). If there was no interior simplex the color of the boundary  $\delta G$  would already directly color  $G$ . Pick an interior point  $x_0 \in V(\hat{G})$ . Color all cycles through  $x_0$  with colors  $A, B$ , by taking the color  $(-1)^{\text{dist}(x, x_0)}$ . For bones (minimal cycles) of odd length, this leads to double points, requiring a third color  $C$ .

We have now colored a sub-manifold  $H$  of  $G$  with 3 colors. If the entire manifold can not be colored, then also  $G \setminus H$  can not be colored and we would have obtained a smaller manifold with boundary for which the boundary is acyclically colored by 3 colors but for which we can not extend the coloring to the interior. This contradicts that  $G$  was the smallest one.  $\square$

**2.9.** Let us add two remarks:

**Remark 1)** The 4-color theorem [2, 11] can not be proven as above. There is no such extension result from the boundary to the interior for manifolds with boundary. And that makes the 4-color theorem difficult. The statement that any 3-coloring of the boundary of a 2-disk can be extended to a 4-coloring of the interior would actually be equivalent to the 4-color theorem. One has tried to use "minimal example" arguments to prove the 4-color theorem and Kempe managed to prove his **5 color theorem** using a minimality argument. In a "cut and conquer argument" we would like to find a path through a 2-ball which only needs 3 colors reducing the coloring problem to two smaller problems. But the boundary extension does not work. The Birkhoff diamond is a small example where the boundary is colored with 3 colors but where the two interior points can not be assigned any color. Proving the 4-color theorem in this way would require to list all counter examples, establish that there are finitely many of them and check that it is possible to color them nevertheless.

**2.10. Remark 2) Grötzsch's theorem** telling that every triangle free planar graph can be colored by 3 colors is stronger than the dual 3-color theorem (when restricted to dimension 2). Not all planar triangle-free graphs are dual to 2-spheres. There are 3-connected planar, triangle free graphs which are not the dual graphs of a 2-sphere. For example, take a cyclic  $C_8$  and connect all even vertices to an additional point  $A$  and all odd vertices to an additional point  $B$ . This graph is maximal among triangle-free planar graphs, is 3-connected but is not the dual graph  $\hat{G}$  of a 2-sphere  $G$  because the vertex degree of the points  $A, B$  in  $\hat{G}$  is 4, while it should be 3 for any dual of a 2-manifold.

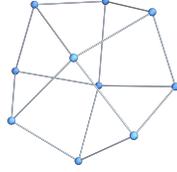


FIGURE 5. A 3-connected planar triangle-free graph that is not the dual graph of a 2-manifold. The reason is that it is not 3-regular. It is a subgraph of the prism graph  $C_8 \oplus \overline{K_2}$ . Grötsch’s theorem applies to this graph, but not to the dual 3 color theorem.

### 3. FISK COMPLEX

**3.1.** The **Fisk complex** of a  $q$ -manifold  $G$  is the  $(q - 2)$ -dimensional simplicial complex generated by the collection of  $(q - 2)$ -simplices  $x$  for which the dual simplex  $\hat{x} = S(x_0) \cap S(x_2) \cap \dots \cap S(x_{q-2})$  (which always is a cyclic graph with 4 or more elements) has an odd number of vertices. If  $G$  is simply connected and oriented and the Fisk complex of  $G$  is empty, then we can color  $G$  with the minimal  $q + 1$  colors. The Barycentric refinement destroys the entire Fisk complex as it doubles the size of the dual spheres. For soft Barycentric refinement however, this is not the case and the refinement process also refines the Fisk complex.

**3.2.** For a 2-manifold, the **Fisk complex** is the collection of vertices on which the vertex degree is odd. For general  $q$ , it can be written as a union of  $(q - 2)$ -manifolds. For an icosahedron for example, all vertices belong to the Fisk complex. The Fisk complex is empty if and only if the graph is Eulerian. The Barycentric refinement of a 2-manifold doubles the vertex degree of any given vertex and introduces new vertices of vertex degree 4 or 6. The soft Barycentric refinement does not change because the number of odd degree vertices stays the same. For a 3-manifold  $G$ , the Fisk complex is a one dimensional complex which is a union of closed curves in  $G$ .

**3.3.** For a **3-manifold**  $G$ , the Fisk complex  $F$  is the set of edges  $x = (a, b)$  for which the dual sphere  $x' = S(a) \cap S(b)$  has an odd number of vertices. The set of these simplices union of closed curves because by the Euler Handshake formula, the number of odd degree vertices is even in any graph. Also for 3-manifolds, the edge degree spectrum in the form of the curvature values at these edges does not grow when doing soft Barycentric refinements. Asymptotically, the contributions of the Fisk complex does not matter. Since in the limit we will have the same number of degree 4 and degree 6 edges, the average edge degree converges to 5. For example, if  $G = K_4$ , then the edge degrees of  $\psi(G)$  in the interior are 4,5 or 6. For  $\psi^4(G)$ , there are only 576 degree 5, but 35184 degree 4 and 34384 degree 6 edges. (There are also 5184 degree 3 edges but they are at the boundary of  $\phi^4(G)$ ).

**3.4.** For a 4-manifold, the Fisk complex is the set of **triangles**  $x$  for which the dual sphere  $x' = S^+(x)$  of  $x$  (a circular graph with 4 or more vertices) has an odd number of vertices. For a general  $q$ -manifold, the Fisk complex is generated by  $(q - 2)$ -simplices for which the dual circle has odd length. Take a triangle  $x$  in the Fisk complex. At every of the edges, there is at least one other triangle of the Fisk complex attached. Continue building like this a manifold. Now start with an other triangle not yet covered. There are many ways how we can build up the Fisk complex. The total Euler characteristic can depend on the decomposition.

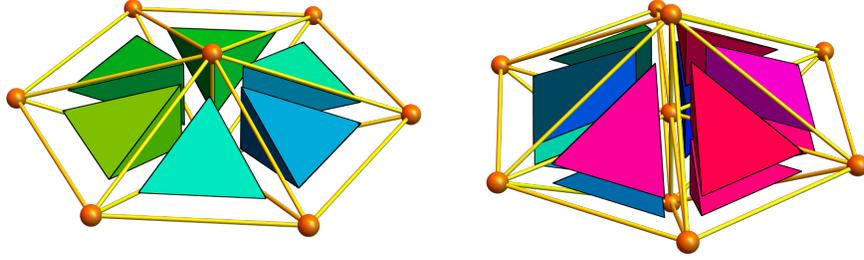


FIGURE 6. For a 3-manifold, a codimension-2 simplex is an edge  $x$ . The dual  $\hat{x}$  is a cyclic polytop  $C_n$ , where  $n$  is the number of tetrahedra in  $G$  that contain  $x$ . After a soft Barycentric refinement, the edge  $x$  has doubled, but each has the same dual. In the interior, the Barycentric refinement is more tame.

**3.5.** Let us remark that the Fisk complex of a  $q$ -manifold is a union of  $(q - 2)$ -manifolds but that the decomposition into  $(q - 2)$  manifolds is not unique in general. Lets illustrate this for  $q = 3$ : given an edge  $(a, b)$  in the Fisk complex  $F$ , it defines a point  $a$  in  $S(b)$  which has odd degree. There must be an other point  $c$  of odd degree so that we can continue the path  $(a, b, c)$ . For  $q = 4$ , given a triangle  $x = (a, b, c)$  in the Fisk complex  $F$ , it defines an edge  $y = (a, b)$  in the 3-sphere  $S(c)$ . The dual circle to  $y$  in  $S(c)$  is the same and odd. There is therefore a continuation of the curve in  $S(c)$  which produces a new triangle  $z = (a, b, d)$  in the Fisk complex  $F$ . We can do this for any of the three sides.

**3.6.** Given a 4 manifold and a set  $T$  of triangles which have the property that at every boundary edge there are two or more triangles attached, Then  $T$  generates a simplicial complex that has the property that it is the union of finitely many 2-manifolds  $M_j$ . The intersection of any two of them is either empty or a curve. It follows that the Euler characteristic of  $T$  is the union of Euler characteristics of  $M_j$ . Is it possible that some  $M_j$  are orientable while some others are not?

**3.7. Example:** the join  $G$  of the icosahedron graph and a circular graph  $C_n$  is a 4-sphere. If  $n$  is even, the Fisk complex  $F$  is a union of 12 octahedra. The union of all edges of  $F$  generates the bipartite graph  $K_{2,12}$ . If  $n$  is odd, then the Fisk complex  $F$  is dense in  $G$  in the sense that all edges are covered.

#### 4. THE TWO DIMENSIONAL CASE

**4.1.** A 2-manifold is a graph such that every unit sphere is a circular graph with 4 or more vertices. The **Barycentric refinement** of a 2-manifold  $G = (V, E)$  takes  $V' = V \cup E \cup F$  as vertices and takes as  $E'$  the set of pairs  $(x, y)$  such that  $x \subset y$  or  $y \subset x$ . If  $f = [|V|, |E|, |F|]$

is the  $f$ -vector of  $G$ , then  $f(G') = Af(G) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} f(G)$ . Because the eigenvalues of  $A^T$

is the vector  $[1, -1, 1]$ , the Euler characteristic of  $G'$  and  $G$  are the same. The vertex degrees double in each step. The linear rule  $A$  holds independently of whether  $G$  is a manifold or not.

**4.2.** The **soft Barycentric refinement** of a 2-manifold takes  $V' = V \cup F$  as vertices and  $E'$  as the set of pairs  $(x, y)$  such that  $x \subset y$  or  $y \subset x$  or  $x \cap y$  is in  $E$ . Now the  $f$ -vectors

transform as  $f' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} f$  but this is only true if there are no boundary faces. The Euler

characteristic satisfies  $\chi(G) = \chi(G')$ . Unlike for Barycentric refinement, the soft Barycentric refinement does not have a universal matrix  $A$  which works for all complexes.

**4.3.** The soft Barycentric refinement preserves the class of flat tori and makes them larger and larger. In general, unlike for the Barycentric refinement, the vertex degrees do not grow in dimension 2. In the limiting case, we approach the hexagonal lattice as almost all vertices have vertex degree 6. We can describe the limiting measure:

**Lemma 2.** *For the hex region (an infinite graph), the Laplacian is diagonal after applying a Fourier transform. It is equivalent to the multiplication operator with  $f = 6 - 2 \cos(x) - 2 \cos(y) - 2 \cos(x + y)$ .*

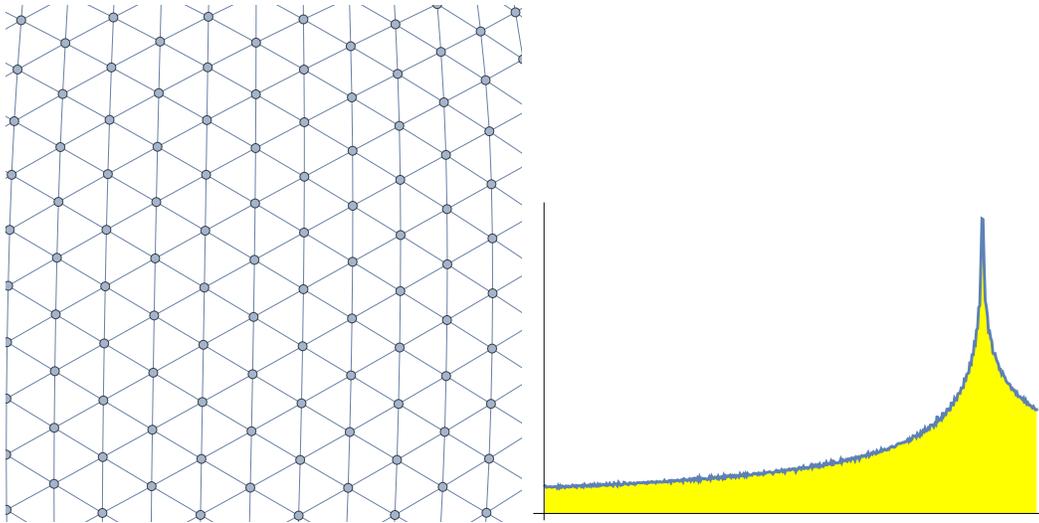


FIGURE 7. The **infinite hexagonal lattice**  $\Lambda$  can be seen as the soft Barycentric limit in two dimensions. Its Laplacian is a bounded linear operator on  $l^2(\Lambda)$  equivalent to the multiplication operator with  $f = 6 - 2 \cos(x) - 2 \cos(y) - 2 \cos(x + y)$  on  $L^2(\mathbb{R}^2/\Lambda)$ . The function  $f$  has range  $[0, 9]$  and has critical points at the three points  $\{0, 8, 9\}$ . The hyperbolic critical points of the function  $f$  lead to the **Van Hove singularity** in the density of states of the hexagonal lattice. Van Hove [3] already linked in 1953 such singularities with the Morse theory of  $f$ .

**4.4.** The universal limiting measure in the 2-dimensional soft Barycentric limit case has compact support like the 1-dimensional standard Barycentric limit. We see experimentally that the universal Barycentric measure  $\mu_q$  in dimension  $q$  has affinities with the spectral type of the universal soft Barycentric measure  $\nu_{d+1}$  in dimension  $q + 1$ . This holds in dimension  $q = 1$ ,

where Fourier theory allows to describe both measures. We can define  $\mu \leq \nu$  on measures on the real line, if there exists  $f \in L^1$  and a homeomorphism  $g$  such that  $\mu(g(x)) = f(x)\nu(x)$ . Then define  $\mu \sim \nu$  if  $\mu \leq \nu$  and  $\nu \leq \mu$ . This is an equivalence relation. It honors the Lebesgue decomposition. If one of them has an (ac) component, the other has, if one of them has a pure point (pp) component, the other has. It follows from the Lebesgue decomposition theorem that if one of them has a (sc)-component, then the other has.

**4.5.** Let us look now at the chromatic number of 2-spheres. For example, by the 4-color theorem, any 2-sphere  $G$  has chromatic number 3 or 4 and the chromatic number 3 appears if and only if the sphere is **Eulerian**, meaning that all vertex degrees are even. The vertex arboricity of a graph being 1 is equivalent to the graph being a tree. The chromatic number of a circular graph is either 2 or 3 and the vertex arboricity of a circular graph is always equal to 2. For any 2-sphere  $G$ , the chromatic number of  $G$  and the chromatic number of the soft Barycentric refinement  $G'$  is the same. Proof: By the 4-color theorem, the chromatic number of the refinement  $G'$  is again 3 or 4. If  $G$  is Eulerian, then  $G'$  is Eulerian because the vertex degrees only can add a possible vertex degree 6. So, if  $c(G) = 3$ , then  $G$  is Eulerian and so  $G'$  is Eulerian so  $c(G') = 3$ .

**4.6.** In dimension  $q = 2$ , we can explicitly color the soft Barycentric refinement  $G' = (V', E', F')$ , if a coloring of the 2-sphere  $G = (V, E, F)$  is known: we have  $V' = V \cup F$  and  $E' = \{(v, f), f \in F, v \in V, v \subset f\} \cup \{(f, g), f, g \in F, f \cap g \in E\}$ . The 4-color theorem assures that we can color with 3 or 4 colors. If  $c : V \rightarrow \{0, 1, 2, 3\}$  is a coloring, then define  $c'(v') = 3$  for all  $v' \in V$  then start assigning  $c(v')$  to one of the  $v' \in V' \cap F$ . (i) Lets first assume that the chromatic number of  $G$  is 4. Now if and other  $w' \in V' \cap F$  is adjacent, there are two possibilities: either  $v'$  and  $w'$  as faces carry the same color triples in the coloring of  $G$ , then define  $c'(w') = c'(v') + 1 \pmod 3$  otherwise  $c'(w') = c'(v') - 1 \pmod 3$ . We have colored  $G'$  with 4 colors. (ii) If the chromatic number is 3, the graph is Eulerian. Color the vertices of  $G'$  with the color 0. If color one of the faces with  $c(v') = 1$ . If  $v'$  and  $w'$  are adjacent faces, let  $c'(w') = -c'(v') \pmod 3$ . Because the graph is Eulerian, this will color all faces without conflict. We have colored  $G'$  with 3 colors.

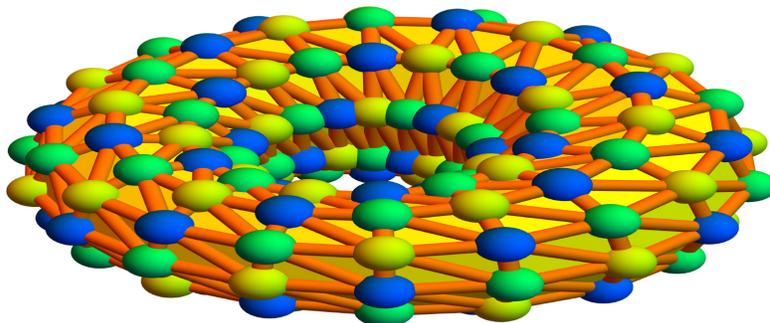


FIGURE 8. A coloring of a flat Clifford torus. Clifford tori are left invariant under soft Barycentric refinements.

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