Definitions

Let $G = (V, E)$ be a **finite simple graph** with vertex set V and edge set E. The **f-vector** $v(G) = (v_0(G), v_1(G), \dots, v_d(G))$ contains as coordinates the number $v_k(G)$ of complete subgraphs K_{k+1} of G. These subgraphs are also called **k-simplices** or **cliques**. The **f-matrix** $V(G)$ has the entries $V_{ij}(G)$ counting the number of intersecting pairs (x, y), where x is a i-simplex and y is a j-simplex in G. The **Euler characteristic** of G is $\chi(G) = \sum_i (-1)^i v_i(G)$. The **Wu characteristic** of G is $\omega(G) = \sum_{ij} (-1)^{i+j} V_{ij}(G)$. If A, B are two graphs, the **graph product** is a new graph which has as vertex set the set of pairs (x, y), where x is a simplex in A and y is a simplex in B. Two such pairs $(x, y), (a, b)$ are connected in the product if either $x \subset a, y \subset b$ or $a \subset x, b \subset y$. The product of G with K_1 is called the **Barycentric refinement** G_1 of G. Its vertices are the simplices of G. Two simplices are connected in G_1 if one is contained in the other. If W is a subset of V, it **generates the graph** (W, F), where F is the subset of $(a, b) \in E$ for which $a, b \in W$. The **unit sphere** $S(v)$ of a vertex v is the subgraph generated by the vertices connected to v. A function $f : V \to R$ satisfying $f(a) \neq f(b)$ for $(a, b) \in E$ is called a **coloring**. The minimal range of a coloring is the **chromatic number** of G. Define the **Euler curvature** $\kappa(v) = \sum_{k=0}^{\infty} (-1)^k v_{k-1}(S(v))/(k+1)$ and the **Poincaré-Hopf index** $i_f(v) = 1 - \chi(S_f^-(v))$, where $S_f^-(x)$ is the graph generated by $S_f^-(v) = \{w \in S(v) \mid f(w) < f(v)\}$. Fix an orientation on **facets** on G simplices of maximal dimension. Let $\Omega^{-1} = \{0\}, \Omega^k(G)$ be the set of functions $f(x_0, \dots, x_k)$ from the set of k-simplices of G to R which are anti-symmetric. The **exterior derivatives** $d_k f(x_0, \dots, x_k) = \sum_{j=0}^{k} (-1)^j f(x_0, \dots, \hat{x}_j, \dots x_k)$ defines linear transformations. The orientation fixes **incidence matrices**. They can be collected together to a large $n \times n$ matrix d with $n = \sum_{i=0}^{d} v_i$. The matrix d is called the **exterior derivative**. Since $d_{k+1} d_k = 0$, the image of d_k is contained in the kernel of d_{k+1}. The vector space $H^k(G) = \ker(d_k)/\mathrm{im}(d_{k-1})$ is the **k'th cohomology group** of G. Its dimension $b_k(G)$ is called the **k'th Betti number**. The symmetric matrix $D = d + d^*$ is the **Dirac operator** of G. Its square $L = D^2$ is the **Laplacian** of G. It decomposes into blocks L_k the **Laplacian on k-forms**. The **super trace** of L is $\mathrm{str}(L) = \sum_k (-1)^k \mathrm{tr}(L_k)$. A **graph automorphism** T is a permutation of V such that if $(a, b) \in E$, then $(T(a), T(b)) \in E$. The set of graph automorphisms forms a **automorphism group** $\mathrm{Aut}(G) = \mathrm{Aut}(G_1)$. If A is a subgroup of $\mathrm{Aut}(G)$, we can form G_k/A which is for $k \geq 2$ again a graph. Think of G as a **branched cover** of G/A. For a simplex x, define its **ramification index** $e_x = 1 - \sum_{1 \neq T \in A, T(x) = x} (-1)^{\dim(x)}$. Any $T \in \mathrm{Aut}(G)$ induces a linear map T_k on $H^k(G)$. The **super trace** $\chi_T(G) = \sum_k (-1)^k \mathrm{tr}(T_k)$ is the **Lefschetz number** of T. For $T = 1$ we have $\chi_{\mathrm{Id}}(G) = \chi(G)$. For a simplex $x = T(x)$, define the **Brouwer index** $i_T(x) = (-1)^{\dim(x)} \mathrm{sign}(T|x)$. An integer-valued function X in Ω is called a **divisor**. The **Euler characteristic** of X is $\chi(X) = \chi(G) + \sum_v X(v)$. If $v_2(G) = 0$, it is $1 - b_1 + \deg(X)$, where $b_1 = g$ is the **genus**. If f is a divisor, then $(f) = Lf$ is called a **principal divisor**. Two divisors are **linearly equivalent** if $X - Y = (f)$ for some f. A divisor X is **essential** if $X(v) \geq 0$ for all $v \in V$. The **linear system** $|X|$ of X is $\{f \mid X + (f) \text{ is essential }\}$. The **dimension** $l(X)$ of X is -1 if $|X| = \emptyset$ and else max $k \geq 0$ so that for all $m \leq k$ and all $\chi(Y) = m$ the divisor $X - Y$ is essential. The **canonical divisor** K is for graph without triangles defined as $K(v) = -2\kappa(v)$, where $\kappa(v)$ is the curvature. The **dimension** of a graph is -1 for the empty graph and inductively the average of the dimensions of the unit spheres plus 1. For the rabbit graph G, the f-vector is $v(G) = [5, 5, 1]$, the f-matrix is $V(G) = [[5, 10, 3], [10, 21, 5], [3, 5, 1]]$. Its dimension is $\dim(G) = 7/5$, the chromatic number is 3, the Euler Betti numbers are $b(G) = (1, 0)$ which super sums to Euler characteristic $\chi(G) = \omega_1(G) = \sum_i (-1)^i v_i(G) = \sum_i (-1)^i b_i(G) = 1$. The Wu Betti numbers are $b(G) = (0, 2, 6, 1, 0)$. The super sum is the Wu characteristic $\omega_2(G) = \sum_{i,j} (-1)^{i+j} V(G)_{ij} = 3$. The cubic Wu Betti numbers are $(0, 2, 16, 34, 16, 1, 0)$ which super sums to cubic Wu $\omega_3(G) = -5$.

Theorems

Here are adaptations of theorems for Euler characteristic to graph theory. Take any graph G. The left and right hand side are always **concrete integers**. The theorem tells that they are equal and show: "**The Euler characteristic is the most important quantity in all of mathematics**".

Gauss-Bonnet: $\chi(G) = \sum_{v \in V} \kappa(v)$

Poincaré-Hopf: $\chi(G) = \sum_{v \in V} i_f(v)$

Euler-Poincaré: $\chi(G) = \sum_i (-1)^i b_i(G)$

McKean-Singer: $\chi(G) = \mathrm{str}(e^{-tL})$

Steinitz-DeRham: $\chi(G \times H) = \chi(G)\chi(H)$

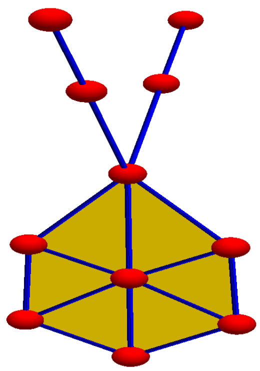
Brouwer-Lefschetz: $\chi_T(G) = \sum_{x=T(x)} i_T(x)$

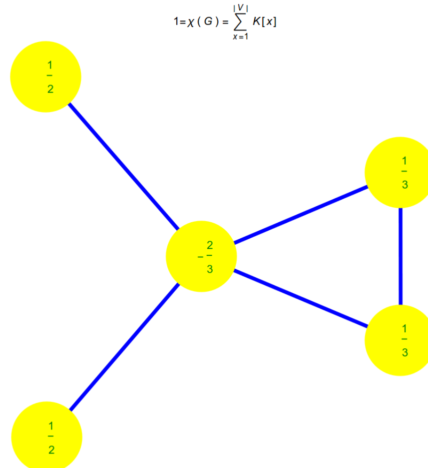
Riemann-Hurwitz: $\chi(G) = |A| \, \chi(G/A) - \sum_x (e_x - 1)$

Riemann-Roch: $\chi(X) = l(X) - l(K - X)$

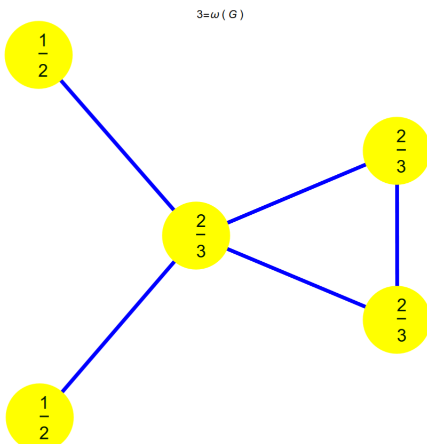
All theorems except Riemann-Roch hold for general finite simple graphs. The complexity of the proofs are lower than in the continuum. In continuum geometry, lower dimensional parts of space are access using "sheaves" or "integral geometry" of "tensor calculus". For manifolds, a functional analytic frame works like elliptic regularity is needed to define the **heat flow** e^{-tL}. We currently work on versions of these theorems to all the **Wu characteristics** $\omega_k(G) = \sum_{x_1 \sim \cdots \sim x_k} \omega(x_1)\omega(x_2)\cdots\omega(x_k)$, where the sum is over all ordered k tuples of simplices in G which intersect. Besides proofs, we built also computer implementation for all notions allowing to compute things in concrete situations. There are Wu versions of curvature, Poincaré-Hopf index, Brouwer-Lefschetz index and an interaction cohomology. All theorems generalize. Only for Riemann-Roch, we did not complete the adaptation of **Baker-Norine theory** yet, but it looks very good (take Wu curvature for canonical divisor and use Wu-Poincaré-Hopf indices). What is the significance of Wu characteristic? We don't know yet. The fact that important theorems generalize, generates optimism that it can be significant in physics as an **interaction functional** for which extremal graphs have interesting properties. **Tamas Reti** noticed already that for a triangle free graph with n vertices and m edges, the Wu characteristic is $\omega(G) = n - 5m + M(G)$, where $M(G) = \sum_v \deg(v)^2$ is the **first Zagreb index** of G. For Euler characteristic, we have guidance from the continuum. This is no more the case for Wu characteristics. Nothing similar appears in the continuum. Related might be **intersection theory**, as Wu characteristic defines an **intersection number** $\omega(A, B) = \sum_{x \sim y, x \subset A, y \subset B} \omega(x)\omega(y)$ for two subgraphs A, B of G. To generalize the parts using cohomology, we needed an adaptation of simplicial cohomology to a cohomology of interacting simplices. We call it **interaction cohomology**: one first defines discrete **quadratic differential forms** $F(x, y)$ and then an exterior derivative $dF(x, y) = F(dx, y)$. As $d^2 = 0$, one gets cohomologies the usual way.

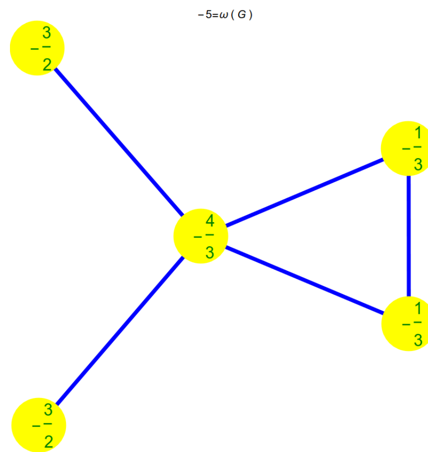
The rabbit graph



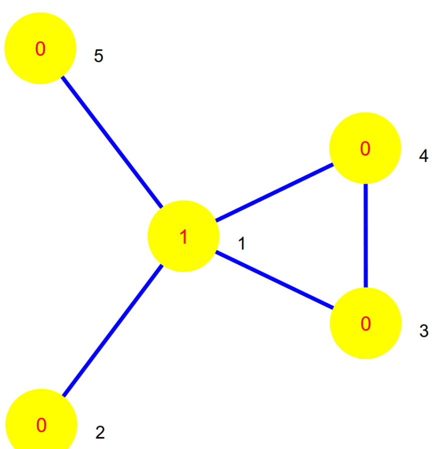


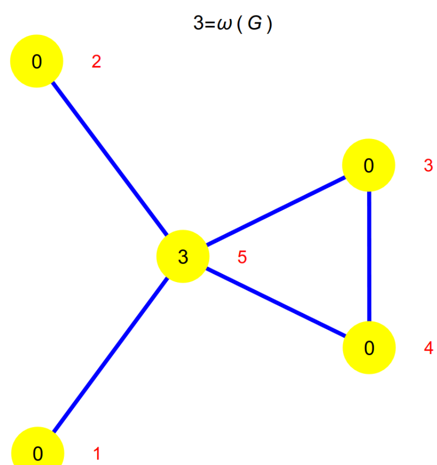
1. Barycentric refinement of the **rabbit graph** R. **2.** Euler curvatures.



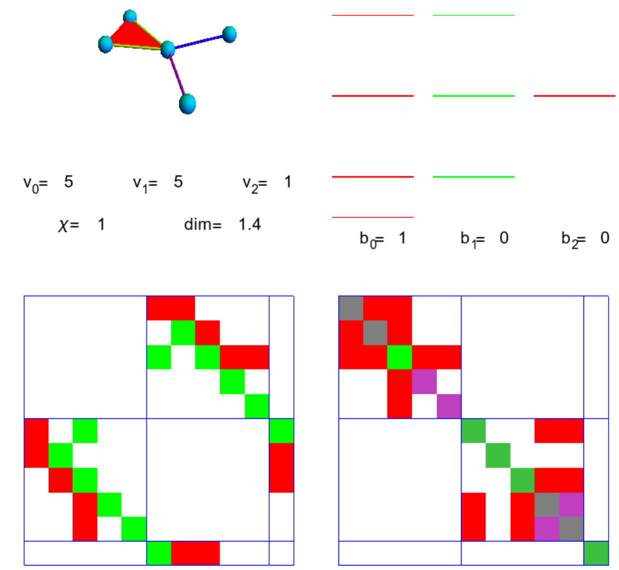


3. Wu curvatures adding up to ω. **4.** Cubic Wu curvatures adding up to ω_3.

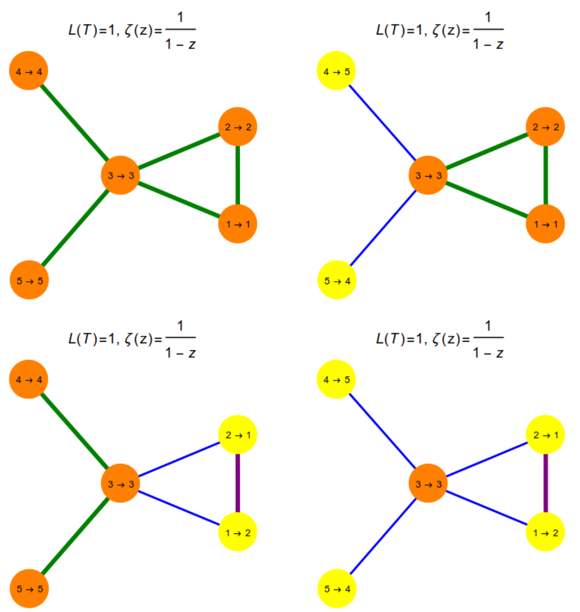




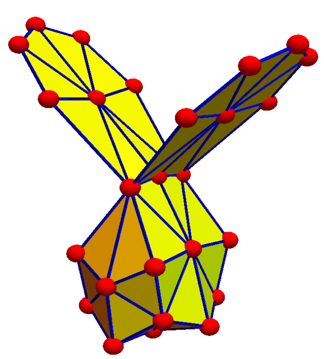
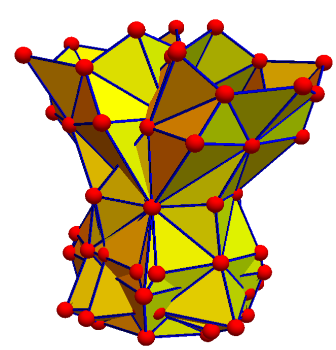
5. Poincaré-Hopf indices. **6.** Wu-Poincaré-Hopf indices



7. Spectrum $\sigma(L_0) = \{5,3,1,1,0\}$,$\sigma(L_1) = \{5,3,3,1,1\}$, $\sigma(L_2) = \{3\}$, Betti numbers=dim(ker(L_k)), Dirac operator D and Laplacian L of R with 3 blocks. **Super symmetry**: **Bosonic spectrum** $\{5,3,1,1\} \cup \{3\}$ agrees with **Fermionic spectrum** $\{5,3,3,1,1\}$.



8. Lefschetz numbers of the 4 automorphisms in Aut(R) $= Z_2 \times Z_2$.

9. 3D rabbit $R \times K_2$ with $\omega = -3$, dim $= 2.59433 \geq \dim(R) + 1 = 2.4$.
10. curled rabit $G = R \times C_4$ with $\chi(G) = \omega(G) = 0$, $b_0 = 1, b_1 = 1$.

Calculus

Fix an orientation on simplices. The exterior derivatives are the **gradient**

$$\mathrm{grad} = d_0 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

the **curl**

$$\mathrm{curl} = d_1 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

which satisfies $\mathrm{curl}(\mathrm{grad}) = 0$ and the **divergence**

$$\mathrm{div} = d_0^* = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The **scalar Laplacian** or **Kirchhoff matrix** $L_0 = d_0^* d_0 = \mathrm{div}(\mathrm{grad})$ is

$$L_0 = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

It does not depend on the chosen orientation on simplices and is equal to $B - A$, where B is the diagonal matrix containing the **vertex degrees** and A is the **adjacency matrix** of G. Has eigenvalues $\sigma(L_0) = \{5, 3, 1, 1, 0\}$ with one-dimensional kernel spanned by $[1, 1, 1, 1, 1]^T$. Also the **1-form Laplacian** $L_1 = d_1^* d_1 + d_0 d_0^*$ does not depend on the orientation of the simplices:

$$L_1 = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 1 \\ -1 & 0 & -1 & 1 & 2 \end{bmatrix}$$

with eigenvalues $\sigma(L_1) = \{5, 3, 3, 1, 1\}$ with zero dimensional kernel. This reflects that the graph is **simply connected**.

The 2-form Laplacian $d_1 d_1^*$

$$L_3 = \begin{bmatrix} 3 \end{bmatrix}$$

has a single eigenvalue $\{3\}$.

Dirac operator $D = d + d^*$ is

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

The form Laplacian or **Laplace Beltrami operator** $L = D^2 = (d + d^*)^* = dd^* + d^*d$ is

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Partial difference equations

Any continuum PDE can be considered on graphs. They work especially on the rabbit. The Laplacian $L = (d + d^*)^2$ can be either for the exterior derivative d for forms or quadratic forms. Let λ_k denote the **eigenvalues** of the form Laplacian L:

The **Wave equation**

$$u_{tt} = -Lu$$

has **d'Alembert solution** $u(t) = \cos(Dt)u(0) + \sin(Dt)D^{-1}u'(0)$. Using $u(0) = \sum_n u_n f_n$ and $v(0) = u'(0) = \sum_n v_n f_n$ with **eigenvectors** f_n of L, one has the solution

$$u(t) = \sum_n \cos(\sqrt{\lambda_n} t) u_n f_n + \sin(\sqrt{\lambda_n} t) v_n f_n / \sqrt{\lambda_n} \; .$$

The **Heat equation**

$$u_t = -Lu$$

has the solution $u(t) = e^{-Lt}u(0)$ or $\sum_n e^{-\lambda_k t} u_n f_n$, where $u(0) = \sum_n u_n f_n$ is the eigenvector expansion of $u(0)$ with respect to eigenvectors f_n of L.

The **Poisson equation**

$$Lu = v$$

has the solution $u = L^{-1}v$, where L^{-1} is the pseudo inverse of L. We can also assume v to be perpendicular to the kernel of L.

The **Laplace equation**

$$Lu = 0$$

has on the scalar sector Ω^0 only the locally constant functions as solutions. In general, these are **harmonic functions**.

Maxwell equations

$$dF = 0, d^*F = j$$

for a 2-form F called **electromagnetic field**. In the case of the rabbit, F is a number attached to the triangle. j, the **current** is a function assigned to edges. If $F = dA$ with a **vector potential** A. If A is Coulomb gauged so that $d^*A = 0$, then we get the Poisson equation for one forms

$$L_1 A = j \; .$$

This solves the problem to find the electromagnetic field $F = dA$ from the current j. In the rabbit case, since it is simply connected, there is no kernel of L_1 and we can just invert the matrix.

The electromagnetic field $F = dA$ defined on the rabbit is a number attached to the triangle:

Let there be light!