A RIEMANN-HURWITZ THEOREM IN GRAPH THEORY

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Abstract. If G is a finite simple graph on which an arbitrary finite group A of order n acts by automorphisms. When we look at G as a ramified cover over the chain $H = G/A$, the Riemann-Hurwitz formula $\chi(G) = n\chi(G/A) - \sum_{x \in G}(e_x - 1)$ holds, where $e_x = 1 + \sum_{a \neq 1, a(x)=x}(-1)^{k(x)}$ and where $\chi(G/A)$ is the Euler characteristic of the chain. For a class of actions which we call simple, the quotient $H = G/A$ is a finite simple graph and Riemann-Hurwitz relates the Euler characteristic of G and H. Cohomologically $\chi(G/A)$ is the average $\frac{1}{n}\sum_{a \in A} L(a)$ where $L(a)$ is the Lefshetz number of the automorphism a.

1. Introduction

The classical Riemann-Hurwitz formula relates the Euler characteristic of two surfaces G and H, where G is a cover of H, where n is the number of sheets and where e_x the ramification index, assumed to be nonzero only for finitely many points $x \in G$. The theorem is used in complex analytic [2] or algebraic-geometric settings [5]. While folding is natural for Riemannian manifolds, the construction is not restricted to surfaces only. For "branched-and-folded" coverings, where the folds can be more general [8]. In discrete settings the situation arrizes when groups act on graphs. [1] have considered the Riemann-Hurwitz formula in a discrete setting using Urakawa harmonic morphisms on graphs with multiple edges. They look at graphs as discrete analogues of Riemann surfaces in which case the Euler characteristic is $g - 1$, where $g = v_1 - v_0 + 1$ is the genus of the graph and $v_0 = |V|, v_1 = |E|$ are the number of vertices and edges. In the continuum, orbifolds $H = G/A$ are obtained by factoring out a finite group A on a manifold G. The Riemann-Hurwitz formula relates there the Euler characteristic of G with the one of H. A simple example is to take two copies of a manifold H with boundary R and glue them along R, leading to a manifold G without boundary on which one has a Z_2 action interchanging the two copies. The boundary R consists then of ramification points so that $\chi(G) = 2\chi(H) - \chi(R)$. Every compact two dimensional manifold is a branched cover of the two sphere; already Riemann saw genus g Riemann surfaces as compact 2-branched covers of the projective line which are compactifications of a plane curve $y^2 = p(z)$ for a complex polynomial p of degree $2g + 1$. Branched covers are used in topology. A theorem of Hilden-Montesinos [6] assures that every compact three manifold M is a $n = 3$ branched cover of a three dimensional sphere S^3 branching along a knot K. The Riemann-Hurwitz formula $\chi(M) = 3\chi(S^3) - \chi(K)$ is trivial

Date: December 26, 2012 (unfinished draft notes).
1991 *Mathematics Subject Classification.* 58J20,47H10,37C25,05C80,05C82,05C10,90B15,57M15,55M20.
Key words and phrases. Graph theory, graph endormorphisms, Lefschetz number, Euler characteristic.

there because all involved manifolds have zero Euler characteristic.

We explore the situation in a combinatorial graph theoretical framework [3] or more generally for chains. Unlike in [1] which emulates complex analytic structures, we look at the Euler characteristic $\sum_{k=0}(-1)^k v_k$ where v_k are the k dimensional complete graphs K_{k+1} in G. This means that the ramification index involves arbitrary complete graphs. Already [1] have to look at vertical and horizonal multiplicities because in their situation ramifications can occur along vertices or edges. Since graphs are more intuitive than chains, we will put some effort to make sure that the quotient G/H is also a graph.

In general, if a group A acts on a graph G, then G/A is only a chain. Take C_n and let Z_n act on it, then the quotient is the chain $x+e$, where x one vertex and e is the edge. There is no second vertex so that G/A is not a graph. The quotient chain has Euler characteristic 0 and there are no ramification points. For K_2 with Z_2 action RR we get the same $H = G/A$ and $0 = \chi(G) = 1 = 2\chi(H) - (-1)$ because the edge is a ramification point with ramification index -1. For a triangle with Z_3 action, only the triangle is a ramification point with ramification index 1 fixed by 2 automorphisms of A different from the identity so that $1 = \chi(G) = 3\chi(H) - 2$. These examples show that we can not avoid chains when looking at quotients. Chains also naturally appear when taking boundaries of graphs: the outside star graph S_4 with $V = \{x, a, b, c, d\}$ for example has as a boundary the chain $-4x + a + b + c + d$. For certain classes of graphs we can assure that boundaries are graphs again. These geometric graphs are closer to the geometric graphs we know in the continuum. A similar thing happens when forming quotients: if we add a bit more "space" to make sure that different parts of an orbit do not come to close, then G/H is a graph again and G can be seen a n-fold cover of H for which at some points the cover collapses. Ramification points x are vertices, edges or higher dimensional cliques in G for which the orbit does not have the maximal cardinality $n = |A|$. If A is an involution, then ramification points are the fixed points of the involution. In the discrete, when taking quotients, an "emerging dimension phenomenon" appears which is not present in the continuum. Taking quotients can increase the dimension of the graph. The complete bipartite graph $G = K_{3,3}$ where the fixed edges of the evolution are removed is one dimensional having no triangles but admits a natural $A = Z_2$ action flipping the two sides. The quotient G/A is a triangle. For the Riemann-Hurwitz formula to hold, we have to ignore the emergence of the triangle which has amalgamated by the action but to see H as a chain with 3 vertices and 3 edges, then $\chi(G) = -3$ and $\chi(H) = 0$ and G/H is a chain with three vertices and three edges and not a graph.

Chains are to graphs what orbifolds are to manifolds, or algebraic stacks are to schemes. Chains extend the category of graphs in such a way that it is closed under forming boundary or taking quotients of group actions. Chains have been introduced very early in topology because they are closed under the boundary operation δ which is dual to taking the exterior derivative d as integration $\int_X f = \langle \sum_i a_i x_i, \sum_j f_j x_j \rangle = \sum_i a_i f_j$ gives Stokes theorem $\int_X df = \int_{\delta x} f$ for chains which also works for geometric graphs and is very close to the classical Stokes theorem in the continuum. While graphs are known since Euler and chains since Poincaré, it seems not yet explored, which group actions A on a graph G produce graphs. We start to address this question here and explore especially the relation between Euler characteristics, which is the Riemann-Hurwitz formula.

The Riemann-Hurwitz identity is already a combinatorical result if no edges are present. For a graph without edges, where the Euler characteristic is the cardinality of the vertex set, it becomes the trivial identity $|G| = 2|H| - |F|$ for the cardinality of sets. Also more general "exclusion-inclusion formulas" can be seen as Riemann-Hurwitz formulas: the Burnside lemma or the Polya enumeration theorem. We will also mention an application on sizings in group theory [9]: one can look at the smallest Euler characteristic $\chi(B)$ which a triangle free Cayley graph of a group B can have. The Riemann-Hurwitz formula readily implies that if A is a subgroup of B then $\chi(A) \geq \chi(B)$.

2. Riemann-Hurwitz for chains

A finite simple graph $G = (V, E)$ is a finite graph without multiple connections and no loops. An automorphism a of the graph G is a permutation a of V which preserves edge relations. Let A be a finite subgroup of the automorphism group $\mathrm{Aut}(G)$. Let $\mathcal{G}_k$ denote the set of k-simplices in G. This means that $\mathcal{G}_k$ is the set of subgraphs K_k of G. We can write G as a chain $\sum_{x \in \mathcal{G}} a_x x$ with $a_x \in \{0,1\}\mathbf{Z}$, where $\mathcal{G} = \bigcup_j \mathcal{G}_j$. For graphs, the Euler characteristic is defined as $\chi(G) = \sum_j (-1)^j v_j$, where $v_j = |\mathcal{G}_j|$ is the cardinality of $\mathcal{G}_j$. For a chain $G = \sum_x a_x x$, define $\chi(G) = \sum (-1)^{k(x)} a_x$ where $k(x)$ is the dimension of x. The action of A on G induces an action on $\mathcal{G}$ as follows: if $x = (x_0, \dots, x_k)$ is a clique, then $a(x) = (ax_0, \dots, ax_k)$. We could but do not assume chains to have the property that any subsimplex is included. For example, if a chain contains a triangle (x, y, z) we do not require that the chain also contains the edges (x, y) for example. If a chain is a graph, the naturally, this is the case.

Definition. *The* **quotient** G/A *is a chain* $\sum_j \sum_{x \in \mathcal{G}_j/A} x$*, where* $\mathcal{G}_j/A$ *is the set of equivalence classes of* $\mathcal{G}_j$ *modulo* A*. We call* G *a* **branched cover** *of* G/A*.*

A chain can for example consist of a triangle t with no vertices and one edge e. We would write $G = t + e$. This chain has Euler characteristic $(-1)^2 + (-1)^1 = 0$.

Definition. *A chain automorphism is a map* a *which permutes every of the simplex sets* $\mathcal{G}_j$*.*

Again, we also do not require that a respects inclusion: if $x \subset y$, then $a(x)$ does not need to be a subset of $a(y)$.

Definition. *Given a chain automorphism and* $x \in \mathcal{G}_k$ *we define the* **ramification index** $e_x = 1 + \sum_{a \neq 1, a(x) = x} (-1)^{k(x)}$*, where* $k(x) = k$ *if* $x \in \mathcal{G}_k$*. The sum* $R = \sum_x (e_x - 1)$ *is called the* **ramification sum***.*

Definition. *A cover* G *of* G/A *is called* **unramified** *if the ramification index is zero everywhere. In this case, the cover* $G \to H$ *can be seen as a discrete* **fibre bundle** *with structure group* A*.*

Theorem 2.1 (Riemann-Hurwitz for chains)**.** *Let G be a simple graph and let A be an action on G of order n, then*

$$\chi(G) = n\chi(H) - \sum_x (e_x - 1) ,$$

where $e_x = 1 + \sum_{a \neq 1, a(x)=x} (-1)^{k(x)}$ and $k(x)+1$ is the dimension of x.

Proof. By the Burnside lemma, we know for every k that

$$\sum_{a \in A} \sum_{x \in \mathcal{G}_k, a(x)=x} 1 = n|\mathcal{G}_k| .$$

Taking the alternate sum over k gives

$$\sum_{a \in A} \sum_{x, a(x)=x} (-1)^k = n\chi(H) .$$

This gives

$$\sum_{a \neq 1} \sum_{x \in G} (-1)^{k(x)} + \sum_{x \in G} (-1)^{k(x)} = n\chi(H) .$$

and so $\chi(G) = n\chi(H) - \sum_{x \in G} \sum_{a \neq 1} (-1)^{k(x)} = n\chi(H) - \sum_{x \in G} (e_x - 1)$. ☐

Remarks.
1) The Euler characteristic of a group is sometimes defined as $1/|A|$. The Riemann-Hurwitz theorem can therefore be read as $\chi(G) = \chi(G/A) * \chi(A) - R$.
2) While chains are natural and the proof is easy, chains are not as intuitive as graphs. A drawing of a graph defines all the higher dimensional structures it contains, to visualize chains we would have to write integers a_x beside each simplex x it contains.
3) There is a similarity between the problem to form moduli spaces, where the existence of automorphisms produces problems and needs to enlarge the category. This was the emergence of stacks.

3. Simple action

Under which conditions is G/A again a finite simple graph? The following is similar to a definition done in [4], where quotient graphs are used to find isospectral graphs.

Definition. *An action A on a graph G is called* **simple** *if (i) for any adjacent vertices x, y in G the images $[x], [y]$ in G/A are adjacent but different. (ii) for any edges x, y which intersect in a vertex but are not included in a triangle, the images $[x], [y]$ still intersect in one vertex but are not included in a common triangle.*

Lemma 3.1. *For a simple action, the quotient chain G/A is a simple graph and $\chi(G/A)$ is the Euler characteristic as a graph.*

Proof. Every simplex in $\mathcal{G}/A$ is defined as an equivalence class of simplices in $\mathcal{G}$. We can form a graph $H = (W, F)$, where W is the set $[G_0]$ of equivalence classes $\mathcal{G}_0/A$ and where F is the set $[G_1]$ of equivalence classes in $\mathcal{G}_1/A$. We claim that the set H_k of k-simplices in H corresponds to the set $[G_k]$ of simplices in $\mathcal{G}_k/A$. Let $([x_1], \dots, [x_k])$ be in H_k. Pick a representative x_1 of $[x_1]$ then chose a representative x_2 near x_1 which maps to $[x_2]$ then pick a representative both adjacent to x_1, x_2 which maps to $[x_3]$ etc. Because the action is simple, every edge $([x], [y])$ in the triangle comes from a lifted edge. The triangle (x, y, z) maps to a triangle T in $[G_2]$. And the map from $[G_k] \to \mathcal{G}_k/A$ defined by $[(x_1, \dots, x_k)] \to ([x_1], \dots, [x_k])$ is a bijection. □

There are three basic mechanisms which make an action fail to be simple: forming loops, forming double edges and forming triangles. Here are examples of actions which are not simple:

Examples.
1) $A = Z_2$ acting on $G = K_2$ produces a one point graph $H = G/A$ with a loop.
2) $A = Z_2$ acting on $G = C_4$ produces a two point graph $H = G/A$ which has a double edge.
3) Z_4 acting on the wheel graph W_4 produces a chain with a triangle, two vertices, one edge and one self loop. The quotient is not a graph any more.
4) The antipodal involutive group on the octahedron produces a quotient which is not a graph any more.
5) Z_2 acting on the utility graph produces a triangle with three loops.
6) The Z_5 action rotating an octahedron G around a symmetry axes through a pentagonal face has as as quotient a graph H which contains two self-loops and a double connection. We have $\chi(G) = -10$ and $\chi(H) = -2$. There are no ramification points.

Remark. If the quotient produces a triangle which does not lift to triangles in G, then this must come through an addition of an edge to two connected edges. It is not possible to have three pairwise disconnected edges in G which project down to a triangle.

Theorem 3.2 (Riemann-Hurwitz for graphs). *Let G be a simple graph and let A be a simple action on G of order n, then*

$$\chi(G) = n\chi(H) - \sum_x (e_x - 1) ,$$

where $e_x = 1 + \sum_{a \neq 1, a(x)=x} (-1)^{k(x)}$.

Proof. The result follows directly from Theorem (2.1) because graphs are chains. The only thing which has changed is that $\chi(H)$ is the Euler characteristic of a graph which however agrees with the Euler characteristic as a chain. □

(1) $A = Z_n$ acting on $G = C_n$ is not simple. The quotient $H = G/A$ is a graph with a loop and $\chi(H) = 0$. There are no fixed points.
(2) $A = Z_n$ acting on $G = C_{2n}$ is not simple. The quotient $H = G/A$ is a double connection $* = *$. There are two edges of distance 2 which afterwards have distance 0 but are distinct.
(3) $A = Z_n$ acting on $G = C_{3n}$ is not simple. The quotient $H = G/A$ is a triangle. There are two edges of distance 2 which afterwards have distance 1.
(4) $A = Z_n$ acting on $G = C_{kn}$ is simple for any $k \geq 4$ The quotient is the circular graph C_k. This is an example of an unramified cover.
(5) $A = Z_2$ acting on G obtained from C_8 with 4 diagonals is not simple (Figure). The quotient contains a tetrahedron.

(6) $A = Z_2$ acting on an octahedron flipping poles is simple. The quotient is a wheel graph W_4. We have 4 fixed vertices with $e_x - 1 = 1$ and 4 fixed edges with $e_x - 1 = -1$. We have $\chi(G) = 2 = 2 * \chi(H)$. This generalizes: take two copies of a graph H and share them along a subgraph B. Then $\chi(G) = 2\chi(H) - \chi(B)$.
(7) Z_n acting on a star graph S_n is simple. The quotient is K_2. The center point has a ramification index $e_x = n$. We have $1 = \chi(G) = n\chi(H) - (n-1)$.
(8) Z_n acting on a wheel graph W_n is not simple. The quotient H is a chain with two vertices, two edges and one triangle. The Euler characteristic is 1. The center point has ramification $n - 1$.

The next lemma is a simple criterion for an action to be simple. It tells that if we avoid different points of an orbit to come too close to each other, then the quotient is simple:

Lemma 3.3. *A group A acting on graph G is simple if the minimal distance of two distinct vertices in an orbit is 2.*

Proof. It includes loops, double connections and the appearance of new triangles. □

4. Lifting a quotient to graphs

In order to avoid chains, we can use a triangularisation G' of G for which $G'/A = H'$ is a graph. We will do this in such a way that the Euler characteristic of G agrees with the one of G' and such that the ramification sum R does not change. This shows that $\chi(G) - R$ is equal to $n\chi(H')$.

Lemma 4.1 (Subdivision of simplex)**.** *Given a complete graph K_n we can form a refined graph K_n' which is a triangularization of K_n and such that the distance between any two points on the boundary on different faces is at least 4 and such that every edge has been subdivided into 4 edges. The Euler characteristic of each K_n' is still 1.*

Proposition 4.2 (Subdivision of graph)**.** *Given any graph G we can form a refined graph in which every K_n is triangulated. The Euler characteristic of G' is the same than the Euler characteristic of G. We can refine in such a way that the automorphism group of G' is the same than G.*

Proof. To break any additional symmetries we can make subdivisions at the simplices of the corner. □

It is custom to call G' homeomorphic to G in the case when the refinements are done for graphs without triangles. The refinement procedure is topological in the sense that other properties like dimension are preserved.

Definition. *The* **ramification sum** *of an action A on G is defined as the sum $\sum_x (e_x - 1)$, where $e_x = 1 + \sum_{a \neq 1, a(x) = x} (-1)^k$.*

Lemma 4.3. *Given a finite graph G and a group A of automorphisms of G, we can find a homeomorphic graph G' which admits the same A action and for which $\chi(G) = \chi(G')$ and so that the ramification sum of (G, A) is the same than the ramification sum of (G', A).*

Proof. The refinement of each simplex x is done in such a way that if x had been a fixed point then the ramification index of this simplex is the sum of the ramification simplices of all subsimplices. □

From the result on simple actions, we see that

Corollary 4.4. *The number $\chi(G) - \sum_x (e_x - 1)$ is divisible by n. It is independent of the simplicial refinement. It agrees with the Euler characteristic of G/A as a chain.*

Theorem 4.5 (Riemann-Hurwitz)**.** *Given a finite simple graph and an automorphism group A of order n on G, then*

$$\chi(G) = n\chi(H) - \sum_x (e_x - 1) \ ,$$

where $H = G'/A$ is any of the regularized quotient graphs.

Remarks.
1) The Euler characteristic for chains has all properties of Euler characteristic: if G_i admit both a A action, then the union, the intersection, the product are defined and a) If G_i are disjoint then $\chi(G_1 \cup G_2/A) = \chi(G_1/A) + \chi(G_2/A)$, b) $\chi(G_1 \cup G_2/A) = \chi(G_1/A) + \chi(G_2/A) - \chi(G_1 \cap G_2/A)$, c) $\chi((G_1 \times G_2)/A) = \chi(G_1/A) \times \chi(G_2/A)$, d) If G_2 is an unramified cover of G_1 of order k, then $\chi(G_2/A) = k\chi(G_1/A)$.
2) How can we see whether a chain H can be written as G/A with some graph G? Chains without vertices can not be written as G/A. Any graph H with additional simple selfloops is an orbifold.
3) We can interpret $\chi(G/A)$ as the average Lefshetz number $L(a)$ of all automomorphisms on G [7].

5. Examples

If applied to sets, graphs without edges, ramification is the notion of compensating over-counting when covering a space. Since Euler characteristic generalizes counting, it applies to situations like Burnside or Polya.

1 **Inclusion-Exclusion.** Let G be the union of two sets U, V which have the same cardinality. Let T be a permutation of G which exchanges U, V. The Euler characteristic of G is the cardinality of $\chi(G) = |U \cup V|$. The group A generated by the involution T has two elements. The quotient $H = G/A$ has the Euler characteristic $\chi(H) = |U| = |V|$. The ramification index of a point is $e_x = 2$ if x is a fixed point and 1 else. The Riemann Hurwitz formula $\chi(G) = 2\chi(H) - \sum_{x \in G}(e_x - 1)$ is nothing else than the inclusion-exclusion formula $|U \cup V| = |U| + |V| - |U \cap V|$. This goes over to graphs. Assume a graph G is the union of two isomorphic graphs U, V which intersect in a graph $U \cap V$. The group A is again the involution and $H = G/A$ is isomorphic to U. The Riemann-Hurwitz formula tells

$$\chi(G) = 2\chi(U) - \sum_x (e_x - 1)$$

Every simplex in the intersection is a ramification point and we get the inclusion-exclusion formula for Euler characteristic $\chi(G) = \chi(U) + \chi(V) - \chi(U \cap V)$.

2 **Examples with involutions**. Take a line graph of length n which has $n+1$ vertices and n edges and reflect it. If n is even, there is fixed vertex, for odd n there is a fixed edge. The involution group A has a fixed vertex or edge. The ramification index of a fixed vertex x is $e_x = 2$. We have $\chi(G) = 1$ and $\chi(H) = 1$ and $n = 2$ and $\chi(G) = 2\chi(H) - 1$. If n is odd, then the quotient G/A is a line graph with one loop of Euler characteristic 0. The loop edge is a fixed point with ramification index -1. Then $\chi(G) = 2\chi(H) + 1$. Take now two copies of a line graph which are glued together along some subgraph U. The quotient graph G/A is a line graph. The graph G has $\chi(G) = 1 - g$ where g is the number of holds. Each vertex or edge in U is a fixed point and the ramification sum is $\chi(U) = g + 1$. Now $1 - g = \chi(G) = 2\chi(H) - \chi(U) = 2 \cdot 1 - (g+1)$. If G is a circular graph G of even length and A acts by involutions fixing vertices, we have a line graph H as a quotient and two ramification points and $0 = \chi(G) = 2\chi(H) - 2$. If G is an octahedron and A is the reflection group at the equator, then The quotient H is the wheel graph W_4 and $2 = \chi(G) = 2\chi(H) - (4-4)$. The ramification sum is the Euler characteristic of the fixed point set which is a circular graph. For $G = K_2$, then $A = S_2$ is the full automorphism group. H is the one vertex loop of Euler characteristic 0.

3 **Burnside lemma.** Since the Burnside lemma is used to prove Riemann-Hurwitz, it is no surprise that one can prove Burnside with Riemann-Hurwitz: assume a group A of order n acts on a graph G without edges. The Euler characteristic of the set G is $\chi(G) = |G|$. The Riemann-Hurwitz formula

$$|G| = n|G/A| - \sum_{x \in G} (e_x - 1)$$

simplifies to

$$0 = n|G/A| - \sum_{x \in G} e_x$$

which is $n|G/A| = \sum_{x \in G} e_x$. Since $e_x = \sum_{a \in A} X^a(x)$ where $X^a(x) = 1$ if x is fixed by a or 0 else and X^a is the number of elements fixed by a is $\sum_{x \in G} X^a(x)$, we have

$$\begin{aligned} \sum_{x \in G} e_x &= \sum_{x \in G} \sum_{a \in A} X^a(x) \\ &= \sum_{a \in A} \sum_{x \in G} X^a(x) \\ &= \sum_{a \in A} |X^a| \, . \end{aligned}$$

Therefore $n|G/A| = \sum_{a \in A} |X^a|$ which gives the Burnside lemma

$$|G/A| = (1/|A|) \sum_{a \in A} |X^a| \, .$$

4 **Redfield-Polya enumeration theorem**. Colored sets can be modeled by sets $G = X^Y$ which are graphs without edges. Over each vertex we have a fibre Y whose elements are called colors. Let A be an automorphism subgroup of $Aut(X)$ and assume that it has n elements. The Riemann-Hurwitz formula simplifies to $n|G/A| = \sum_{x \in G} e_x = \sum_{a \in A} |X^a|$. In this case, $|X^a| = |Y|^{c(a)}$ where $c(a)$ is the number of cycles of a. This leads to the simplest version of the Redfield-Polya enumeration theorem:

$$|G/A| = \frac{1}{n} \sum_{a \in A} |Y|^{c(a)} .$$

This popular theorem in combinatorics allows to count colorings of things modulo some symmetries. For example if G is the cube and we have $|Y| = 3$ colors, and if A be the set of rotations of the cube. Then $n = 24$ and the formula counts the number of cubes modulo rotations.

5 **Complete graphs.** Let G be the complete graph K_3. The automorphism group is S_3. Now $H = G/A$ is a chain with 1 vertex, one edge, one triangle. The Euler characteristic is 1. The ramification index of a vertex x it is $e_x = 2!$. For the complete graph K_n with automorphism group S_n, the quotient chain has Euler characteristic 1 if n is odd and 0 if n is even. In this case, the ramification index of every k-simplex is $e_x = k!(n-k)!$. The Riemann-Hurwitz formula is for odd n,

$$1 = n! \cdot 1 - \sum_k \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^{k-1} [k!(n-k)! - 1]$$

and for even n,

$$1 = n! \cdot 0 - \sum_k \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^{k-1} [k!(n-k)! - 1] .$$

6 **Cyclic graphs.** For a cyclic graph $G = C_n$ with $n \geq 4$ the automorphism group is the dihedral group D_n. The quotient graph is a one loop graph. For $p, q > 3$, the cyclic graph C_{pq} is an unramified cyclic cover of degree q over the cyclic graph $H = C_p$ with $A = Z_q$. The Euler characteristic is zero for both. The group A is generated by $T(x) = x^p$ on $G = C_{pq}$. If we let the cyclic group Z_n act on C_n, then the quotient graph G/A is a one loop graph of Euler characteristic 0. The cyclic graph C_n is a n fold ramified cover over the one loop graph $1 \to 1$ which contains one edge and one vertex and has Euler characteristic 0. The Riemann-Hurwitz formula is $0 = \chi(G) = n \cdot 0 - 0$. Now let Z_n act on C_{2n} by rotation. The quotient graph is now a two point graph with two edges. This example shows that the G/A can have multiple connections. Let now Z_n act on C_{3n} by rotation. The quotient graph is a three point graph with three vertices and three edges. The quotient graph G/H is not a graph any more, even if we allow multiple connections and loops. The reason is that as a graph, the G/H would have a triangle. But the triangle should not be included. This is the simplest example where G/H falls into the larger category of chains.

7 **Two dimensional geometric graphs.** A graph G is two dimensional geometric, if every unit sphere is a cyclic graph. We also assume that the

graph is connected and that there is an orientation on triangles which induces an orientation on edges which cancel on adjacent triangles. This is a purely graph theoretical description of a two dimensional compact orientable manifold. If we embed every wheel graph in G with a two dimensional disc we can consider them as charts of a two dimensional compact manifold M and G is a triangularization of M. It is well known that M can be seen as a quotient S^2/A for some finite group of homeomorphisms of order 2 with $2g+2$ ramification points. We do not need the continuum however and stay in the graph theoretical setup and show that every two dimensional oriented geometric graph can be deformed to be a two dimensional geometric graph G which permits a Z_2 action A such that G/A is a two dimensional geometric sphere and such that the involution has $2g+2$ fixed vertices and no other fixed simplices. The Riemann-Hurwitz formula is

$$2 - 2g = 2 * 2 - (2g + 2) .$$

8 **Examples of discrete varieties** Graphs for which the unit sphere is one or zero dimensional at every point are models for algebraic curves. Points for which the unit sphere is a circular graph are regular points and other points are singular points. For a figure 8 graph for example, most points have unit spheres of Euler characteristic 2 but there is a singular point x where the unit sphere $S(x)$ has Euler characteristic 4. Take a wedge sum G of n spheres, joined at a single vertex. Take a cyclic group $A = Z_n$ acting on G has one fixed point. The quotient $H = G/A$ is a sphere. Now $\chi(G) = n\chi(H) - (n-1) = n+1$. This reflects the fact that $H^0 = 1, H^1 = 0, H^2 = n$. We could take n three dimensional spheres and glue them along a circle. The ramification sum in this case is zero.

9 A discrete d dimensional sphere $G = S^d$ as a branched cover of degree $n = 2$ over the d dimensional disc H. The ramification points form a $(d-1)$-dimensional sphere $R = S^{d-1}$. The sum over all ramification indices is $\chi(R)$. The Riemann-Hurwitz formula shows

$$\chi(S^d) = 2\chi(D^d) - \chi(S^{d-1}) = 2 - \chi(S^{d-1}) .$$

This is the recursion formula for the Euler characteristic of the sphere.

10 **The projective plane.**
The sphere G is a unramified two fold cover of the projective plane H. This is the same for graph versions. Indeed $\chi(G) = 2 = 2 \cdot \chi(H)$. The simplest discrete example is the octahedron, on which the north-south reflection acts. The equator C_4 consists of fixed points. The Riemann-Hurwitz formula tells that the quotient has Euler characteristic 1. It is a disc. An other involution reflects around the center. This action is not simple. The quotient is a chain which has 4 triangles, 6 edges and 3 vertices. It is not even a simple graph. There are no ramification points. After doing the refinement G', the quotient H' is a discretisation of the projective plane H, a finite simple graph.

11 **Hyperbolic graphs.**
A graph is one dimensional, if every unit sphere is a discrete graph. It is called hyperbolic if the degree at every point is larger than 2. Since the

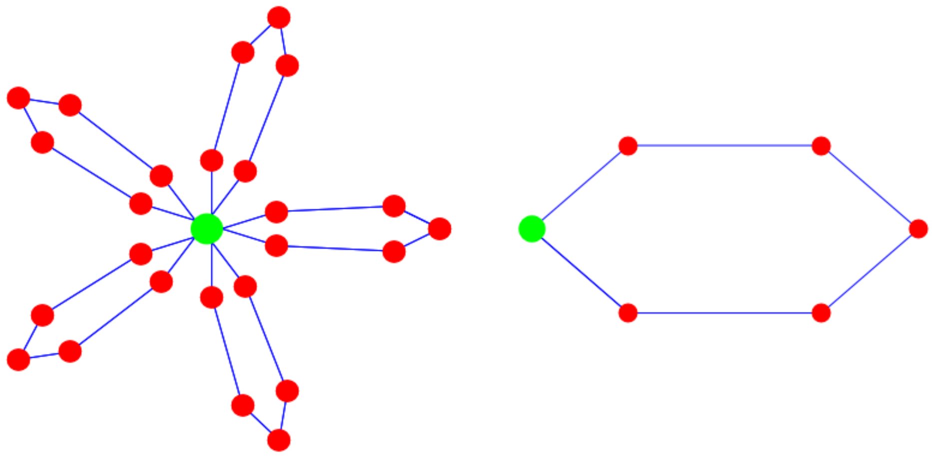

FIGURE 1. An example of a cyclic ramified cover G over $H = C_6$. The automorphism a generating A has order 5. We have $\chi(G) = 26 - -30 = -4$ because there are no triangles in G. We have $\chi(H) = \chi(C^6) = 0$. There is one ramification point x of G where T has order 5. Therefore $e_x = 5$. The Riemann-Hurwitz formula gives $\chi(G) = 5\chi(0) - (5-1) = -4$.

sum of the degrees is by Euler's handshaking lemma twice the number of edges, the Euler characteristic is $\chi(G) = v - e = v - \sum_x d(x)/2$ and since $d(x) > 2$ we have $\chi(G) < 0$. Examples: For a cube G, we have $\chi(G) = -4$. If A of the cube generated by rotations around the diagonal. There are two fixed points and the ramification sum is 4. It has order $n = 3$. The simple quotient graph H has Euler characteristic 0. The fixed vertices have ramification index 2 and indeed

$$-4 = \chi(G) = 3\chi(H) - 2 \cdot 2 \ .$$

An other example is the dodecahedron G. Take the cyclic group generated by a rotation by $2\pi/5$ around an axes through the center of the faces. We have $\chi(G) = 20 - 30 = -10$ and $n = 5$. The quotient $H = G/A$ is not a simple graph. It is line graph with two loops at the end and a double connection. Its Euler characteristic is $\chi(H) = -2$.

12 **Discrete torus.** A discrete torus $G = C_p \times C_q$ has translational symmetries. Assume $p, q \geq 4$. Since rotation in one direction does not have fixed point, it can be seen as an unramified cover over $H = C_p$ with structure group $A = C_q$.

13 **Star and wheel graphs.** Let G be a star graph with $n+1$ vertices and n edges. It carries a cyclic automorphism group of order n turning the spikes. The center point is the ramification point with index $e_x = n$. The quotient graph H is a two point graph K_2 with Euler characteristic 2. The Riemann-Hurwitz formula gives $1 = \chi(G) = n\chi(H) - (n-1)$.
The wheel graph $G = W_n$ is a discrete analogue of the unit disc. It also has

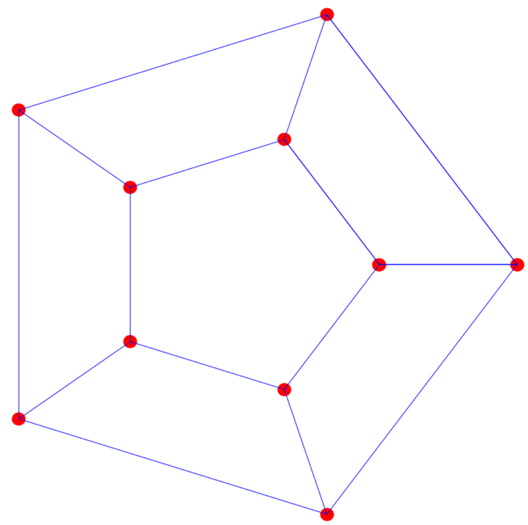

FIGURE 2. The Cayley graph G of the dihedral group $D(5,2)$ which is a finitely presented group $\langle u, v | u^5 = v^2 = 1 \rangle$. The automorphism $a : x \to vx$ shows that G can be thought of as a $n = 2$ fold cover of $H = G/A = C_5$. The ramification points are the connecting edges. They have the ramification index $e_x = 2$. The Riemann-Hurwitz formula gives

$$-5 = 10 - 15 = \chi(G) = 2\chi(H) - \sum_x (e_x - 1) = 2 \cdot 0 - 5 \; .$$

Euler characteristic 1 and a cyclic group $A = Z_n$ acting on it. The quotient $H = G/A$ is a chain of Euler characteristic 1 consisting of 1 triangle, two vertices, one edge and one loop. There is one branch point with ramification index $n-1$. Now let Z_p act on W_{pq} with $q > 3$, then the quotient is a wheel graph W_q of Euler characteristic 1. Again we have $\chi(G) = n\chi(G/A) - (n-1)$.

6. Applications

1 Spheres A convex polyhedron H defines a two dimensional graph if it is a triangularization of a sphere.

Corollary 6.1. *There is no connected unramified cover of a convex polyhedron H.*

Proof. We have $\chi(H) = 2$. If there are no ramifications, we have $\chi(G) = n\chi(H)$ which is impossible for two dimensional graphs for which the Euler-Poincaré formula shows $\chi(G) = v - e + f = b_0 - b_1 + b_2 = 2 - b_1$. In particular, the Euler characteristic can not be larger than 2. The Riemann-Hurwitz formula shows that we need ramification points to get from $n\chi(H) = 4$ to $\chi(G) \leq 2$. □

Corollary 6.2. *A triangularization of a sphere can only be an unramified cover of order 2. In that case H is a triangularization of the projective plane.*

Proof. Let G be the cover. We have

$$2 = \chi(G) = n\chi(H)$$

which forces n to be be two and $\chi(H)$ to be 1. Now we have a discrete sphere with a ZZ_2 action without fixed points. This is a discretization of the projective plane. □

2 **Sizings** Groups can be represented in a vector space, presentated using generators and relations or realizes as subgroups of a finite permutation group. Groups can also be presented as symmetry groups of graphs. A theorem of Frucht assures that any finite group as an automorphism group of a finite simple graph. The Riemann-Hurwitz point of view is to look at a group A as a subgroup of the automorphism group and to look at $G/A = H$.

Definition. *Let B be a finite group and S a set of generators which are symmetric $S = S^{-1}$ and do not contain 1, then the Cayley graph of B is a finite simple graph.*

Definition. *Let B be a group. Define $\chi(B)$ to be the smallest Euler characteristic which a Cayley graph G of B can have. Let $\chi_2(B)$ be the smallest Euler characteristic which a triangle free Cayley graph G of B can have.*

Examples:
1) For a trivial group B we have $\chi(B) = 1$. **2)** For a cyclic group C_n we have $\chi(B) \leq -n$. **3)** For a dyadic group D_n we have $\chi(B) \leq -n$. **4)** If B is a group of n elements and if S is the set of nonzero elements, then the Cayley graph is the complete graph. We can therefore always have a Cayley graph with $\chi(G) = 1$.

Definition. *A* **sizing** *is an integer valued function which satisfies $f(A) \leq f(B)$ if A is isomorphic to a subgroup of B.* [9].

Proposition 6.3 ($-\chi_2$ is a sizing)**.** *$\chi_2(A) \geq \chi_2(B)$ if A is a subgroup of B.*

Lemma 6.4 (Consequence of Riemann-Hurwitz)**.** *If G is a triangle-free Cayley graph for a group B and A is a subgroup of B of order n, then the chain $H = G/A$ satisfies $\chi(H) \geq \chi(G)/n$.*

Proof. The RH result shows $\chi(G) = n\chi(H) - R$ where $R \geq 0$. □

Now to the proof of the proposition

Proof. Let G be a triangle free Cayley graph of B with minimal Euler characteristic. For a connected triangle free graph, the Euler characteristic is $1 - b_1 \leq 1$ where b_1 is the first Betti number. The case when the Euler characteristic is 1 means that the graph G is contractible and then $H = G/A$ is contractible too, meaning that $\chi(H) = \chi(G)$ and implying $\chi(G) \leq 0$.

Let A be a subgroup of B. Assume we have a Cayley graph H for A with smaller Euler characteristic than $\chi(G)$ that is $\chi(H) < \chi(G) \leq 0$. But then $\chi(H)n < \chi(G)$ contradicting the lemma. □

Remarks.
1) $\chi(G)$ is not a sizing. The trivial group has $\chi(G) = 1$ and for $G = S_3$ the range of Euler characteristics of Cayley graphs reaches 2 as the octahedron can be realized as the Cayley graph of S_3.
2) If G is the Cayley graph of a group of order n and the graph has a vertex with valence $> n/2$ then G has a triangle.
3) Small Euler characteristic can be obtained by taking many generators trying to avoid triangles. How small can $\chi(G)$ or the sizing $\chi_2(G)$ become?

3 Hurwitz automorphism theorem

Definition. *A graph G is called a "two dimensional geometric" if it is simple, connected and orientable and the unit sphere to every point is a circular graph C_d at every vertex.*

Lemma 6.5. *The Euler characteristic $\chi(G)$ of a two dimensional orientable geometric graph is $2-2g$ where g is called the genus of the surface.*

Proof. The graph G is a triangularization of a two dimensional compact manifold for which a classification by genus is well known. □

Definition. *The* **curvature** *at an interior vertex of a two dimensional graph is $K = 1 - d/6$. A graph has constant curvature if $K(x)$ is the same for every vertex.*

Again, one can see by embedding the graph as a triangularization of a constant curvature manifold that any two dimensional constant negative constant curvature graph G can be written as $G = M/B$ where M is the infinite constant curvature graph and B is an automorphism group of M for which M/B is finite. The graph M is a discrete analogue of the hyperbolic plane.

Theorem 6.6 (Hurwitz Automorphism theorem)**.** *The maximal order of an automorphism group A acting simply and without ramification points on a connected orientable two-dimensional constant negative curvature graph is $84(g-1)$.*

Proof. The graph G has a discrete constant curvature plane as a universal cover. The graph $H = G/A$ is a constant curvature graph too and has has a fundamental domain is a connected subgraph which is geometric with boundary. The boundary can be seen as a polygon with vertices consisting of points where the geodesic curvature is not zero. At each of those corner points, the angle is $1/k_i$ of the full circle. Let A be the group of order n acting on G. The Euler characteristic of $H = G/A$ is $\chi(G)/n$ as a graph. The Euler characteristic is 1 for the fundamental domain which is a contractible graph with boundary. By Gauss-Bonnet, the number a of vertices of in the interior satisfies

$$a(1 - d/6) + \sum_{i=1}^{m} (1 - 2/k_i) = 1 .$$

Because of Gauss-Bonnet applied to the graph G we have from Riemann-Hurwitz:

$$n * a(1 - d/6) = 2 - 2g \ .$$

Putting these two equations together gives

$$(2g - 1)/n = (m - 1) - \sum_{i=1}^{m} 2/k_i \geq 1/21 \ ,$$

where the right estimate is an elementary arithmetic fact. Therefore $n \leq 21(2 - 2g)$. Because we can have an additional involution, the estimate doubles. ☐

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