GREEN FUNCTIONS OF ENERGIZED COMPLEXES

OLIVER KNILL

1. Results

1.1. A function $h : G \to \mathbb{K}$ from a finite abstract simplicial complex G to a ring $\mathbb{K}$ with conjugation x^* defines $\chi(A) = \sum_{x \in A} h(x)$ and $\omega(G) = \sum_{x,y \in G, x \cap y \neq \emptyset} h(x)^* h(y)$. Define $L(x,y) = \chi(W^-(x) \cap W^-(y))$ and $g(x,y) = \omega(x)\omega(y)\chi(W^+(x) \cap W^+(y))$, where $W^-(x) = \{z \mid z \subset x\}$, $W^+(x) = \{z \mid x \subset z\}$ and $\omega(x) = (-1)^{\dim(x)}$ with $\dim(x) = |x| - 1$.

1.2. The following relation [8] only requires the addition in $\mathbb{K}$

Theorem 1. $\chi(G) = \sum_{x,y \in G} g(x,y)$

1.3. The next new **quadratic energy relation** links simplex interaction with multiplication in $\mathbb{K}$. Define $|h|^2 = h^* h = N(h)$ in $\mathbb{K}$.

Theorem 2. $\omega(G) = \sum_{x,y \in G} \omega(x)\omega(y)|g(x,y)|^2$.

1.4. The next determinant identity holds if h maps G to a division algebra $\mathbb{K}$ and det is the **Dieudonné determinant** [1]. The geometry G can here be a finite set of sets and does not need the simplical complex axiom stating that G is closed under the operation of taking non-empty finite subsets.

Theorem 3. $\det(L) = \det(g) = \prod_{x \in G} h(x)$.

1.5. If $h : G \to \mathbb{K}$ takes values in the **units** $U(\mathbb{K})$ of $\mathbb{K}$, like i.e. $\mathbb{Z}_2, U(1), SU(2), \mathbb{S}^7$ of the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, the **unitary group** $U(H) \cap \mathbb{K}$ of an operator C^*-algebra $\mathbb{K} \subset B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ or the units in a ring $\mathbb{K} = O_K$ of integers of a number field K, and if G is a simplicial complex, then:

Theorem 4. *If $h(x)^* h(x) = 1$ for all $x \in G$, then $g^* = L^{-1}$.*

Date: October 18, 2020.

1991 *Mathematics Subject Classification.* 05C10, 57M15.

Key words and phrases. Geometry of simplicial complexes.

1.6. For an overview of simplicial complexes and references, see [5]. Except for Theorem (2), the results were known [8, 7] in special cases like the topological $h(x) = \omega(x)$, where $\chi(G) = \sum_{x \in G} \omega(x)$ is the **Euler characteristic** and $\omega(G) = \sum_{x \sim y \in G} \omega(x)\omega(y)$ is the **Wu characteristic** [11, 2]. The pair $(G, \mathbb{K})$ is an example of a **ringed space** or a sheaf. For $\mathbb{K} = \mathbb{Z}$ we might think of h as a **divisor** and $h(G)$ as its degree, for $\mathbb{K} = \mathbb{C}$ as a quantum mechanical wave and $|\omega(h)|^2 = h^* h$ as a **probability amplitude**, for $\mathbb{K} = \mathbb{R}^n$, we might interpret h as a section of a vector bundle or as an embedding of G in $\mathbb{R}^n$ like for example when doing a geometric realization of G, where $h(x)$ is the location of the simplex in space.

1.7. When taking $\mathbb{K} = \mathcal{G}$ as the ring generated by simplicial complexes and $h(x) = X(x)$, the complex generated by $x \in G$, we can see $G \in \mathbb{K} \to \exp(G) = \det(L(G)) \in \mathbb{K}$ as an **exponential map** because $\exp(G_1 + G_2) = \exp(G_1)\exp(G_2)$ as addition $G_1 + G_2$ is the disjoint union and Theorem (3) shows that we get from a sum a product.

2. Proofs

2.1. We reprove Theorem (1) algebraically. The setup in[8] was harder, because we did not start with the explicit expressions for $g(x, y)$ yet. Let $\{x_1, \dots, x_n\}$ enumerate the elements of $V = \bigcup_{x \in G} x$ and h take values in the **free algebra** (monoid ring) generated by the variables $x_1, \dots, x_n$. If $\mathbb{K}$ is commutative, we can work with the **polynomial algebra** $\mathbb{Z}[x_1, \dots x_n]$. The algebraic picture is now transparent:

Proof. Write $h(x) = x$ and have x be the variable associated to the set $x \in G$. The matrix entries of L and g are linear expressions:

$$L(u, v) = \sum_{x \in G, x \subset u \cap v} x \; ,$$

$$g(u, v) = \sum_{x \in G, u \cup v \subset x} \omega(u)\omega(v) x \; .$$

Seen as such, the claim is the algebraic relation

$$\sum_{x \in G} x = \sum_{x \in G} [\sum_{u, v \in G, u \cup v \subset x} \omega(u)\omega(v)] x \; .$$

Because x is a simplex of Euler characteristic 1, we have $\sum_{u \subset x} \omega(u) = 1$ and $\sum_{v \subset x} \omega(v) = 1$ so that also

$$[\sum_{u, v \in G, u \cup v \subset x} \omega(u)\omega(v)] = [\sum_{u \in G, u \subset x} \omega(u)]^2 = 1 \; .$$

□

2.2. Theorem (2) can also be seen algebraically. While one needs to distinguish xy and yx in the non-commutative case, associativity does not yet factor in because only products of two elements occur. The Theorem also so holds also for octonions $\mathbb{K} = \mathbb{O}$ or Lie algebras $x * y = [x, y]$ or if $xy = \langle x, y \rangle$ is considered to be an inner product.

Proof. When writing the expressions algebraically, $\omega(G) = \sum_{x \cap y \neq \emptyset} x^* y$ is a **generating function** for all intersection relations in the complex G. Take a pair of sets x, y which do need to be different and look at the expression $x^* y$ on the left. On the right, the term $x^* y$ appears if we consider $g(u, v)$ for any pair $u, v \subset x \cap y$. We see especially that x and y need also to have a non-empty intersection to the right. We have to show

$$x^* y = \sum_{u,v \subset x \cap y} \omega(u)\omega(v) x^* y \ .$$

We get the same term $x^* y$ on the right because

$$\sum_{u \cup v \subset x \cap y} \omega(u)\omega(v) [\sum_{u \subset x \cap y} \omega(u)][\sum_{v \subset x \cap y} \omega(v)]$$

which is $\chi_{top}(x \cap y)^2 = 1$. □

2.3. Theorem (3) holds more generally for any set G of non-empty sets, where also the empty set $\emptyset$ (= void) is allowed. Unlike for simplicial complexes, the class of sets of sets has an involution $x \leftrightarrow x' = V = \bigcup_{x \in G} x \setminus x$, assigning to x its complement $x' \in V$. The proof makes use of this duality switching $W^+(x)$ and $W^-(x)$ as to establish linearity of det in one variable, we need both a proportionality factor 1 as well as the affinity factor 0 in each variable.

Proof. Because $L^+(u, v) = \sum_{x \in G, x \subset u \cap v} x$ and $L^-(u, v) = \sum_{x \in G, u \cup v \subset x} x$ are dual to each other in the category of sets of sets, we only need to verify the identity for $L = L^-$ or L^+. We can use induction with respect to the number of elements n in G and use that if we we lift a property for L^+ from $(n-1)$ to n, we have also shown it for L^-. For $n = 1$, the situation is clear as then $L^+ = L^- = [x]$ is a 1×1 matrix. In general because the matrix entries of L are linear in each variable, a Laplace expansion will show in the induction that the determinant is affine $a_k x_k + b_k$ in each variable x_k. We need then to establish multi-linearity. The induction assumption is that for any set of $(n-1)$ sets like $\{x_2, \dots, x_n\}$ or we have $\det(L^+) = \det(L^-) = \prod x_i$, which is a multi-linear expression in each of the variables. Lets assume that x_1 is a minimal element as a set then L^- has zero column and row entries if its value is zero and deleting these rows and columns produces the connection matrix L^+ of a set of sets without the x_1 set in which some entries are changed. Still as a row is zero, the expression $\det(L^+(x_1))$ is linear ax_1 in x_1 for some a and not affine $ax_1 + b$. To fix the proportionality factor a we use duality and look at x_1 in $L^+(x)$ which corresponds to take a maximal element x_n in $L^-(x)$ which means that it is a minimal element in the dual picture. Given G with n elements, and $x = x_n$ is maximal, we look at $\det(L^-(x))$. In that matrix L^-, only the corner entry $L^-_{n,n}$ contains a linear expression in x and x does not appear anywhere else. A Laplace expansion shows then that the determinant is of the form $x\det(A) + b$, where A is the $n-1 \times n-1$ matrix in which the last column and row is deleted and b is some constant. Together, these two insights show adding a new set, the determinant is a linear function in the energy $h(x)$ of that set so that $\det(L) = \prod_{i=1}^n x_i$. □

2.4. To see Theorem (4), we order G so that if $|x| < |y|$, then the set x comes before y in the listing of G. Also this theorem needs the simplicial complex assumption for G.

Proof. With the elements in G ordered according to dimension, the matrix g^*L is (i) upper triangular, (ii) contains terms $|x|^2 = x^*x$ in the diagonal and (iii) contains only sums of terms of the form $|y|^2 - |z|^2$ in the upper triangular part. If all $|x|^2 = 1$, these three properties (i),(ii),(iii) then show that g^*L is the identity matrix. Now to the proof of the three statements: the product $(g^*L)(x,y) = \sum_{z \in G} g^*(x,z)L(z,y)$ with $g^*(x,z) = \sum_{u, x\cup z \subset u} \omega(x)\omega(z)u^*$ and $L(z,y) = \sum_{v, v \subset z \cap y} v$ is

$$\sum_z \omega(z) g^*(x,z)L(z,y) = \sum_z \sum_{u,v,x\cup z\subset u, v\subset z\cap y} \omega(x)\omega(z)u^*v$$

which is 0 if $y \subset x$ and equal to $x^*y = x^*x = |x|^2 = 1$ if $x = y$ and which is a sum of terms $\sum_z \sum_{x\subset z\subset y} \omega(x)\omega(z)|z|^2 = 0$ if $x \subset y$. The last follows from $\sum_{x\subset z\subset y} \omega(z) = 0$ rephrasing that the reduced Euler characteristic $1 - \chi_{top}(X)$ of a simplex $X \subset G$ is zero. □

2.5. Let us formulate the last step as a lemma

Lemma 1. *If X is a complete complex with n elements and $Y \subset X$ is a complete sub-complex with $0 < m < n$ elements, then the number of odd and even dimensional simplices in X containing Y are the same.*

Proof. Assume X is the complex generated by its largest element x and Y is the complex generated by its largest element y. Define a map $\phi : z \to z \setminus y$ and build the set of sets $\phi(X) = \{\phi(z), z \in X\}$. It is an extended complete simplicial complex containing also the void $\emptyset$, a set of dimension -1 satisfies $\omega(\emptyset) = (-1)$. The f-vector $(f_{-1}, f_0, f_1, \dots, f_{n-m-1})$ has the Binomial coefficients $f_k = B(n-m, k+1)$ as components. Because $f(t) = \sum_{k=0}^{n-m} f_{k-1} t^k = (1+t)^{n-m}$ satisfies $f(-1) = 0$ for $n > m$, the number of odd and even dimensional simplices are the same. $\square$

2.6. For $m = 0$, there is one more even dimensional simplex and odd dimensional and $\sum_{y \subset x} \omega(x) = 1$ for $m = n$, there is only the simplex x and $\sum_{x \subset y \subset x} \omega(x) = \omega(x)$. For X is the complex generated by x which is $X = \{x = \{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}\}$ and $Y = \{\{y = \{1,2\}\}\}$ then $\{\{1,2\}, \{1,2,3\}\}$ is the set of sets in x containing y. It contains one even and one odd dimensional simplex. If $Y = \{y = \{1\}\}$, then $\{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ contains two even and two odd dimensional simplices.

3. Remarks

3.1. Theorem (3) justifies to see $g(x,y)$ as **Green function entries** or potential energy values between x and y. The notation $N(g(x,y))$ for arithmetic norm or the real **amplitudes** is commonly used if the ring $\mathbb{K}$ is a **number field** or **ring of integers** in a number field like $N(a+ib) = a^2 + b^2$ in $\mathbb{K} = \mathbb{Z}[i]$. In the case if $\mathbb{K}$ is a C^* algebra, then $N(x) = |h(x)|^2$ is the square of the norm of the operator $h(x)$ which is the **spectral radius** of the self-adjoint operator $x^* x$.

3.2. If $\sum_{x \in G} L(x,x)$ denotes the **trace** of L then

$$\mathrm{str}(L) = \sum_{x \in G} \omega(x) L(x,x)$$

is called the **super trace**. With the **checkerboard matrix** $S(x,y) = \omega(x)\omega(y)$ one can write $\mathrm{str}(L) = \mathrm{tr}(SL)$. Our first proof of Theorem (1) used the following identity in Corollary (2) which had been a key when proving the energy theorem in the topological case without the Green-Star formula. It actually identified with the Green function entries $K(x) = \omega(x) g(x,x)$ as **curvature** which add up by Gauss-Bonnet to Euler characteristic. Theorem (3) then identifies this curvature $K(x)$ as the **potential** which all the simplices (including x) induce on x. When $h(x) = \omega(x)$ we have seen $\omega(x) g(x,x) = 1 - S(x)$ as the reduced Euler characteristic of the unit sphere $S(x)$ in the Barycentric refinement graph of x. Even so this curvature or potential energy is an element in the ring $\mathbb{K}$, Theorem (1) can be interpreted therefore as a **Gauss-Bonnet** formula

Corollary 1. $\chi(G) = \sum_{x \in G} K(x) = \mathrm{tr}(Sg) = \mathrm{str}(g)$.

Proof. We have

$$g(u,v) = \sum_{x \in G, u \cup v \subset x} \omega(u)\omega(v)x \; ,$$

so that

$$\mathrm{str}(g) = \sum_u \omega(u) g(u,u) = \sum_u \omega(u) \sum_{x \in G, u \subset x} x$$

Comparing the coefficient of the expression x in $\chi(G) = \sum_{x \in G} x$ with the expression x appearing in $\mathrm{str}(g) = [\sum_{u \subset x} \omega(u)]x$ gives a match because the topological Euler characteristic of a simplex $x \in G$ is 1. □

3.3. Corollary 2 can be seen as a connection analogue of the discrete McKean Singer formula $\chi(G) = \mathrm{str}(e^{-Ht})$ [9, 3] for the **Hodge Laplacian** $H = (d+d^*)^2 = D^2$, where d are the **incidence matrices** of the simplicial complex. The self-adjoint Dirac operator D and its square $D^2 = H$ act on the same Hilbert space than the matrices L, g and produces a symmetry of the non-zero spectrum of even and odd dimensional forms. The dimension of the kernel of the blocks of $H = D^2$ are the Betti numbers. We should see L as the **connection analog** of D and L^*L as the analogue of H. We do not have kernels of L and no Hodge theory for L. There are some relations although. The matrix $(L+g)^*(L+g) = L^*L + g^*g + 2$ and for one-dimensional complexes, $L + g$ is the sign-less Hodge Laplacian.

Corollary 2. $\omega(G) = \mathrm{tr}(Sg^*Sg))$

Proof. Define

$$A(x,y) = (Sg^*S)(x,y) = \omega(x)\omega(y)g^*(x,y)$$

so that we can see this as a **Hilbert-Schmidt inner product**

$$\mathrm{tr}(Sg^*Sg) = \mathrm{tr}(Ag) = \sum_{x,y} A(x,y)g(x,y) = \sum_{x,y} \omega(x)\omega(y)|g(x,y)|^2 \; .$$

Now use Theorem (2). □

3.4. We still have a relation for the **cubic Wu characteristic** $\omega_3(G) = \sum_{x,y,z \in G} h(x)^* h(y) h(z)$, where the sum is over all triples which pairwise interact. We have $\omega_3(G) = \mathrm{tr}((Sg)^3)$ but this then starts to fail for $\mathrm{tr}((Sg)^4)$. We still need to investigate more these higher Wu characteristic $\omega_n(x)$ for $n \geq 3$. While for ω we look only at pair interactions, for ω_3 we look at three point interactions Because the Green function entries $g(x,y)$ can be through of as the interaction energy between x and y, it is likely that some "tensor quantity" like $L^+(x,y,z) = \chi(W^+(x) \cap W^+(y)) + \chi(W^+(x) \cap W^+(z)) + \chi(W^+(y) \cap W^+(z))$ will capture the three point interaction of the three simplices x, y, z better.

3.5. If h takes values in a real or complex Hilbert space $\mathcal{H}$, one could replace the pairing $h(x)^*h(y) \in \mathbb{K}$ with some **inner product** $h(x)^* \cdot h(y) = \langle h(x), h(y) \rangle \in \mathbb{C}$. If h takes values in unit spheres of a Hilbert space, one gets then close to an **Ising** or a **Heisenberg type model** (i.e. [10]). By the classification of real division algebras, this is natural for $\mathbb{R}$ (Ising), $\mathbb{C}$ (2D Heisenberg) and $\mathbb{H}$ (3D Heisenberg). Also the octonion case $\mathbb{O}$ or any linear space with Hilbert space works. For non-commutative cases like $\mathbb{H}, \mathbb{O}$, the determinant becomes the Dieudonné determinant which in the non-commutative division algebra case happens to agree with the **Study determinant** and in our case $\prod_x |h(x)|$. If we do not insist on working with determinants, we can have $h(x)$ take values in the unit sphere of any Hilbert space and still have $g^*L = 1$. With the dot product as "multiplication", the right hand side 1 in $g^*L = 1$ does then have real entries 1 and operator 1 entries like the matrices g and L.

3.6. If $A, B \subset G$ are any subsets with k elements, we can look at **minors** $\det(g_{A,B})$ which are matrix entries of the **exterior product** $g \wedge g \cdots \wedge g$. The **Fredholm energy** $\det(1 + g^*g)$ is a sum over all possible amplitudes $|\det(g_{A,B})|^2$, where $A, B \subset G$ have the same cardinality. This is a **generalized Cauchy-Binet** formula [4]

$$\det(1 + F^T G) = \sum_P \det(F_P)\det(G_P)$$

which holds for all $n \times m$ matrices F, G and also extends to Dieudonné determinants. We can think of a subset $A \subset G$ with $|A| = k$ as a k**-particle state** and $\chi(A)$ as a sort of momentum and $\omega(A)$ as a sort of kinetic energy. For two A, B of cardinality k, the minor $g(A, B) = \det(g_{A,B})$ is a matrix entry of $\wedge_{j=1}^k g$. The Cauchy-Binet relation $g^2(A, B) = \sum_C g(A, C)g(C, B)$ and more generally the n'th matrix power $g^n(A, B)$ sums over all paths

$$g(A, C_1)g(C_1, C_2) \dots g(C_{n-1}, B) .$$

We mention this to illustrate that there is a **multi-particle interpretation** of the set-up. The determinant $\det(L)$ is then an n-particle quantity. The additive energy $\chi(G)$ the quadratic energy $\omega(G)$ and the **Fredholm energy** $\sum_j 1 + |\lambda_j|^2$ are now all natural notions.

3.7. Unrelated to the **intersection calculus** described in Theorems (1) to (4) is an **incidence calculus** defined by **incidence matrices** d defining an **exterior derivative** satisfying $d^2 = 0$. The **Dirac matrix** $D = d + d^*$ and the **Hodge Laplacian** $H = (d + d^*)^2$ are like L, g finite matrices of the same size $n \times n$ than L or g. When doing a **Lax deformation** of D, we deform the exterior algebra the matrix entries of d become then ring valued. The Hodge matrix H is block diagonal with blocks H_i for which $\dim(\ker(H_i) = b_i$ are still **Betti numbers** defining the Poincaré polynomial $p(t) = \sum_{j=0} b_j t^j$. This information uses the topological $h(x) = \omega(x)$ and by Euler-Poincaré, the topological Euler characteristic $\chi(G) = \sum_x \omega(x)$ is the Poincaré polynomial evaluated at $t = -1$. For Wu characteristic, there is also a **quadratic incidence calculus** by defining the exterior derivative $dF(x, y) = F(dx, y) - F(x, dy)$ leading to Betti numbers and a **Wu-Poincaré polynomial** $q(t)$, where $q(-1)$ is the Wu characteristic $\omega(G) = \sum_{x \sim y} \omega(x)\omega(y)$. Also the **Wu-Poincaré map** $q : \mathcal{G} \to \mathbb{Z}[t]$ is a ring homomorphism. Unlike simplicial cohomology associated with $\chi(G)$, the **quadratic incidence cohomology** associated with $\omega(G)$ is not a homotopy invariant. But it can distinguish the cylinder from the Möbius strip. Also here, the Dirac operator can be deformed in a $\mathbb{K}$-valued frame work (for associative $\mathbb{K}$) without changing the quadratic cohomology.

3.8. For subset $A \subset G$, the sum $\omega(A) = \sum_{x,y \in A} h(x)h(y)$ does in general not relate to the Green function entries $g(x, y)$, where $g(x, y)$ is the Green function of the entire complex G. This also was the case for $\chi(A) = \sum_{x \in A} h(x)$ which is in general not the sum over all green function entries of G, nor of A (as the energy theorem requires that A is a simplicial complex). For sub-complexes $A \subset G$ we can take the Green functions of the sub-complex and ignore the outside $G \setminus A$. With $\overline{A} = \bigcup_{x \in A} W^+(x)$ as some sort of closure, we tried to see whether $\chi(\overline{A})$ agrees with $\sum_{x,y \in A} g_A(x, y)$ or $\omega(\overline{A}) = \sum_{x,y \in A} |g_A(x, y)|^2 \omega(x)\omega(y)$ but this also does not seem to work. The quantities $\omega(A)$ depends on how A is embedded in G. There are interaction energies between A and places outside A if A is not a simplicial complex itself. The boundary is crucial. We know that for a discrete manifold G without boundary in the topological case $\omega(G) = \chi(G)$ and for a discrete manifold G with boundary δG one has $\omega(G) = \chi(G) \setminus \chi(\delta G))$. This implies that $\omega(B) = (-1)^d$ for a closed ball B of dimension d and so $\omega(x) = \omega(X) = (-1)^{\dim(x)}$ if X is the complete simplicial complex generated by a simplex x. This is the reason why we denoted the Wu characteristic with ω.

3.9. If h takes values in $\{-1, 1\}$, then L, g are inverses of each other by Theorem (4) are **real integral quadratic** forms for which the number of negative eigenvalues agree with the number of negative h values. This follows from the relation $\det(L) = \det(g) = \prod_{x \in G} h(x)$ holding for all $\mathbb{C}$-valued h and which when comparing arguments shows that $\sum_j \arg(\lambda_j) = \sum_{x\in G} \arg(h(x))$. More generally:

Corollary 3. *If $\mathbb{K} = \mathbb{R}$ and $h(x) \neq 0$ for all x, then the number of elements in G with $h(x) > 0$ agrees with the number of positive eigenvalues of L or g.*

3.10. In the constant case $h(x) = 1$, the matrices L, g are **integral positive definite quadratic forms** L, g which are inverses of each other $L^{-1} = g$ and which have a **symplectic property** in that they are iso-spectral [6]. The reason for the association is that symplectic matrices have the property that the inverse of a matrix has the same eigenvalues than the matrix itself. It is known by a **theorem of Kirby** that if n is even and a $n \times n$ matrix has this spectral symmetry of $\sigma(L) = \sigma(L^{-1})$, then L is conjugated to a symplectic matrix A (meaning $A^T JA = J$ with the standard symplectic matrix J satisfying $J^2 = -1$ and $J^T = J^{-1} = -J$. The spectral property follows from the definition $A^{-1} = J^T A^T J$.) In general, since $L, g = L^{-1}$ are self-adjoint, it follows from the spectral theorem that there is an orthogonal U such that $L^{-1} = U^T LU$. In the symplectic case, the unitary matrix is $U = J$. Kirbi's observation is just that that if n is even, there is a coordinate system in which $U = J$ and that if n is odd we have an eigenvalue 1, there is a coordinate system in which the unitary U decomposes into a $(n-1) \times (n-1)$ symplectic block J and a 1×1 block 1.

3.11. Still in the case $h(x) = 1$, the **spectral Zeta function** of L $\zeta(s) = \sum_{j=1}^n \lambda_j^{-s}$ is an entire function in s satisfying the functional equation $\zeta(a + ib) = \zeta(-a + ib)$. The reason is that there is not only the symmetry $\zeta(z) = \zeta(-z)$ but also the symmetry $\zeta(z) = \zeta(z^*)$, where z^* is the complex coordinate. The same functional equation $\zeta(a + ib) = \zeta(-a + ib)$ for the zeta function holds if $h(x) = \omega(x)$ and if G is one-dimensional. In general, if h is complex valued, the zeta function needs to be defined properly as it is not clear which branch of the logarithm to use for each λ. It should then be considered for the matrix $L^*L = |L|^2$ or its inverse $g^*g = |g|^2$ which are positive definite self-adjoint matrices and so have real eigenvalues.

3.12. If $h : G \to \mathbb{K}$ takes values in the units of a ring of integers $\mathcal{O}$ in a number field $\mathbb{K}$, then g^*g is a positive definite integer quadratic form over $\mathcal{O}$ and L^*L is the inverse of g^*g. They are both positive definite $\mathcal{O}$-valued quadratic forms. We could also take the iso-spectral gg^* rather than g^*g but selfadjoint cases like $g + g^*$ or $(L + g)^*(L + g)$ do not have an inverse in general. For $\mathbb{K} = \mathbb{C}$, and $h(x) \neq 0$, we get positive definite Hermitian forms g^*g and L^*L. There is a unique

Hermitian matrix A such that $e^{-A} = g^*g$ and $e^A = L^*L$. One can get them by finding the unitary matrix U with diagonal $U^*(g^*g)U = D$ and $U^*(L^*L)U = D^{-1}$ then defining $A = U\log(D)U^*$. Now we can define for $t \in \mathbb{C}$ the one-parameter group e^{At} of operators which for $t = 1$ gives L^*L and for $t = -1$ gives g^*g.

3.13. If λ_j are the eigenvalues of $A = g^*g$, the zeta function $\zeta(s) = \sum_{j=1}^n \lambda_j^{-s}$ which can be rewritten as $\mathrm{tr}(g^*g)^s) = \mathrm{tr}(L^*L)^{-s})$. It makes sense for all $s \in \mathbb{C}$. The **Schrödinger equation** $iu' = -Au$ has the solution $u(t) = U(t)u(0) = e^{-iAt}u(0) = (g^*g)^{it}u(0)$ so that $\mathrm{tr}(U(t)) = \zeta(it)$. The zeta function is therefore both interesting for the random reversible walk $(L^*L)^n$ (when taking integer n) and for the unitary Schrödinger flow $(g^*g)^{it}$. We need only that $h(x)$ takes values in some unitary group of an operator algebra, so that Theorem 4 applies. We have now an action of the complex plane $\mathbb{C}$ which leads to a trace interpretation of the zeta function:

Corollary 4. *For $H = L^*L = e^A$ the flow H^s is defined for all complex $s \in \mathbb{C}$ and $\zeta(s) = \mathrm{tr}(H^s)$.*

With classical Laplacians this is not possible. The zeta function of the circle is related to the quantum harmonic oscillator and is the **Riemann zeta function**. The trace of the evolution in negative time only exists by analytic continuation and one has to disregard the zero energy. For classical Laplacians Δ on functions or Hodge Laplacians $(d + d^*)^2$ on forms, the heat flow can not be evolved backwards due to the existence of harmonic forms leading non-invertibility. Also discrete random walks defined by stochastic matrices not be reversed as there are always zero eigenvalues.

3.14. We could also define a **non-linear Schrödinger flow** as follows. Let $h(t) = u(t)$ define $L(t)$, then look at the differential equation $u'(t) = iH(t)u(t)$ in which the energy operator $H(t) = L(t)L(t)^*$ is defined by the wave $u(t)$. A discrete version is to start with $u(0)$, then define $u(1) = L_0^*L_0u(0)$ then $u(2) = L_1^*L_1u(1)$, where always L_k are defined by the functions $u(k) : G \to \mathbb{K}$. This flow still defines a zeta function $\zeta(s) = \mathrm{tr}(H^{-s})$ but now the eigenvalues $\lambda_k(t)$ move with time and we might have to analytically continue to define $\zeta(s)$. We have not yet explored that. The possibility to attach operators L to a wave $h : G \to \mathbb{C}$ and then **let these operators L act on the wave** is an interesting case, where fields h become **quantized** in the sense that we attach an operator to a field and let this operator propagate the field. This is an ingredient of **quantum field theories**. Only that in this combinatorial settings, it only in involves combinatorics and linear algebra, leading to non-linear ordinary differential equations. Because the dynamics does not change the norm of the operators or fields, there is a globally defined dynamics. We still need to investigate this flow and study its long term properties depending on the geometry G.

3.15. We will elaborate elsewhere more on the arithmetic of complexes G as the current work is heavily motivated by that. Complexes generate a natural ring $\mathcal{R}$ in which the addition is the disjoint union and the multiplication is the Cartesian product. There is a natural norm on this Abelian ring $\mathcal{R}$ given in terms of the **clique number** $c(G)$ of the graph complement of the connection graph of G then defining $|G| = \min_{G=A-B}|c(A)+c(B)|$ in the group completion of the monoid given by disjoint union. This works as $c(A+B) = c(A)+c(B), c(A \times B) = c(A)c(B)$ for simplicial complexes. This defines a norm satisfying the Banach algebra property $|G_1 \times G_2| \leq |G_1||G_2|$ so that we can complete the ring to a **commutative Banach algebra** and with a conjugation even to a C^***-algebra** $\mathcal{K}$ extending the Banach algebra of complex numbers $\mathbb{C}$ we know for our usual arithmetic constructs. Actually, the complex plane is a sub algebra generated by 0-dimensional complexes, leading to a complex scaling multiplication $G \to \lambda G$ for complex λ. So, the base space G is in $\mathbb{K}$ but also the target ring $\mathbb{K}$ can be that space. Now take $\mathbb{K} = \mathcal{K}$. For example, we can look at $h(x) = X$, where X is the complex generated by the set x. This function defines $\chi : \mathcal{K} \to \mathcal{K}$ given by $\chi(X) = \sum_{x \in X} h(x)$. The spectral properties of L and g are such that the spectra are the union of spectra under addition and the product of the spectra under multiplication. This shows that for every fixed complex number s, the value $G \to \zeta_G(s)$ is a character and so an element in the Gelfand spectrum of the ring $\mathbb{K}$ which by the Gelfand isomorphism is $C(K)$ for some compact topological space K (it is compact because $\mathbb{K}$ is unital). The zeta map $s \to \zeta_G(s)/n(G) \in K$, where $n(G) = \zeta_G(0) = \mathrm{tr}(L(G))^0)$ is for connected finitely generated simplices the number of elements in G, which extends to a character in $\mathcal{K}$, now embeds the complex line in the compact space K. We don't know whether this **zeta curve** is dense in the Gelfand spectrum K. We can for example ask whether $n : \mathcal{K} \to \mathbb{C}$ defined by extending cardinality to $\mathcal{G}$ or Euler characteristic $\chi_{top} : \mathcal{K} \to \mathbb{C}$ which are known to be a characters correspond to points in the spectrum K of $\mathcal{K}$, can be approximated by a zeta curve. This is related to the open question whether we can read off the topological Euler characteristic $\chi(G)$ from the spectrum of a natural connection Laplacian L like in the topological case when $h(x) = \omega(x) \in \{-1, 1\}$.

4. Examples

4.1. Lets take the example, where $\mathbb{K}$ is the free algebra generated by the variables $x_1, x_2, \dots, x_n$ augmented by conjugated entries x_k^* defining $|x_k|^2 = x_k^* x_k$ in an enumeration of $V = \bigcup_{x \in G} x = \{x_1, x_2, \dots, x_n\}$. For $G = K_2 = \{\{1\},\{2\},\{1,2\}\} = \{x_1, x_2, x_3\}$ we have $\chi(G) = x_1+x_2+x_3$ and $\omega(G) = x_1^*x_1+x_2^*x_2+x_3^*x_3+x_1^*x_3+x_3^*x_1+x_2^*x_3+x_3^*x_2$. The matrices

$$L = \left[\begin{array}{ccc} x_1 & 0 & x_1 \\ 0 & x_2 & x_2 \\ x_1 & x_2 & x_1+x_2+x_3 \end{array}\right], g = \left[\begin{array}{ccc} x_1+x_3 & x_3 & -x_3 \\ x_3 & x_2+x_3 & -x_3 \\ -x_3 & -x_3 & x_3 \end{array}\right]$$

multiply to

$$g^* L = \left[\begin{array}{ccc} |x_1|^2 & 0 & |x_1|^2-|x_3|^2 \\ 0 & |x_2|^2 & |x_2|^2-|x_3|^2 \\ 0 & 0 & |x_3|^2 \end{array}\right].$$

4.2. For the next example $G = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ lets use variables $G = \{x, y, z, a, b, c\}$. Now,

$$L = \left[\begin{array}{cccccc} x & 0 & 0 & x & x & 0 \\ 0 & y & 0 & y & 0 & y \\ 0 & 0 & z & 0 & z & z \\ x & y & 0 & a+x+y & x & y \\ x & 0 & z & x & b+x+z & z \\ 0 & y & z & y & z & c+y+z \end{array}\right],$$

$$g = \left[\begin{array}{cccccc} a+b+x & a & b & -a & -b & 0 \\ a & a+c+y & c & -a & 0 & -c \\ b & c & b+c+z & 0 & -b & -c \\ -a & -a & 0 & a & 0 & 0 \\ -b & 0 & -b & 0 & b & 0 \\ 0 & -c & -c & 0 & 0 & c \end{array}\right].$$

One can check that $\sum_{x,y} g(x,y) = a+b+c+x+y+z = \chi(G)$. We have $\omega(G) = ab+ba+ac+ca+ax+xa+ay+yz+bc+ca+bx+xb+bz+zb+cy+yc+cz+zc+|a|^2+|b|^2+|c|^2+|x|^2+|y|^2+|z|^2$, the generating function for the intersection relations. We compute $\sum_{x,y} \omega(x)\omega(y)|g(x,y)|^2 = |a+b+x|^2+|a+c+y|^2+|b+c+z|^2-|a|^2-|b|^2-|c|^2$ and can check that this is the same. We have $\det(L) = \det(g) = abcxyz$. Finally, we see

$$gL = \begin{bmatrix} |x|^2 & 0 & 0 & |x|^2-|a|^2 & |x|^2-|b|^2 & 0 \\ 0 & |y|^2 & 0 & |y|^2-|a|^2 & 0 & |y|^2-|c|^2 \\ 0 & 0 & |z|^2 & 0 & |z|^2-|b|^2 & |z|^2-|c|^2 \\ 0 & 0 & 0 & |a|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & |b|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & |c|^2 \end{bmatrix} .$$

If all entries have length 1, we get the identity matrix.

4.3. Lets look at the example $G = \{\{1\},\{2\},\{1,2,3\}\}$ which is not a simplicial complex. Denote the energy variables by $G = \{x, y, z\}$. Now,

$$L = \begin{bmatrix} x & 0 & x \\ 0 & y & y \\ x & y & x+y+z \end{bmatrix}, g = \begin{bmatrix} x+z & z & z \\ z & y+z & z \\ z & z & z \end{bmatrix} .$$

We have $\omega(G) = |x|^2 + xz + zx + |y|^2 + yz + zy + |z|^2$ and

$$\sum_{x,y} \omega(x)\omega(y)g(x,y)^2 = |x+z|^2 + |y+z|^2 + 7|z|^2$$

which are not the same. We need the simplicial complex structure. Also the energy $\chi(G) = x+y+z$ does not agree with $\sum_{x,y\in G} g(x,y) = x+y+9z$ so that Theorem (1) does not hold. We have however $\det(L) = \det(g) = xyz$. The determinant identity Theorem (3) holds in general, also if G is not a simplicial complex.

References

[1] J. Dieudonné. Les determinants sur un corps non commutatif. *Bulletin de la S.M.F.*, 71:27–45, 1943.

[2] B. Grünbaum. Polytopes, graphs, and complexes. *Bull. Amer. Math. Soc.*, 76:1131–1201, 1970.

[3] O. Knill. The McKean-Singer Formula in Graph Theory. http://arxiv.org/abs/1301.1408, 2012.

[4] O. Knill. Cauchy-Binet for pseudo-determinants. *Linear Algebra Appl.*, 459:522–547, 2014.

[5] O. Knill. The amazing world of simplicial complexes. https://arxiv.org/abs/1804.08211, 2018.

[6] O. Knill. The counting matrix of a simplicial complex. https://arxiv.org/abs/1907.09092, 2019.

[7] O. Knill. Energized simplicial complexes. https://arxiv.org/abs/1908.06563, 2019.

[8] O. Knill. The energy of a simplicial complex. *Linear Algebra and its Applications*, 600:96–129, 2020.

[9] H.P. McKean and I.M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.
[10] B. Simon. *The statistical mechanics of lattice gases*, volume Volume I. Princeton University Press, 1993.
[11] Wu W-T. Topological invariants of new type of finite polyhedrons. *Acta Math. Sinica*, 3:261–290, 1953.

Department of Mathematics, Harvard University, Cambridge, MA, 02138