

DEHN SOMMERVILLE MANIFOLDS

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ABSTRACT. Dehn-Sommerville manifolds are a class of finite abstract simplicial complexes that generalize discrete manifolds. Despite a simpler definition in comparison to manifolds, they still share most properties of manifolds. They especially satisfy all Dehn-Sommerville symmetries telling that half of the f -vector entries are redundant. They also share other properties with q -manifolds: for every Dehn-Sommerville q -manifold G and any function $g : V(G) \rightarrow A_k = \{0, \dots, k\}$ with $k \geq 1$, the set of x such that $A_k \subset g(x)$ is a Dehn-Sommerville $(q - k)$ -manifold if not empty. We also see that for Dehn-Sommerville q -manifolds, all higher characteristics $w_m(G)$ agree with Euler characteristic $w_1(G) = \chi(G)$. We also see that the chromatic number of a Dehn-Sommerville q -manifold is bounded above by $2q + 2$ and that odd-dimensional Dehn-Sommerville manifolds are flat and form a monoid under the join operation. In general, Dehn-Sommerville manifolds are invariant under edge refinement, Barycentric refinement and Cartesian products.

1. INTRODUCTION

1.1. The Dehn-Sommerville symmetry [8, 54, 18] (see also [13, 5, 4, 53, 51, 48, 6, 14, 16, 56]) can best be understood by introducing a class of finite abstract simplicial complexes which all share this symmetry. The symmetry states that half of the combinatorial data f_k counting k -dimensional part of space are redundant. This class, where this works includes manifolds but goes much beyond the usual manifolds. The class of simplicial complexes is here called the class of **Dehn-Sommerville q -manifolds**. In order to appreciate this, one should note that most literature in the context of Dehn-Sommerville is about convex polyhedra [13, 56] (and so about discrete spheres). The difficulties of defining polyhedra is a fascinating showcase in the general epistemology of science [11, 47, 13, 12]. The fact that Dehn-Sommerville identities like $-7898f_1 + 11847f_2 - 14360f_3 + 16155f_4 - 17528f_5 + 18627f_6 = 0$ hold for all 6-manifolds (and similar explicit identities for arbitrary dimensions) seems not have been realized before [29]. But there are three such identities and they hold for all 6-manifolds. They hold even for more general 6-dimensional objects like the suspension G of any 5-manifold M , which is no more a manifold if M is not a 5-sphere. The suspension of a non-sphere 5-manifold is an example of a Dehn-Sommerville 6-manifold that is not a 6-manifold. It still has the Dehn-Sommerville symmetry.

1.2. The definition is as follows: a **Dehn-Sommerville q -manifold** is a q -dimensional finite abstract complex for which all unit spheres $S(x) = \delta U(x)$, (the topological boundaries of the smallest open set $U(x) = \{y \in G, x \subset y\}$ containing $x \in G$ in the complex equipped with the finite **Alexandrov topology** [2, 39]) are Dehn-Sommerville $(q - 1)$ -manifolds of Euler characteristic $1 - (-1)^q$. By the **Euler gem formula** for the Euler characteristic of traditional discrete spheres, every discrete q -manifold is a Dehn-Sommerville manifold. The sphere formula $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$ [40] (holding for all simplicial complexes) shows that

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every odd-dimensional Dehn-Sommerville manifold is a Dehn-Sommerville sphere. For any Dehn-Sommerville q -manifold, there is a $\lfloor (q+1)/2 \rfloor$ dimensional space of valuations that vanish. In other words, half of the **combinatorial f-data** of any manifold are redundant. And this is independent of whether we have **Poincaré duality** or not. Most Dehn-Sommerville q -manifolds do not satisfy the Poincaré duality - even if they are orientable. In the next few paragraphs, we sketch the various pictures. Some of these topics are reviewed later in more detail. As mentioned in the abstract, there are also new results.

1.3. Picture 1: first of all, there are the historical valuations $X_{k,q}$ on simplicial complexes. Every **valuation** (a functional X defined on the set of subsets of G satisfying $X(A \cup B) + X(A \cap B) = X(A) + X(B)$) has its own **Gauss-Bonnet formula** $X(G) = \sum_{v \in V} K(v)$. It writes $X(G)$ as a sum of curvatures $K(v)$ on vertices $v \in V$, where V are the set zero-dimensional elements in G . (The dimension of a set $x \in X$ is $|x| - 1$ where $|x|$ is the cardinality.) The Dehn-Sommerville curvatures are special because the curvature at a vertex v agrees with $X_{k-1,q-1}$, applied to the unit sphere $S(v)$: we have $X_{k,q}(G) = \sum_{v \in V} X_{k-1,q-1}(S(v))$. The valuation $X_{-1,q} = \chi$ is the **Euler characteristic**. It is zero in the odd-dimensional case, implying $X_{0,q}$ is zero in even dimension q . So, all $X_{k,q}$ are zero for $k \geq 0$. One can then see that half of them are independent. The space of valuations on a simplicial complex of dimension q has dimension $q+1$ by the **discrete Hadwiger theorem** [17]. The functionals $G \rightarrow f_k(G)$ form a basis for $k = 0, \dots, q$ if the maximal dimension is q .

1.4. Picture 2: the second picture explains how many of these invariants are redundant. The **simplex generating function** $f_G(t) = \sum_{k=-1}^q f_k t^{k+1}$, (where the convention $f_{-1} = 1$ is used for notational reason), encodes the combinatorial **f-vector** (f_0, f_1, \dots, f_k) of G . The **functional Gauss-Bonnet theorem** $f'_G = \sum_v f_{S(v)}$ [29] which holds for every simplicial complex G , implies that Dehn-Sommerville q -manifolds have the property that f_G is either even or odd with respect to the point $t_0 = -1/2$, depending on whether the maximal dimension q of G is even or odd. By looking at roots, this can be seen to be equivalent to the statement that the polynomial $h(t) = (t-1)^q f(1/(t-1))$ has **palindromic coefficients**. The later polynomial property in turn shows that $\lfloor (q+1)/2 \rfloor$ valuations are redundant. This implies that at least half of the $q+1$ eigenvectors $Y_{k,q}$ of the transpose **Barycentric refinement operator** A_q must be zero. It will imply that at least half of all the classical valuations $X_{k,q}$ are linearly independent.

1.5. Picture 3: the third picture uses **Barycentric refinement**. The Dehn-Sommerville symmetry is a **duality**. It is a combinatorial analog of **Poincaré duality** but it is rather complementary: most Dehn-Sommerville manifolds do not satisfy Poincaré duality for cohomology. Let us look at an example which was noted in [25, 29]: the **f-vector** of **every 4-manifold** satisfies $-22f_1 + 33f_2 - 40f_3 + 45f_4 = 0$. The classical Dehn-Sommerville story as developed by the pioneers only assumes simplicial complexes to be “polyhedra”, assuming in particular that we have a sphere manifold structure. The class of Dehn-Sommerville manifolds is much larger: any suspension of an arbitrary 3-manifold for example is a Dehn-Sommerville 4-manifold. There is also no mystery about the valuations. The vector $Y_{2,4} = [0, -22, 33, -40, 45]$ is an eigenvector of the Barycentric refinement operator (a 5×5 matrix) in dimension $q = 4$. The other linearly independent Dehn-Sommerville invariant in dimension $q = 4$ is the trivial valuation $[0, 0, 0, -2, 5]$ and eigenvector of A^T which restates that 5 times the number of 4-simplices is twice the number of 3-simplices. It is rephrasing that every 3-simplex in a 4-manifold is contained in exactly two 4-simplices. This manifold property was the only one that was necessary

to define geodesic flow and sectional curvature. The work [44, 45] was the reason to revisit the Dehn-Sommerville story here.

1.6. Picture 4: the relation with the discrete version of the Gauss-Bonnet-Chern curvature provides an other picture. In [19] already, we had simplified the curvature of a 5 manifold as $K(v) = -f_1(S(v))/6 + f_2(S(v))/4 - f_3(S(v))/6$. We had used there that the unit sphere $S(v)$ has the Euler characteristic $f_0 - f_1 + f_2 - f_3 = f_4 = 2$ and that $2f_4 = 6f_5$ in the unit sphere, but we could not yet verify then (back in 2010) that this curvature was always zero for all 5-manifolds. Today we would see this curvature as a linear combination of classical Dehn-Sommerville invariants like $K(v) = X_{1,4}/12 - X_{3,4}/6$ and so verify that it is zero. As pointed out in [21] based on [20, 22, 32, 31], it is much simpler however to see curvature $K(v)$ as the expectation of symmetrized **Poincaré-Hopf indice** $j_g(v)/2 = [i_g(v) + i_{-g}(v)]/2$ which for odd-dimensional Dehn-Sommerville q -manifolds G is equal to $-\chi(M_g(v))/2$, where $M_g(v)$ is the level surface $\{g = c\} = \{w, g(w) = c = g(v)\}$ in the even dimensional manifold $S(v)$ which is the Euler characteristic of an odd-dimensional manifold and so zero (we discuss level surfaces later on). It is here that our new result that level surfaces in Dehn-Sommerville q -manifolds are Dehn-Sommerville $(q - 1)$ -manifolds comes in. The complete picture of valuations in Dehn-Sommerville manifolds illustrates the Gauss-Bonnet story, that started the research in [19].

1.7. Picture 5: there is an other line of research which fits into the Dehn-Sommerville story. This is the **connection calculus story** that has lead to **higher characteristic invariants** $w_m(G)$ [37], where Euler characteristic $\chi(G) = w_1(G)$ is the first invariant. These invariants go back to [55, 11] and are defined for any subset $A \subset G$ and given by $w_m(A) = \sum_{x \in A^m, \cap_j x_j \in A} \prod_{j=1}^m \omega(x_j)$, where $\omega(x_j) = (-1)^{\dim(x_j)}$ and $U(x) = \{y, x \subset y\}$ is the star of x . These invariants parallel the first characteristic = Euler characteristic, where simplicial cohomology satisfies the **Euler-Poincaré** and the sphere formula $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$ hold. We have seen that in general, the **sphere formula** $\sum_{x \in G} \omega(x)w_m(S(x)) = 0$ holds [40]. This immediately implied that all odd- dimensional Dehn-Sommerville manifolds G have zero higher invariants $w_m(G) = 0$. The connection calculus story is quite general [26, 35, 30, 33, 36].

1.8. As we will see here, more is true. This is something that goes beyond review and is new. All higher characteristic $w_m(G)$ are the same $w_m(G) = \chi(G)$ if G is a Dehn-Sommerville q -manifold. We knew that before only for q -manifolds, special Dehn-Sommerville manifolds in which the unit spheres are classical spheres. The subject of higher characteristic invariants is exciting. Each of these quantities can be seen as a total potential theoretic energy. There are **k-point Green function identities**: consider any k -tuple of points $X \in G^k$ which do not necessarily have to intersect. The **m'th order potential energy** of the **k-particle configuration** X is defined as $V(X) = w(X)w_m(U(X))$, where $w(X) = \prod_{j=1}^k \omega(x_j)$ and $U(X) = \bigcap_{j=1}^k U(x_j)$, and where $U(x_j)$ are the **stars**, the smallest open sets in the Alexandrov topology \mathcal{O} of the complex. The topology \mathcal{O} is a finite topology that is non-Hausdorff in positive dimensions, similarly to the Zariski topology in algebraic geometry. One of the theorems proven in [37] is that the **total k-point Green function energy** is equal to the total energy $\sum_{X \in G^k} w(X)w_m(U(X)) = w_m(G)$. For $k = 1$ already, the formula $\sum_{x \in G} w(x)w_m(U(x)) = w_m(G)$ is very useful as it is a fast way to compute the invariant w_m as a sum of curvatures = potential energies of elements $x \in G$. For $k = 2$, where the potential energy $g(x, y) = V(x, y)$ between two particles x, y defines a $n \times n$ matrix g (where n is the number of elements in G), g is the inverse of the **connection**

Laplacian matrix L . The discovery of these identities for $k = 1$ and $k = 2$ is documented in in [26, 35, 34]. It was generalized to larger k 's in [37].

1.9. Picture 6: the last picture is a relation with the **arithmetic of geometric objects**. Dual to the **disjoint union** of geometries is the **join** operation $A \oplus B = A \cup B \cup \{a \cup b, a \in A, b \in B\}$ which generates from two complexes of dimension a, b a new object of dimension $a + b + 1$. The **zero** element is the **void** $0 = \{\}$, the empty complex (which of course is again a simplicial complex as it is closed under the subset operation of its elements and because it does not contain the empty set). A **group completion** defines then the Abelian group of **signed complexes**. The join operation is useful because joining two spheres produces a new sphere. One can so for example upgrade a given q -sphere G to a $(q + 1)$ -sphere $G \oplus S^0$ by joining the 0-sphere $S^0 = \{\{1\}, \{2\}\}$. This is called the **suspension**. There is also the **Euler gem monoid** consisting of simplicial complexes, satisfying the **Euler Gem formula** $\chi(G) = 1 + (-1)^q$. Then there is the monoid of **varieties** which are inductively defined as complexes G with the property that all $S(x)$ are varieties of dimension 1 less. Also the intersection of two monoids is always again a monoid. The **Dehn-Sommerville sphere monoid** is the intersection of **variety monoid** with the **Euler-gem monoid**. Since all odd-dimensional manifolds are Dehn-Sommerville spheres, the monoid of Dehn-Sommerville spheres contains also the submonoid of all odd-dimensional Dehn-Sommerville manifolds. We therefore can “calculate” with all odd-dimensional Dehn-Sommerville manifolds. What is amazing is that Dehn-Sommerville manifolds share so many properties of actual manifolds, despite their simpler definition and despite that they are much more general.

1.10. Of course, we should also wonder about the **physical relevance** of Dehn-Sommerville manifolds. Every physical measurement that has been performed so far indicates that our physical space is a 3-dimensional space. No single measurement has indicated that our physical space has spheres $S_r(x)$ that are not topological 2-spheres or that is a higher dimensional fiber bundle in which the fibers are too small to be detected. Attempts to merge quantum mechanics and general relativity suggest that non-trivial topologies could occur at the **Planck scale** 10^{35} meters but such scales are many order of magnitudes off from experimental reach. There have been **speculations** (comparable to Demokritus' guesses) that space is quantized. The evidence of ancient philosophers, was to look at the smallest possible particles like gypsum and speculate from limitations that it is too small to be further divided. It turned out to be false; only from the 19th century on (Brownian motion) revealed that atoms have a scale of 10^{-10} meters. They themselves turned out have structure, to be divisible. Whether the smallest known quark constituents of matter at a scale of 10^{-22} are divisible (like having a **preon structure**) is again only a playground for speculations. The question whether physical space can have a Dehn-Sommerville nature is interesting. A simple starting point is the question to classify Dehn-Sommerville q -spheres. Every connected Dehn-Sommerville 2-sphere is a 2-sphere. There are plenty of disconnected Dehn-Sommerville 3-spheres, like a suspension of a disjoint union M of 2-manifolds whose sum of Euler characteristics $\chi(M)$ is 2 so that M is a Dehn-Sommerville 2-sphere.

1.11. The **Vietoris-Ribbs picture** is to take an continuum compact Riemannian q -manifold M and pick n random points on M . Given $\epsilon > 0$ and a point cloud in M that is $\sqrt{\epsilon}$ dense. Define the graph in which the n points are the vertices and where two points v, w are connected if $d(v, w) < \epsilon$. The Whitney complex of this graph is an example of a **Vietoris-Ribbs complex**. Its dimension goes to infinity if $\epsilon \rightarrow 0$ and the number of points is adapted accordingly. The

k -simplices x of this complex consist of $k+1$ points which all have distance less than ϵ from each other. Now start **melting points away** by successively removing vertices from the graph for which $S(v)$ is contractible. What we tried unsuccessfully to prove ¹, we end up with a discrete q -manifold. We could prove this only in dimension 1, where it is related to the **”machine graph”**,², where V is the finite set of the real numbers represented by a computer and ϵ is determined by machine precision, same tolerance and equal tolerance of the implemented arithmetic (for simplicity, we can identify the largest and smallest possible machine number to be topologically on a circle to have no boundary). But dimension $q = 1$ is special in that classically it is the only case, where unit spheres are disconnected. Because every connected Dehn-Sommerville 1-manifold is a circle, the failure to extend the Vietoris-Rips conjecture to higher dimensions $q > 1$ could also be founded on the fact that the reduction process could end up not in a q -manifold but Dehn-Sommerville q -manifold. It could be the case that in higher dimensions, the melting process of removing points with contractible unit spheres ends up with a Dehn-Sommerville $(q-1)$ spheres and not with $(q-1)$ sphere. We still think that the original conjecture has a chance to be true, especially in small dimensions like $q = 2$, where the Jordan curve theorem kicks in preventing unit spheres to contain a disconnected union of circles. In any case, an easier task is to ask whether that every Vietoris-Ribbs graph of a compact Riemannian q -manifold M defines for large enough n a finite Dehn-Sommerville q -manifold obtained, after successfully melting away points v which have contractible unit spheres $S(v)$.

2. DEHN-SOMMERVILLE MANIFOLDS

2.1. A **finite abstract simplicial complex** is a finite set of non-empty sets closed under the operation of taking non-empty subsets [9]. Every such geometry G carries a **finite topology** \mathcal{O} [2, 49, 23] in which the **open sets** are unions of stars $U(x) = \{y \in G, x \subset y\}$ and where the **closed sets** agree with the sub-simplicial complexes of G . Like the Zariski topology in algebraic geometry, this topology is non-Hausdorff, if the maximal dimension q of G is positive. Indeed, if v, w are zero-dimensional points contained in a simplex x , then $x \subset U(v) \cap U(w)$ so that v and w can not be separated by open sets in \mathcal{O} . Everything in this article is finite. Geometric realizations of finite abstract simplicial complexes [9] only appear for visualization purposes. No infinity axiom is ever involved.

2.2. The **unit sphere** of $x \in G$ is defined to be the topological boundary $S(x) = \delta U(x) = \overline{U(x)} \setminus U(x)$ of the **star** $U(x) = \{y \in G, x \subset y\}$ of x . The unit sphere is a sub-simplicial complex because it is the intersection of the **closed ball** $B(x) = \overline{U(x)}$ and the complement of the **open ball** $U(x)$. The empty set $0 = \emptyset = \{\}$ is a simplicial complex by definition. It is called the **void**. If x is a maximal simplex in G , then $U(x) = \{x\}$ and $\overline{U(x)} = \{y \in G, y \subset x\}$ is the **core** of x . The boundary $\delta U(x)$ is in that case the **boundary complex** $\{y \in G, y \subset x, y \neq x\}$ of the simplex x which is a closed set, a sub-simplicial complex of G .

2.3. The **void**, the empty complex 0 is the **zero** 0 in the **monoid** of all simplicial complexes with **join addition** $A \oplus B = A \cup B \cup \{a \cup b, a \in A, b \in B\}$, where \cup is **disjoint union**. The **Euler characteristic** of an arbitrary subset $A \subset G$ is defined as $\chi(A) = \sum_{x \in A} \omega(x)$, where $\omega(x) = (-1)^{\dim(x)}$ and $\dim(x) = |x| - 1$. Here, $|x|$ denotes the cardinality of $x \in X$. The maximum $\max_{x \in G} \dim(x)$ is the **maximal dimension** of G and denoted by q .

¹See our talk ”Homotopy manifolds from May 3, 2021

²See our talk ”The machine graph” from Oct 2, 2022

2.4. Elements in G of dimension k are the sets of cardinality $k + 1$ and are called k -**simplices** or k -**faces**, the 0-simplices are also called **vertices**, the q -simplices **facets**, the 1-simplices **edges**, the $(q - 1)$ simplices **walls**, the 2-simplices are **triangles** and the $(q - 2)$ -simplices **bones**. The Euler characteristic is a **valuation**, a map from subsets of G to the integers so that $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ for all $A, B \subset G$. As we review below, Euler characteristic is the only valuation that is invariant under Barycentric refinement and for which $\chi(1) = 1$ for the **one point complex** $1 = \{\{1\}\}$.

2.5. Inductively, a complex G is called a **Dehn-Sommerville q -sphere** if all unit spheres $S(x)$ are Dehn-Sommerville $(q - 1)$ -spheres and the Euler-gem formula $\chi(G) = 1 + (-1)^q$ holds. The **zero complex** = empty complex = void $0 = \{\}$ is the Dehn-Sommerville (-1) sphere. Inductively, a complex G is called a **Dehn-Sommerville q -manifold** if every unit sphere $S(x)$ is a Dehn-Sommerville $(q - 1)$ -sphere. A Dehn-Sommerville 0-manifold is a non-empty finite set of vertices. The set of 0-dimensional Dehn-Sommerville manifolds form, together with the (-1) manifold 0 , a sub-monoid, a monoid isomorphic to $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. In dimension 1, every Dehn-Sommerville 1-manifold is also a Dehn-Sommerville 1-sphere, a finite union of circles, where a single circle is a cyclic complex C_n with $n \geq 3$. For $n \geq 4$ it is the **Whitney complex** = **flag complex** = **order complex** of the graph C_n , for $n = 3$, this is not the same than $K_3 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. It is its 1-skeleton complex $C_3 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. To summarize this section, all 1-manifolds are Dehn-Sommerville manifolds and Dehn-Sommerville spheres. This generalizes to all odd dimensions q :

Theorem 1. *Every odd dimensional Dehn-Sommerville q -manifold has zero Euler characteristic. Therefore, every odd-dimensional Dehn-Sommerville manifold is also a Dehn-Sommerville sphere.*

Proof. Every complex can be partitioned into two sets $G = G_e + G_o$, where $G_e = \{x \in G, \dim(x) \text{ even}\}$ and $G_o = \{x \in G, \dim(x) \text{ odd}\}$. Euler characteristic is $\chi(G) = |G_e| - |G_o|$. The **sphere formula** [40] is $\sum_x \omega(x) \chi(S(x)) = \sum_{x \in G_e} \chi(S(x)) - \sum_{x \in G_o} \chi(S(x)) = 0$. For Dehn-Sommerville q -manifolds, the unit spheres all have the same Euler characteristic. It is a constant in $\{0, 2\}$. By the sphere formula, an odd-dimensional Dehn-Sommerville q -manifold must have zero Euler characteristic and so by definition is a Dehn-Sommerville sphere. \square

2.6. Because every odd-dimensional Dehn-Sommerville manifold is a Dehn-Sommerville sphere, the suspension of any odd dimensional Dehn-Sommerville q -manifold M is an even dimensional $(q + 1)$ -Dehn-Sommerville manifold G . If M is an even dimensional Dehn-Sommerville manifold that is not a sphere, it sill can happen that the suspension is Dehn Sommerville. An example is when M is a disjoint union of two projective planes. This is a Dehn-Sommerville 2-sphere, because it is a 2-manifold with Euler characteristic 2. Its suspension therefore is a Dehn-Sommerville 3-manifold that is not a 3-manifold. It is even a Dehn-Sommerville sphere, because every Dehn-Sommerville manifold is also a Dehn-Sommerville sphere.

3. A CLASS OF EXAMPLES

3.1. A q -**manifold** is a simplicial complex for which we require that every unit sphere $S(x)$ is a $(q - 1)$ -**sphere**. A **q -sphere** G is a q -manifold such that there exists $x \in G$ such that $G \setminus U(x)$ and $U(x)$ are both contractible. A simplicial complex G is called **contractible**, if it is either the 1-point complex $1 = \{\{1\}\} = K_1$ or then if there exists a x such that both the star

$U(x)$ and the sub-complex $G \setminus U(x)$ are contractible. Since q -spheres have Euler characteristic $1 + (-1)^q$ by the **Euler-Gem formula**, every q -sphere is also a Dehn-Sommerville q -sphere and every q -manifold is also a Dehn-Sommerville q -manifold.

3.2. The proof that a q -sphere G satisfies the **Euler-Gem formula** follows directly from the **valuation formula** $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ applied to the case, where $A = G \setminus U(x)$ and $B = \overline{U(x)}$; in this case, $A \cap B = \overline{U(x)} \setminus U(x) = S(x)$. The contractibility of A and B implies $\chi(B) = 1$ and $\chi(A) = 1$. The induction assumption then gives $\chi(A \cap B) = (1 + (-1)^{q-1})$ so that we can conclude using the valuation formula that $\chi(A \cup B) = 1 + 1 - (1 + (-1)^{q-1}) = 1 + (-1)^q$.

3.3. An example of Dehn-Sommerville 2-sphere G that is not a 2-sphere is the union of k disjoint 2-spheres identified along corresponding north and south poles. Such a complex can be seen in various ways: G is the suspension of a union of k circles. It is a **branched cover** of a 2-sphere with two branch points. As it carries an action of the group \mathbb{Z}_k with two fixed points, and because the quotient M/T is the 2-sphere, it is a **ramified cover** of a 2-sphere. It is a Dehn-Sommerville 2-sphere. More generally, the suspension of any odd-dimensional Dehn-Sommerville q -manifold is a Dehn-Sommerville $(q + 1)$ -manifold and also a Dehn-Sommerville q -sphere.

3.4. We have seen that every odd-dimensional Dehn-Sommerville manifold is also a Dehn-Sommerville sphere. One can use the join operation to produce larger odd-dimensional Dehn-Sommerville manifolds. This will be discussed in a section below. The following construction is analog to a construction for manifolds, where one picks a q -manifold for each vertex and uses a **connected sum construction** for each edge. The connected sum construction will be considered in a separate section below.

3.5. So, here is a construction of odd-dimensional Dehn-Sommerville manifolds that are not manifolds. Assume q is even. Pick an arbitrary finite simple graph $\Gamma = (V, E)$ and replace every edge $e = (a, b) \in E$ with a suspension $M_e \oplus \{a, b\}$ of a $(q - 1)$ -dimensional Dehn-Sommerville sphere M_e . Then replace every vertex v with with neighbors $w_1, \dots, w_{d(v)}$ with a Dehn-Sommerville q -manifold M_v in which $d(v)$ different points are chosen to connect it with the neighboring points of v in the graph. Alternatively, assume M_v is a single point and glue all the edge manifolds at this point. In both cases, we get a new larger dimensional Dehn-Sommerville q -manifold.

3.6. The simplicial cohomology of $G_{\Gamma, M}$ can be computed explicitly from the cohomology of M and the combinatorial data of the 1-dimensional complex Γ .

Theorem 2. *Given $\Gamma = (V, E)$ and an odd-dimensional Dehn-Sommerville manifold M with $b(M) = (1, b_1, b_2, \dots, b_q)$, Then $G_{\Gamma, M}$ has Betti vector $(1, |E| - |V| + 1, |E|b_2, \dots, |E|b_q)$ and $\chi(M_{\Gamma, M}) = |V| + |E|b_1$.*

Proof. The simplicial cohomology of the 1-skeleton complex is $(1, |E| - |V| + 1)$. The simplicial cohomology of $G_{\Gamma, M}$ is $(1, |E| - |V| + 1, |E|b_2, \dots, |E|b_q)$. Since $\chi(M) = 0$, the Euler characteristic of the new complex is $\chi(G_{\Gamma, M}) = |V| + |E|\chi(M) = |V| + |E|b_1$. \square

3.7. This shows that if we have an odd-dimensional Dehn-Sommerville manifold with a given b_1 and n vertices, we can construct an even dimensional Dehn-Sommerville manifold with $m * n$ vertices and $\chi(M) = m + m(m - 1)b_1/2$.

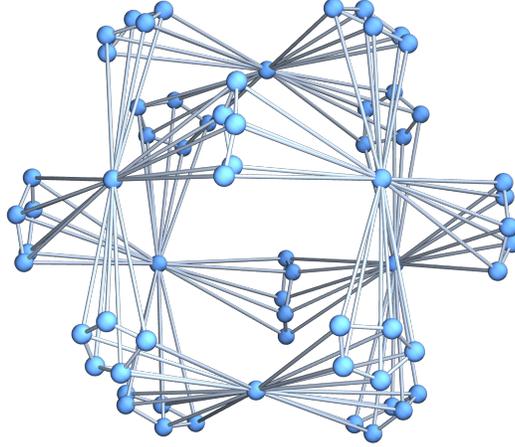


FIGURE 1. A Dehn-Sommerville 2-manifold $G = M_{\Gamma, M}$ with $\Gamma = K_{2,2,2}$ and $M = C_5$ that is not a manifold. Its 1-skeleton graph is a non-Hamiltonian graph. We have $b(G) = (1, 7, 12)$ and $\chi(G) = 6 + |E|b_1 = 6$.

3.8. Let us look at some examples of invariants $Y_{k,q}$, the $(k+1)$ 'th eigenvector of A_q understood as the valuation $Y_{k,q}(G) = Y_{k,q} \cdot f_G$.

Example 1: The **K3 surface** G is one of most famous 4-manifolds. It is complex 2-manifold and a Calabi-Yau surface. It can be seen as a quartic surface $x^4 + y^4 + z^4 + w^4 = 0$, a (real co-dimension 2) sub-manifold in the complex 3-manifold $\mathbb{C}P^3$ (a complex 3-manifold is a real 6-manifold). There is a rather small implementation with f -vector $f = (16, 120, 560, 720, 288)$. The dot product with the Barycentric valuation $Y_{2,4} = (0, -22, 33, -40, 45)$ is zero. The K3 surface satisfies Poincaré duality because it is orientable. Indeed, its Betti vector is $b = (1, 0, 22, 0, 1)$.

Example 2: An other nice 4-manifold is the **real projective space** $\mathbb{R}P^4$. It is a 4-manifold that can carry positive curvature. There is a small implementation with $f = (16, 120, 330, 375, 150)$. Again, this is perpendicular to the Barycentric Dehn-Sommerville eigenvector $Y_{2,4} = (0, -22, 33, -40, 45)$. The $\mathbb{R}P^4$ manifold is not orientable and does not satisfy Poincaré duality. Its Betti vector is $b = (1, 0, 0, 0, 0)$.

Example 3: Let G be a **twisted circle bundle** over the 3-sphere. While the Betti vector of the product $S^3 \times S^1$ is $b = (1, 1, 0, 1, 1)$ and satisfies Poincaré duality, the twisted version does have the Betti vector $b = (1, 1, 0, 0, 0)$. It is not orientable. There is a small implementation with f -vector $f = (12, 60, 120, 120, 48)$. Again, $f \cdot (0, -22, 33, -40, 45) = 0$.

Example 4: Let G be the **complex projective plane** $\mathbb{C}P^2$. It has Betti vector $b = (1, 0, 1, 0, 1)$. There is a small implementation with f -vector $f = (9, 36, 84, 90, 36)$. It satisfies the Dehn-Sommerville symmetry $f \cdot Y_{2,4} = f \cdot (0, -22, 33, -40, 45) = 0$.

Example 5: Lets take an example of a suspension G of the 3-manifold $M = \mathbb{R}P^3$. A suspension of a 3-manifold M is only a manifold if M is a sphere. Since $M = \mathbb{R}P^3$ is not a sphere, $G = M \oplus 2 = M \oplus S^0$ is no more a manifold. But it is a Dehn-Sommerville 4-manifold. The Betti vector of G is the same than the Betti vector of the 4-sphere, but it is not a 4-sphere.

There is a small implementation of G with f-vector $(13, 73, 182, 200, 80)$. Of course, it satisfies the Dehn-Sommerville symmetry.

Example 6: Start with the 4-manifold from example 4. Since it has Euler characteristic 2, it actually is a **Dehn-Sommerville 4-sphere**. Now take the suspension again to get a 5-manifold. It is the double suspension of $\mathbb{R}P^3$. It can also be seen as the join of $\mathbb{R}P^3$ with the 2-sphere S^2 . The f -vector of a small implementation of this 5-manifold is $f = (15, 99, 328, 564, 480, 160)$. As a 5-manifold, this f-vector is perpendicular to the Euler characteristic vector $(1, -1, 1, -1, 1, -1)$. Beside the other "trivial" Dehn Sommerville vector $(0, 0, 0, 0, -1, 3)$, we have the non-trivial Dehn-Sommerville symmetry $Y_{2,5} = (0, 0, -19, 38, -55, 70)$ in dimension 5. And indeed, $(15, 99, 328, 564, 480, 160) \cdot (0, 0, -19, 38, -55, 70) = 0$.

Example 7: Lets look at the suspension $S^0 \oplus K^3$ of the $K3$ surface. It is a 5-dimensional simplicial complex. There is a small implementation with f-vector $f = (18, 152, 800, 1840, 1728, 576)$. Now, $(18, 152, 800, 1840, 1728, 576) \cdot (0, 0, -19, 38, -55, 70) = 0$. But this was just an accident. The suspension of the $K3$ surface is neither a 5 manifold nor a Dehn-Sommerville 5 manifold. Indeed, the Euler characteristic of this 5-manifold is $\chi(G) = -22$ and not $\chi(G) = 0$ as it should be for any Dehn-Sommerville 5-manifold. But the other Dehn-Sommerville symmetries still hold.

Example 8: If we look at the join $G = T^2 \oplus S^2$ of the 2-torus T^2 and the 2-sphere S^2 , we get a 5-dimensional complex. A small implementation has the f -vector $f(G) = (14, 84, 264, 448, 384, 128)$. It is no more perpendicular to $Y_{3,5} = (0, 0, -19, 38, -55, 70)$. It did not help that S^2 was a Dehn-Sommerville sphere. This already fails in smaller dimension. The join $T^2 \oplus S^1$ of T^2 with S^1 is a 4-manifold which has an f-vector not perpendicular to $Y_{2,4} = (0, -22, 33, -40, 45)$.

Example 9: Let us look at the double suspension $S^1 \oplus (S^2 + T^2)$ of a disjoint union of a 2-torus \mathbb{T} and a 2-sphere S^2 . This is a Dehn-Sommerville 4-manifold but not a 4-manifold. There is a small implementation with $f = (18, 96, 224, 240, 96)$. It is perpendicular to $Y_{2,4} = (0, -22, 33, -40, 45)$.

Example 10: Lets look at the double suspension of a **homology 3-sphere**. It is a Dehn-Sommerville 5-manifold but not a 5 manifold. This is an interesting example because its geometric realization is actually homeomorphic to the 5 sphere by the double suspension theorem. There is a small implementation of G with f-vector $f = (28, 254, 972, 1786, 1560, 520)$. This is perpendicular to the Barycentric Dehn-Sommerville vector $Y_{3,5} = (0, 0, -19, 38, -55, 70)$. The other invariant is the Euler characteristic (which is zero) and the trivial valuation $Y_{5,5} = (0, 0, 0, 0, -2, 6)$.

4. LEVEL SETS

4.1. As in the continuum, where manifolds can be constructed as level surfaces on a classical manifold, we have see that in the discrete if G is a q -manifold and $g : V(G) \rightarrow A_k = \{0, \dots, k\}$ is a function write $g(x) = \{g(w), w \subset x, \dim(w) = 0\}$. First extend g from $V(G) \subset G$ to G by $g(\{x_0, \dots, x_j\}) = \{f(x_i), i = 0, \dots, j\}$ so that $g(x) \subset g(y)$ if $x \subset y$. Then define

$$G_g = \{x \in G, g(x) = A_k\} .$$

The set G_g is an open set. But we can look at its Barycentric refinement to get a simplicial complex. We first developed this for $k = 1$ in [24] then extended it to higher co-dimension in [38, 43].

4.2. Just because G_g is a priori only an open subset of G so that we need to define what we mean it to be a Dehn-Sommerville q -manifold. We say that an open set $U \subset G$ is a **Dehn-Sommerville q -manifold**, if the graph $(V = U, E)$ is a Dehn-Sommerville q -manifold graph, where $E = \{(a, b), a \subset b \text{ or } b \subset a\}$. For example, if $G = K_{2,2,2}$ is the Whitney complex of an octahedron and $f = 1$ on two opposite poles and $f = -1$ on the equator, then $G_f = U$ consists of all edges and faces that hit the poles (this is the collection of all triangles and all edges that are not on the equator). The graph $(V = U, E)$ is a disjoint union of two C_4 graphs. We see therefore the manifold G_g as a disjoint union of two circles.

4.3. We first want to point out that in a Dehn-Sommerville manifold G , there is for every $x \in V$ a decomposition $S(x) = S^-(x) \oplus S^+(x)$, where the **stable sphere** $S^-(x)$ consists of all simplices strictly contained in x and the **unstable sphere** $S^+(x)$ consists of all simplices that strictly contain x . Writing the unit sphere as a join of a stable and unstable sphere is a **hyperbolic structure** on G that is present for all simplicial complexes. It is just that in the Dehn-Sommerville case, both factors are Dehn-Sommerville spheres. From the symmetry of the simplex generating functions $f_{S(x)}(t)$ and $f_{S^-(x)}(t)$, and the relation $f_{S^+(x)}f_{S^-(x)}(t) = f_{S(x)}(t)$ follows that also that $S^+(x)$ is a Dehn-Sommerville sphere.

4.4. The following result - telling that the level set G_g of a function $g : V \rightarrow \mathbb{R}$ is always a $(q - k)$ -manifold if not empty - is the main reason why we decided to look at Dehn-Sommerville manifolds in a more general sense and not look at Dehn-Sommerville spheres.

Theorem 3 (Dehn-Sommerville Level Sets). *If G is a Dehn-Sommerville q -manifold and $g : V(G) \rightarrow A_k$ is an arbitrary function, then G_g is a Dehn-Sommerville $(q - k)$ -manifold, if it is not empty.*

Proof. Let x be a m -simplex in G on which f takes all values. This means $g(x) := \{g(w), w \in x\} = A_k$. The stable sphere $S^-(x) = \{y \subset x, y \neq x\}$ is always a $(m - 1)$ -sphere as can be seen by induction with respect to dimension noting that for a 0-dimensional x the set $S^-(x) = \{\}$ is the (-1) -dimensional sphere. The simplices in $S^-(x)$ on which f still reaches A_k is by induction a $(q - 1 - k)$ -manifold and because it is a submanifold of the simplex boundary $S^-(x)$, it has to be a $(q - 1 - k)$ -sphere. (Also by induction, one can check that every level set in a boundary complex $S^-(x)$ of a simplex x is a boundary complex of dimension k less.) Every unit sphere $S(x)$ in G is a $(q - 1)$ -sphere and the join of $S^-(x)$ with $S^+(x) = \{y, x \subset y, x \neq y\}$. The sphere $S_g^+(x)$ in M_g is the same than $S^+(x)$ in M because every simplex z in M containing x automatically has the property that $g(z) = A_k$. So, the unit sphere $S(x)$ in G_g is the join of a $(m - 1 - k)$ -sphere and the $(q - m - 1)$ -sphere and so a $(q - k - 1)$ -sphere. Having shown that every unit sphere in G_g is a $(q - k - 1)$ -sphere, we see that M_g is a $(q - k)$ -manifold. \square

4.5. There is an analog version for varieties. It is simpler, because there is no need to invoke Euler characteristic. Here is the definition: inductively, the void $0 = \{\}$ is called the (-1) **Dehn-Sommerville variety** and a **Dehn-Sommerville q -variety** is a simplicial complex for which every unit sphere is a Dehn-Sommerville $(q - 1)$ -variety.

Theorem 4 (Dehn-Sommerville Varieties). *If G is a Dehn-Sommerville q -variety and $g : V(G) \rightarrow A_k$ is an arbitrary function, then G_g is a $(q - k)$ -variety, if it is not empty.*

Proof. The proof parallels the proof before. The key of the inductive proof is again the **hyperbolic structure** which exists in any simplicial complex: the unit sphere $S(x) = \{y \in G, x \neq y \text{ and } y \subset xx \subset y\}$ is the join of $S^-(x) = \{y \in G, y \neq x, y \subset x\}$ and $S^+(x) = \{y \in G, y \neq x, x \subset y\}$. If x is a m -simplex, then the stable sphere $S^-(x)$ is the boundary sphere of the simplex x . It is a simplicial complex and a $(m-1)$ -sphere. As for the open set $S^+(x)$, it is a Dehn-Sommerville $(q-k-1)$ -variety as all unit spheres $S(y)$ are Dehn-Sommerville $(q-k-2)$ -varieties. Note that $S(y)$ for $y \in S^+(x)$ is the same than $S^+(y)$ in $S^+(x)$. Now $S_{G_g}(x) = S^-(x) \oplus S_g^+(x)$ which by induction is the join of two varieties and so a Dehn-Sommerville variety. \square

4.6. An example of a Dehn-Sommerville 1-variety that is not a Dehn-Sommerville 1-manifold is the **star graph** S_n or the **cube graph** or the **dodecahedron graph**. Every unit sphere is either the n -point graph n or the 1-point graph 1. More generally, any 1-dimensional simplicial complex which is pure (meaning in the 1-dimensional case that there are no isolated vertices) is a Dehn-Sommerville 1-variety. Dehn-Sommerville varieties are a decent generalization of Dehn-Sommerville manifolds because they form a monoid. We get to this in the next section.

5. MONOIDS

5.1. The suspension $M \oplus S^0$, the join of M with a 0-sphere $S^0 = \{\{a\}, \{b\}\}$ of a q -manifold M is already not a manifold if M is not a q -sphere. It is however a Dehn-Sommerville manifold, if M is a Dehn-Sommerville sphere. The reason is that the two points $\{a\}, \{b\}$ have as unit spheres the original manifold M that is not a sphere. An example of a Dehn-Sommerville 3-manifold that is not a 3-manifold is the suspension of a disjoint copy of two projective planes $M = \mathbb{P}^1 + \mathbb{P}^1$ which has Euler characteristic 2. The suspension of any 3-manifold is a Dehn-Sommerville 4-manifold. The double suspension of a homology 3-sphere is only a **Dehn-Sommerville 5-manifold** and not a classical 5-manifold. However, by the double suspension theorem it is in the continuum homeomorphic to a 5-sphere. But homeomorphisms in the continuum can be very complicated. Still we see from this example that there are Dehn-Sommerville 5-manifolds that are not 4-manifolds but which have geometric realizations that are topological continuum 5-spheres.

Theorem 5. *The set of all Dehn-Sommerville spheres is a submonoid of all simplicial complexes.*

Proof. We use induction with respect to q and the fact that $S_{A \oplus B}(x) = S_A(x) \oplus B$ if $x \in A$ and $S_{A \oplus B}(x) = A \oplus S_B(x)$ if $x \in B$. Since $S_A(x)$ resp. $S_B(x)$ are both one dimension lower, the monoid property for dimension $q-1$ implies the property for dimension q . By Theorem (??) odd-dimensional Dehn-Sommerville manifolds are Dehn-Sommerville spheres. The induction foundation is $q = (-1)$, where both A, B are 0.

The compatibility with the Euler characteristic follows from $f_{A \oplus B}(t) = f_A(t)f_B(t)$ and $\chi(A) = 1 - f_A(-1)$ and the fact that $chi(A) \in \{0, 2\}$ is equivalent to $f_A(-1) \in \{-1, 1\}$. \square

Corollary 1. *The set of all odd dimensional Dehn-Sommerville manifolds is a submonoid of all simplicial complexes.*

Proof. An odd dimensional Dehn-Sommerville manifold is a Dehn-Sommerville sphere. \square

5.2. While the focus is on Dehn-Sommerville manifolds, there is an analog for the already mentioned **Dehn-Sommerville varieties** which is the smallest monoid that is invariant under the property that all unit spheres are in this class. The same formula for the unit sphere of the join imply that:

Theorem 6. *The set of all Dehn-Sommerville varieties form a submonoid of all simplicial complexes.*

5.3. An other monoid is the set of all **contractible complexes**, if we define a complex G to be **contractible**, if it the one point complex or if there exists x such that both $S(x) = \delta U(x)$ and $G \setminus U(x)$ are contractible. It actually has the ideal property that the join of a non-zero contractible complex with an arbitrary complex is contractible. The smallest monoid containing all 0-dimensional complexes is the **partition monoid** [41] whose elements are multi-partite graphs $K_{n_1, \dots, n_k} = n_1 \oplus n_2 \cdots n_k$ with $\sum_{j=1}^k n_j = q$. It contains complete complexes K_n , the utility graph complex $K_{3,3}$ or the octahedron $K_{2,2,2}$ or q -dimensional spheres $K_{2, \dots, 2}$.

5.4. One could consider much more examples: given an arbitrary monoid \mathcal{M} of complexes, one can look at all complexes that are homotopic to an element in such a monoid. One can for example look at the monoid that contains all complexes that contract to an odd dimensional Dehn-Sommerville q -manifold. It contains most Vietoris-Rips complexes of classical continuum q -manifolds: take a random point cloud, remove successive points with contractible unit spheres, until only points remain for which all unit spheres are $(q-1)$ -spheres. The monoid of **Whitney complexes of graphs** is isomorphic to the monoid of graphs and slightly smaller than the monoid of all simplicial complexes. Finally, we should point out that **delta sets** and more generally, **abstract delta sets** given by a triplet (G, D, R) (where D is a Dirac matrix and R a dimension function) form monoids containing the simplicial complex monoid. For the later just note that given exterior derivative d on G and an exterior derivative on H naturally defines an exterior derivative on $G \oplus H$. One can define **Dehn-Sommerville abstract delta sets** in the same way by induction as (G, D, R) defines an order structure which defines a topology and so has a notion of unit sphere.

6. INVARIANTS

6.1. Given an arbitrary simplicial complex G . The number $f_k(G)$ counting the number of k -simplices in G defines the **f -vector** $f = (f_0, f_1, \dots, f_q)$ and the **f -generating function** $f_G(t) = 1 + \sum_{k=0}^q f_k t^k$ which is a polynomial of degree q . An integer vector (X_0, \dots, X_q) in \mathbb{Z}^{q+1} defines a **valuation** $X(G) = X \cdot f = X_0 f_0 + \dots X_q f_q$. By distributing the values X_k attached to each k -simplex in G equally to its $k+1$ vertices, we get the **curvature** $K(v) = \sum_{k=0}^q X_k f_{k-1}(S(x))/(k+1)$ for the valuation X and $\sum_{v \in V} K(x) = X(G)$ is the **Gauss-Bonnet theorem** for the valuation X . In the case $X = (0, 1, 0, \dots, 0)$ it leads to $X(G) = f_1(G)$, and curvature is $K(v) = \deg(v)/2$ and Gauss-Bonnet is the ‘‘Euler handshake’’. If $X = f_q$ is the volume of G , then $K(v) = \text{vol}(U(v))/(q+1)$, The valuation $X = (1, -1, 1, -1, \dots)$ with $X_k = (-1)^k$ gives the **Euler characteristic**.

6.2. Of special interest are **Dehn-Sommerville valuations**, defined by the vectors

$$X_{k,q} = \sum_{j=k}^q (-1)^{j+q} \binom{j+1}{k+1} e_j - e_k,$$

where $e_0 = (1, 0, 0, \dots, 0), \dots, e_q = (0, 0, \dots, 1)$ are basis vectors. We assume $k < q$ because this is zero for $k = q$. If the vectors $(X_{0,q}, \dots, X_{q-1,q})$ are written as row vectors in a matrix X_q , we have

$$X_1 = [-2 \ 2], X_2 = \begin{bmatrix} 0 & -2 & 3 \\ 0 & -2 & 3 \end{bmatrix}, X_3 = \begin{bmatrix} -2 & 2 & -3 & 4 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & -2 & 4 \end{bmatrix}, X_4 = \begin{bmatrix} 0 & -2 & 3 & -4 & 5 \\ 0 & -2 & 3 & -6 & 10 \\ 0 & 0 & 0 & -4 & 10 \\ 0 & 0 & 0 & -2 & 5 \end{bmatrix}.$$

Lemma 1 (Curvature lemma). *For any simplicial complex, the curvature of the valuation $X_{k,q}$ is $K(v) = X_{k-1,q-1}(S(v))$.*

Proof. This is the combinatorial identity

$$X_{k,q}(j+1)/(j+1) = X_{k-1,q-1}(j)/(k+1)$$

which translates into

$$(-1)^{j+q+1}B(j+2, k+1)/(j+1) = (-1)^{j+q+1}B(j+1, k)/(k+1).$$

For $k = -1$ one has to define $B(x, y) = \Gamma(x+1)/\Gamma(y+1)\Gamma(x-y+1)$. In the limit $k \rightarrow -1$ we need to look at $X_{k-1,q-1}(j)/(k+1)$ as a limit. \square

It follows that

Theorem 7. *Dehn-Sommerville valuations $X_{k,q}$ are all zero for Dehn-Sommerville spheres if $k+q$ is even.*

Proof. Use Gauss-Bonnet $\sum_{x \in G} K(x)$ and induction. For $q = 1$, a Dehn-Sommerville manifold is a union of cyclic complexes for which the Dehn-Sommerville valuation $X(0, 1)$ The reason is that $X(-1, q) \cdot f(G)$ is $(-1)^q \chi(G)$. \square

7. EIGENVALUE FUNCTIONALS

7.1. The valuations $X_{k,q}$ invoked Binomial identities. An other approach to the identities is via eigenfunctions of the Barycentric refinement operator. If $f = (f_0, \dots, f_q)$ is the **f -vector**, and $f_G(t) = 1 + \sum_{k=0}^d f_k(G)t^{k+1}$ is the **simplex generating function**, define $F_G(t) = \int_0^t f_G(s) ds$. In [29] we noted that Gauss-Bonnet can be written as a functional identity. Let $V \subset G$ denote the set of vertices, zero-dimensional simplices. We have

Theorem 8 (Functional Gauss-Bonnet). *For all simplicial complexes, $f_G(t) = 1 + \sum_{v \in V} F_{S(v)}(t)$.*

7.2. It follows from this formula that if all $f_{S(v)}$ are odd with respect to $-1/2$ then $F_{S(v)}$ is even with respect to $-1/2$ so that f_G is even with respect to $-1/2$. On the other hand if $f_{S(v)}$ are all even with respect to $-1/2$ then $F_{S(v)}$ and so f_G is odd with respect $-1/2$, provided that $f_G(-1) = -1$ meaning $\chi(G) = 0$.

7.3. Here is the proof of the functional Gauss-Bonnet formula:

Proof. As all simplex generating functions satisfy $f_G(0) = 1$, the statement is equivalent to $f'_G(t) = \sum_{v \in V} f_{S(v)}(t)$ and gives for $t = -1$ the **Levitt Gauss-Bonnet formula** for Euler characteristic. The proof is an observation: attach to every $y \in S(x)$ an energy t^{k+1} so that $f_G(t)$ is the total energy Every k -simplex $y \in S(x)$ defines a $(k+1)$ -simplex z in $U(x) \subset G$ carrying the charge t^{k+2} . It contains $(k+2)$ vertices. Distributing this charge equally to these points gives each a charge $t^{k+2}/(k+2)$. The curvature $F_{S(x)}(t)$ adds up all the charges of z . \square

See [19, 40, 29].

7.4. Given a finite simplicial complex G , the h -**function** is defined as $h_G(t) = (t-1)^q f_G(1/(t-1))$. It is a polynomial $h_G(t) = h_0 + h_1 t + \dots + h_q t^d + h_{q+1} t^{q+1}$, defining a h -**vector** $(h_0, h_1, \dots, h_{q+1})$. Let us say a complex satisfies **Dehn-Sommerville functional relations** if h is **palindromic**, meaning that $h_i = h_{q+1-i}$ for all $i = 0, \dots, q+1$.

Lemma 2. *The following are equivalent:*

- a) $f_G(t) = (-1)^q f_G(-1-t)$.
- b) h is palindromic.
- c) $g(t) = f(t-1/2)$ is either **even** or **odd**.

Proof. The palindromic condition of h -vector means that the roots of the h -function $h(t) = 1 + h_0 t + \dots + h_q t^{q+1} = (t-1)^q f(1/(t-1))$ is invariant under the involution $t \rightarrow -1-t$ which is equivalent to the symmetry $f(-1-t) = \pm f(t)$ and in turn means that $g(t)$ is either even or odd. \square

7.5. Examples: for the 3-sphere G with $f_G(t) = 1 + 8t + 24t^2 + 32t^3 + 16t^4$, we have $f_G(t-1/2) = 16t^4$ which is an even function. For the **Barnette sphere** [3], a 3-sphere with f vector $(8, 27, 38, 19)$ and f -function $f(t) = 1 + 8t + 27t^2 + 38t^3 + 19t^4$, we have $f(t-1/2) = 3/16 - 3t^2/2 + 19t^4$, which is an even function.

7.6. The unitary involutive map $T(f)(x) = f(-1-x)$ on the linear space of polynomials of degree q is a reflection on \mathbb{R}^{q+1} and has eigenvalues 1 and -1 . The spectral theorem of normal operators defines an eigenbasis. In analogy to $\tilde{T}(f)(x) = f(-x)$, call eigenfunctions of 1 of T **even functions** and eigenfunctions of -1 **odd functions**. As T and the Barycentric operation A commute, they have the same eigenbasis. The eigenbasis of A is also an eigenbasis of T : even eigenvectors are eigenfunctions of T to the eigenvalue 1 and odd eigenvectors of A are eigenfunctions of T to the eigenvalue -1 . We have verified:

Theorem 9. *For even q , the even eigenvectors of A^T and for odd d , the odd eigenvectors of A^T define functionals which are zero for all Dehn-Sommerville spheres.*

This is covered in Theorem (1) in [25] and [29]. More details are given also below.

8. REFINEMENTS

8.1. The **Barycentric refinement** G_1 of G is the order complex of G . It is the Whitney complex of the graph $\Gamma(G) = (V = G, E)$, where E is the set of pairs (x, y) with $x \subset y$ or $y \subset x$. The Barycentric refinement operation and the Dehn-Sommerville symmetry are compatible. This is not a surprise, given that one of the descriptions uses the eigenfunctions of the Barycentric refinement operator.

Theorem 10. *If G is Dehn-Sommerville then G_1 is Dehn-Sommerville.*

Proof. We use induction with respect to dimension. The unit spheres of G are either $(q-1)$ -spheres, boundary spheres of simplices or Barycentric refinements of a unit sphere of G . Both cases satisfy Dehn-Sommerville by induction. \square

8.2. The **Barycentric refinement operator** A is defined as $f(G_1) = Af(G)$. If G has maximal dimension q , then the matrix A is a $(q+1) \times (q+1)$ upper triangular $A_{ij} = \text{Stirling}(i, j) i!$, involving the **Stirling numbers of the second kind**. All eigenvalues $\lambda_k = k!$ are distinct.

8.3. The linear unitary reflection involution $T(f)(x) = f(-1 - x)$ on polynomials defines an involution on \mathbb{R}^{d+1} . By the spectral theorem for normal operators, there is an eigenbasis for the reflection. Call eigenfunctions of 1 **even functions** and eigenfunctions of -1 **odd functions**.

8.4. Any eigenvector X of A^T defines a valuation

$$\phi_X G = X \cdot f_G .$$

If λ is an eigenvalue, then $\phi_X G_1 = \lambda \phi_X G$ because $\phi_X G_1 = X \cdot f_{G_1} = X \cdot A f_G = A^T X \cdot f_G = \lambda X \cdot f_G$.

8.5. Most $\phi_X(G_n)$ explode in general under refinements G_n . The only functional that stays invariant is ϕ_X , the Euler characteristic.

Lemma 3. *The eigenbasis of A is also an eigenbasis of T .*

Proof. The linear operators T and the Barycentric operation A commute. and so share an eigenbasis. \square

Corollary 2. *For even d , the even eigenvectors of A^T and for odd d , the odd eigenvectors of A^T define functionals which are zero on Dehn-Sommerville spaces.*

8.6. This was Theorem (1) in [25], where the idea of proving Dehn-Sommerville via curvature has hatched and multi-variate versions of Dehn-Sommerville answered a question of Gruenbaum [11] from 1970. In multi-dimensions, the Dehn-Sommerville symmetry just has to hold for each of the variables appearing in the simplex generating function $f(t_1, \dots, t_m)$. The proof in higher dimensions is identical using Gauss-Bonnet.

8.7. Let $Y_{k,q}$ denote the $k + 1$ 'th eigenvector of A_q . It defines a functional $Y_{k,q}(G) = Y_{k,q} \cdot f_G$. By definition $Y_{k,q}(G_1) = k! Y_{k,q}(G)$.

8.8. Every valuation is a linear combination of eigenvectors of A_q . We have seen that there is a $[q/2]$ dimensional space of valuations which are zero on Dehn-Sommerville spaces. Assume $X = \sum_i a_i Y_i$ is such an invariant, then $X(G_1) = \sum_i a_i i! Y_i$ implying that each valuation that is zero must be a linear combination of eigenvectors for which each is a valuation that is zero. This means that there are $[q/2]$ eigenvectors which are Dehn-Sommerville invariants.

Theorem 11. *The span of $\{X_{k,q}, 0 \leq k \leq q, k + q \text{ odd}\}$ and $\{Y_{k,q}, 0 \leq k \leq q, k + q \text{ odd}\}$ are the same.*

8.9. While we have an equivalence between the functionals $X_{k,q}$ and $Y_{k,q}$, we do not know yet how to make this explicit. For example,

$$Y_{1,6} = aX_{0,6} + bX_{2,6} + cX_{4,6}$$

with $a = 3949, b = -359, c = 169$.

8.10. To summarize: there are four different ways to see the Dehn-Sommerville symmetries:

Theorem 12 (Dehn-Sommerville symmetries). *Let G be a Dehn-Sommerville q -sphere (defined using unit spheres). Then*

- a) *The valuations $X_{k,q}$ are zero if $k + q$ is even (Gauss-Bonnet).*
- b) *The valuations $Y_{k,q}$ are zero if $k + q$ is even (Barycentric refinement)*
- c) *$f_G(t) = (-1)^q f_G(-1 - t)$, meaning f_G is either even or odd (functional Gauss-Bonnet)*
- d) *$h_G(t)$ is palindromic (change of coordinates).*

9. CHROMATIC NUMBER

9.1. In this section we generalize the chromatic number estimate $\chi(G) \leq 2q + 2$ from q -manifolds to Dehn-Sommerville q -manifolds. First of all, we have to say what the chromatic number of a simplicial complex is as it is traditionally defined for graphs. But any simplicial complex G defines a 1-dimensional simplicial complex, the 1-skeleton complex in which the 0-simplices are the vertices and the 1-simplices are the edges. A **coloring** is an assignment of values to the vertices $V = \dim^{-1}(0)$ of the complex G . The **chromatic number** is then the minimal target set which allows to color V in such a way that adjacent vertices have different colors. For a Dehn-Sommerville q -manifold of course the chromatic number is bound below by $q + 1$ as we have q -dimensional simplices there which consist of $q + 1$ vertices all connected to each other. By using a theorem of Whitney one can see that the **4-color theorem** is equivalent to the statement that all 2-spheres have chromatic number 4 or less. Note that this does not work for Dehn-Sommerville 2-spheres, as the disjoint union of two projective planes is a Dehn-Sommerville 2-sphere and that there are projective planes with chromatic number 5. We show here that for all Dehn-Sommerville 2-manifolds, the chromatic number is maximally 6 and more generally that for all Dehn-Sommerville q -manifolds, the chromatic number is maximally $2q + 2$. We have no example yet, for which the chromatic number of a 2-manifold is actually 6. For 2-manifolds, a conjecture of Albertson-Stromquist [1] states that the chromatic number is 5 or less. Any Barycentric refinement of a Dehn-Sommerville q -manifold of course has chromatic number $q + 1$ as this holds for arbitrary simplicial complexes: the dimension functional is a coloring.

9.2. Let $\Gamma = (V, E)$ be a finite simple graph. The **vertex arboricity** $\text{ver}(\Gamma)$ [7] is the maximal number of forests partitioning V such that each forest generates itself in Γ . It originally was also called **point arboricity**. The chromatic number $\text{chr}(\Gamma)$ of a graph Γ is the maximal number of colors that can be assigned to V such that neighboring vertices have different colors. Vertex arboricity sandwiches the chromatic number $\text{chr}(\Gamma)$ with $\text{ver}(\Gamma) \leq \text{chr}(\Gamma) \leq 2\text{ver}(\Gamma)$. Despite name appearance, there is a severe difference between vertex and the **edge arboricity** $\text{arb}(\Gamma)$ (the minimal number of forests partitioning the edge set of Γ). Computing $\text{ver}(\Gamma)$ is a NP-hard, while $\text{arb}(\Gamma)$ is a polynomial task by the **Nash-Williams theorem**.

9.3. The **dual** $\Gamma = \hat{G}$ of a Dehn-Sommerville q -manifold G has the facets of G as vertices and connects two if they intersect in a $(q - 1)$ -simplex. For example, the dual graph of the octahedron complex $K_{2,2,2}$ is the cube graph, the dual graph of the icosahedon complex is the dodecahedron graph.

Lemma 4. *If G is a Dehn-Sommerville q -manifold then \hat{G} is a $(q + 1)$ -regular triangle-free graph.*

Proof. For every k -simplex $x = (x_0, \dots, x_k)$ in a q -manifold, the intersection of all unit spheres $S(x_j)$ is a $(q - k - 1)$ -dimensional sphere. This means that for $k = q - 1$ the intersection is a 0-Dehn-Sommerville sphere which is a 2 point graph. This shows that any of the walls can not be part of more than two maximal simplices meaning that \hat{G} is triangle free. If there is no boundary, then every facet x is surrounded by $q + 1$ other facets, as there are $q + 1$ walls in q q -simplex x . \square

9.4. The vertex arboricity of triangle free graphs can be arbitrarily large, even on the class of triangle-free graphs. The reason is that the chromatic number can be arbitrarily large [52]. This is unlike in the planar case where the chromatic number is ≤ 3 by **Groetsch's theorem**. Dual graphs of manifolds are of a special kind however. Unlike the Mycielski examples, their vertex arboricity is simple:

Theorem 13. *The vertex arboricity of the dual \hat{G} any Dehn-Sommerville q -manifold G with or without boundary is always 2 if $q > 0$.*

9.5. This results from the following proposition. We can cut a manifold G to get two manifolds K, H with common boundary C . By induction in q , the vertex arboricity of C is 2. The proposition in turn then allows to extend the forests to the interiors and so to the entire manifold G . Any forest pair on the boundary of a manifold can grow into the interior of the manifold. First of all, the **boundary** of the dual \hat{G} of a q -manifold G consists of the vertices x (facets in G) for which there is one or more walls with only one facet x attached.

Proposition 1 (Growing forests into the interior). *If G is a Dehn-Sommerville q -manifold with boundary and assume the boundary of \hat{G} has been covered with two forests generating themselves, then the two forests can be extended into the interior of \hat{G} , still generating themselves.*

Proof. Take a minimal counter example G (meaning a manifold with minimal q and for this q with minimal number of vertices in \hat{G}) can not be extended to the interior. As it can not be a ball, by the next lemma, we can cut it into two manifolds with boundary $G \setminus U(x), B(x)$, where x is a wall on the boundary. As the original G was minimal, the proposition applies to $G \setminus U(x)$. If we choose forests in both boundaries, agreeing on their original intersection face, we can extend the forests to the interior of each of the parts. Now glue the two parts together. This covers G with two forests that both generating themselves. This contradicts the assumption that G lacked this property. \square

9.6. Note that a manifold with boundary does not have to have interior zero dimensional points. For example, a single q -simplex is a q -manifold G with boundary. Its dual \hat{G} is a single point.

Lemma 5. *Take $G = B(x)$ with boundary $S(x)$. If $S(\hat{x})$ is covered with 2 trees, this can be extended to the interior $B(\hat{x})$.*

Proof. Assign the center of the ball to either tree color. In both cases, we have two trees covering \hat{G} . \square

Lemma 6. *Given a Dehn-Sommerville q -manifold G with boundary and x is a wall at the boundary then $G \setminus U(x)$ is a Dehn-Sommerville q -manifold.*

Corollary 3. *The chromatic number of a Dehn-Sommerville manifold is $2q + 2$ or less.*

Proof. Since the vertex arboricity of the dual \hat{G} is 2, we can partition the vertices of \hat{G} into two forests A, B . Use $q + 1$ colors to color each facet in G that belongs to a vertex $A \subset V(\hat{G})$ and an other set of $q + 1$ colors to color each facet belonging to B . \square

9.7. We had prove in [28] that manifolds are Hamiltonian. This property however does not extend to Dehn-Sommerville manifolds. The complex defined in Figure 1 is a Dehn-Sommerville 2-manifold that is not Hamiltonian, meaning that the skeleton graph is not a Hamiltonian graph.

10. CONSTRUCTIONS

10.1. The **Cartesian product** of two simplicial complexes A, B is the Whitney complex of the graph (V, E) in which $V = \{(a, b) \mid a \in A, b \in B\}$ are the vertices and where two different vertices $(x, y), (u, v)$ are connected by an edge if either $x \subset u, y \subset v$ or $u \subset x, v \subset y$. This product is also known as the **Stanley-Reisner product**. We already have seen that the Stanley-Reisner product of a p -manifold with a q -manifold is a $(p + q)$ -manifold.

Theorem 14. *If G is a Dehn-Sommerville p -manifold and H is a Dehn-Sommerville q -manifold, then $G \times H$ is Dehn-Sommerville $p + q$ -manifold.*

Proof. Denote by $B_H(y)$ the unit ball of a vertex $y \in H$. Its boundary is the unit sphere $S_H(y)$ in H . The unit sphere $S_{G \times H}(x, y)$ is the union of two generalized cylinders $S_G(x) \times B_H(y)$ and $B_G(x) \times S_H(y)$ glued along $S_G(x) \times S_H(y)$. One can see by induction that this is a Dehn-Sommerville $(q - 1)$ -sphere: if z is a point in this space then its unit sphere is again of this form but of dimension 1 lower. \square

10.2. The **connected sum** of two simplicial complexes A, B of the same maximal dimension q is obtained by removing two isomorphic unit balls $U(x), U(y)$ and identifying along $S(x), S(y)$. This construction does not need A, B to be manifolds. It works for two general manifolds as long as the maximal dimension is the same. There are always ways to do that. A simple example is to take $x \in A, y \in B$ of maximal dimension so that $S(x), S(y)$ are isomorphic $(q - 1)$ -spheres.

Theorem 15. *If two Dehn-Sommerville q -manifolds A, B are Dehn-Sommerville, then $A \# B$ is also a Dehn-Sommerville q -manifold.*

Proof. Assume the connected sum $A \# B$ has been glued along the sphere $H = S(x) \sim S(y)$. If $x \in A \setminus H$ or $y \in B \setminus H$, then the unit sphere in $A \# B$ is the same than the unit sphere in A or B . If $x \in H$, then $S(x)$ consists of three parts $\{y \in S(x) \cap A \setminus H\}, \{y \in S(x) \cap B \setminus H\}$ and $\{y \in S(x) \cap H\}$. But this means that $S(x)$ is essentially the suspension of the $q - 2$ sphere $S_H(x)$. It is obtained by gluing $S(x) \cap A$ with $S(y) \cap B$ along H . In other words $S(x) = S_A(x) \# S_B(x)$. We can use induction to see that this is a Dehn-Sommerville sphere. \square

10.3. An **edge refinement** of a complex picks an edge $e = (a, b)$, adds a new vertex c in the middle and connects c to the bone $S(a) \cap S(b)$.

Theorem 16. *Edge refinement preserves Dehn-Sommerville manifolds.*

Proof. The first one is to increase f_0 and f_1 by 1 (which means adding $t + t^2$ to $f_G(t)$). Then we add $tf_{S(a) \cap S(b)} + 2t^2f_{S(a) \cap S(b)}$, because every k -simplex in $S(a) \cap S(b)$ defines a new $(k + 1)$ -simplex connecting to c and two new $(k + 2)$ -simplices connecting $S(a) \cap S(b)$ to (a, c) and (b, c) . Now, the set of functions satisfying the Dehn-Sommerville symmetry form a linear space. The claim follows as the added part $t + t^2 + tf_{S(a) \cap S(b)} + 2t^2f_{S(a) \cap S(b)}$ satisfy the Dehn-Sommerville symmetry by induction because the space $S(a) \cap S(b)$ is in \mathcal{X}_{d-2} if $G \in \mathcal{X}_d$. \square

11. COHOMOLOGY

11.1. Simplicial complexes G carry a **simplicial cohomology** defined by the **exterior derivative** d which naturally exists on functions on G . If G has n elements, the matrix d is a $n \times n$ matrix. The matrix depends on a given orientation chosen for each simplex but choosing an order on each simplex is simply putting a coordinate system. Interesting quantities like the kernels of $(d + d^*)^2$ do not depend on the coordinate system. Simplicial cohomology is only the first of a sequence of cohomologies. The definitions are reviewed as follows:

11.2. If G with n elements, the **exterior derivative** d is a $n \times n$ matrix. It is defined if every $x \in G$ is totally ordered. The definition is $d(x, y) = \text{sign}(x, y)$, if $|x|$ and $|y|$ differ by one. The sign corresponds to the embedding signature of the permutations. The set of **forms** $G \rightarrow \mathbb{R}$ identifies with \mathbb{R}^n . Functions on p -simplices are **p-forms**. The linear transformation d maps p -forms to $(p + 1)$ -forms. The transpose d_p^* maps $(p + 1)$ -forms to p -forms. The **Kirchhoff matrix** $d_0^* d_0$ is the analog of the scalar Laplacian as d_0 is the analog of the **gradient** and d_0^* is the analog the **divergence**.

11.3. The **Dirac matrix** $D = d + d^*$ defines the **Hodge Laplacian** $L = D^2 = (d + d^*)^2 = dd^* + d^*d$. It is block diagonal $L = L_0 \oplus L_1 \oplus \dots \oplus L_q$. The **Betti vector** $b = (b_0, b_1, \dots, b_q)$ has as components the **Betti numbers** $b_p = \dim(\ker(L_p))$, where q is the **maximal dimension** of G . The linear space of **harmonic p-forms** $\ker(L_p)$ is the **p-th cohomology** space. This is motivated by classical Hodge theory [15] first adapted to the discrete in [10]. The Euler-Poincaré formula $\chi(G) = \sum_{k=0}^q (-1)^k b_k(G)$ is a consequence of McKean-Singer symmetry. We repeat the argument again: define first the **super trace** of a block diagonal matrix $K = \bigoplus_{k=0}^q K_k$ as $\sum_{k=0}^q (-1)^k \text{tr}(K_k)$. Now, $\text{str}(e^{-tL})$ is for $t = 0$ the super trace $\text{str}(1) = \chi(G)$ For the limit $t \rightarrow \infty$ it is $\sum_{k=0}^q (-1)^k b_k(G)$.

11.4. The next cohomology is **quadratic cohomology** [25, 27] which was also called Wu cohomology as its characteristic $w_2(G)$ is Wu characteristic. It defines an exterior derivative on functions on pairs of simplices $(x, y) \in G \times G$. The **quadratic cohomology** defines a **quadratic Betti vector** $b(G)$ which is associated to **quadratic characteristic** $w_2(A) = \sum_{(x,y) \in A^2, x \cap y \in A} \omega(x)\omega(y)$ which satisfies **energy identities** $w_2(G) = \sum_{x \in G^2} \omega(x)\omega(y)w_2(U(x) \cap U(y))$ and even has been generalized to k-point Green function identities for all **higher order characteristics** [37]. The interaction of open and closed sets produces **quadratic fusion inequalities** [42]. This is much richer than the **fusion inequality** for simplicial cohomology [46].

11.5. As for simplicial cohomology of Dehn-Sommerville manifolds, we have more freedom in building manifolds. There is a basic open ended question for simplicial complexes or manifolds: "what Betti vectors are possible for manifolds up to a given size?" In small dimensions, where the classification of manifolds is known, we can tell: 1-manifolds the Betti vector is $b = (k, k)$, where k is the number of circles, for 2-manifolds the possible Betti vectors are $(a+b, (2a+b)g, b)$, where $a + b \geq 1, b \geq 0, g \geq 0$ are integers. The reason is that we have $b = (1, 2g, 1)$ for a connected orientable manifolds that are a connected sum of g tori, and $b = (1, g, 0)$ for a connected non-orientable manifold that is a connected sum of g projective planes. In general, there are always Euler Characteristic constraints as $\sum_{k=0}^q (-1)^k b_k = \chi(G)$ which is zero for odd dimensional manifolds. In the orientable manifold case, there is always the Poincaré duality constraint $b_k = b_{q-k}$ for $k = 0, \dots, q$.

11.6. Let us end this section with a question: is it possible that for every simplicial complex G we can construct a Dehn-Sommerville manifold M such that the cohomology of G and M agree? This is affirmative in dimension 1: for one-dimensional simplicial complex $G = (V, E)$, the Betti number b_0 is the number of connected components and b_1 is $1 - \chi(G)$ which is $b_1 = b_0 - |V| + |E|$. For 2-manifolds we have a classification of 2-manifolds using connected sum construction $M = S^2 \# P^1 \# \dots \# P^1$ for which $b = (1, k)$, where k is the number of copies of P^1 . There non-orientable 2-manifolds of Euler characteristic $(a, b, 0)$ for any $a \geq 1, b \geq 0$.

11.7. For orientable 2-manifold, the possible Betti vectors are $(a, 2b, a)$ with $a \geq 1, b \geq 0$ by the classification of orientable 2-manifolds. As for orientable Dehn-Sommerville 2-manifolds, the constraints that the middle cohomology is even goes away. Take a graph with Betti vectors (a, b) . The above construction gives now a orientable Dehn-Sommerville 2 manifold G such that the cohomology is (a, b, a) . Just attach suspensions of a circle at every edge. The first cohomology is now the same as the one from the graph: the possible Betti vectors for graphs are $(a, b, 0)$. By gluing in projective planes rather than 2-spheres, we can realize any Betti vector (a, b, c) with $a \geq 0, b \geq 0, c \geq 0$.

11.8. The higher cohomology of Dehn-Sommerville spaces can be very rich. Like for q-manifolds, we know that all characteristics of a q-manifold are the same. The quadratic cohomology however can already be complicated however. If we construct for example from a graph with $|V| = 5$ and $|E| = 10$ the Dehn-Sommerville 2-manifold, we have for example $b = (1, 6, 10)$ for simplicial cohomology and $b = (0, 0, 45, 110, 70)$ for quadratic cohomology. This needs to be much more explored.

12. CONNECTION CALCULUS

12.1. In this section we generalize the formula $\omega_m(G) = \chi(G) - \chi(\delta(G))$ from manifolds to Dehn-Sommerville manifolds G with boundary δG . In order to do so, we first define **Dehn-Sommerville manifolds with boundary**. Inductively, a **Dehn-Sommerville q-manifold with boundary** has either Dehn-Sommerville $(q - 1)$ -spheres with boundary = Dehn-Sommerville $(q-1)$ -balls as unit spheres or then Dehn-Sommerville $(q-1)$ -spheres as unit spheres. A $(q - 1)$ -ball is $H \setminus U(x)$, where H is a $(q - 1)$ -sphere and $x \in H$. We can get examples of such manifolds with boundary using level surface construction. The point is here that we get natural examples of manifolds with boundary.

Theorem 17. *Take a function $g : V(G) \rightarrow \{0, 1, \dots, k\}$ and look at all simplices for which $g(x) = \{1, \dots, k\}$. This is Dehn-Sommerville manifold $q - k + 1$ manifold with $(q - k)$ manifold as boundary.*

12.2. The **connection Laplacian** of G is defined as $L(x, y) = \chi(S^-(x) \cap S^-(y))$. By the **unimodularity theorem** [26, 50, 34], this is unimodular matrix with determinant $\prod_{x \in G} \omega(x)$. Its inverse $g(x, y)$ is by the **Green star formula** $g(x, y) = \omega(x)\omega(y)\chi(S^+(x) \cap S^+(y))$, where $\omega(x) = (-1)^{\dim(x)}$. The Euler characteristic is $\chi(G) = \sum_{x \in G} \omega(x)$. The **Wu characteristic**, the quadratic characteristic, is $\omega(G) = \sum_{(x,y) \in G^2, x \cap y \neq \emptyset} \omega(x)\omega(y)$.

12.3. Lets start with a lemma about stars $U(x)$:

Lemma 7 (Star lemma). $w_m(U(x)) = \chi(U(x))^m$ for all $m \geq 1$.

Proof. Just note that any m -tuple $X = (x_1, \dots, x_m)$ of simplices in $U(x)$ all intersect as $x \subset \bigcap X$. Therefore $(\sum_{y \in U} \omega(y))^m = w_m(U(x))$. \square

12.4.

Lemma 8 (Local boundary formula). *If G is a simplicial complex and $x \in G$, then $w_m(B(x)) = w_m(U(x)) - (-1)^m w_m(S(x))$ for all $m \geq 1$.*

Proof. We use double induction. The outer induction goes with m . For $m = 1$ we have where $w_1 = \chi$ is the Euler characteristic, we have $B(x) = U(x) \cup S(x)$, which is a disjoint union and $w_1(B(x)) = w_1(U(x)) + w_1(S(x))$ is the valuation property.

Now assume things have been proven for $m - 1$. We establish it for m . The inner induction goes now with respect to the size of the simplicial complex. This can be proven by building up the simplicial complex $G = B_G(x)$ successively adding maximal simplices such that the new complex is always again a ball $H = B_H(x)$. If G is zero dimensional and $G = B(x)$, then $B(x) = U(x)$ and $S(x) = \emptyset$. Now add edges until the one dimensional skeleton complex is filled, then add triangles (2-simplices) etc. Assume we have established the relation for a complex $G = B_G(x)$ and we add a new q -simplex X so that $H = B_H(x)$. By construction when adding X , the frame $S(X) \cap B_G(x)$ had already been constructed before. The quantities change as follows: We have $B_H(x) = B_G(x) \cup \{X\}$ and $U_H(x) = U_G(x) \cup \{X\}$ and $S_H(x) = S_G(x) \cup \{Y\}$, where $Y = X \setminus x$. The subcomplex $K = B_G(X)$ still contains x . All m -tuples in $U_H(x)$ that are not in $U_G(x)$ must contain X as one element and all other elements in $U_K(x)$. All m -tuples in $B_H(x)$ that are not in $B_G(x)$ must contain X as one element and all other elements in $B_K(x)$. All m -tuples in $S_H(x)$ that are not in $S_G(x)$ must contain Y as one element and all other elements in $S_K(x)$. When going from G to H we have added $w_{m-1}(B_K(x))w_1(X)$ to the left and $w_{m-1}(U_K(x))w_1(X) + w_{m-1}(S_K(x))w_1(Y)$ to the right. As the dimension of Y is one lower than X and $w_1(Y) = -w_1(X)$, and having established the relation for $m - 1$ (and applying it for K), we are done. \square

12.5. We illustrate this remarkable relation in examples.

- a) If x is locally maximal of dimension q , then $U(x) = \{x\}$ and $w_m(U(x)) = (-1)^{qm}$ and $w_m(S(x)) = 1 - (-1)^q$ and $w_m(B(x)) = (-1)^{q(m+1)}$.
- b) For a 1-dimensional complex, where x is a vertex of degree d we have $S(x) = \overline{K_d}$ and $B(x)$ is the complex of a star graph and $U(x)$ is a union of d open K_2 . It is a nice combinatorial exercise to compute $w_m(G)$ if G is the star graph.
- c) For $G = K_2$. $U(x) = \{\{1\}, \{1, 2\}\}$, $B(x) = G$ $S(x) = \{\{2\}\}$. $w_2(U) = 0$, $w_2(S) = 1$, $w_2(B) = -1$. We have $w_2(B(x)) = w_2(U(x)) - w_2(S(x))$ and $w_1(B(x)) = w_2(U(x)) + w_2(S(x))$. More generally, if $B(x) = K_q$, $x = K_k$, $S(x) = K_{q-k}$, then the left hand side is $(-1)^{m+1}$ and the right $(-1)^{mq} - (-1)^m(-1)^{q-k+1}$.

APPENDIX: GRAPH CASE

12.6. In our expositions about this subject in 2019 [29], we had used the language of graphs, rather than the language of simplicial complexes. While complexes coming from complexes are a restriction, this is mainly terminology, because a simplicial complex G defines a graph with G as vertex set where two are connected if one is contained in the other. A graph in turn defines a simplicial complex, the Whitney complex. The composition of these operations is the Barycentric refinement. There are complexes that are not Barycentric refinements of an other complex. Examples are skeleton complexes like the boundary $q - 1$ complex of K_q which is a $(q - 1)$ -sphere simplicial complex but is not the Whitney complex of a graph. Let us nevertheless go through the definitions in the case of graphs as graphs are more intuitive.

12.7. The definition of Dehn-Sommerville graph can also be built within the **monoid of finite simple graphs**, where the join is the **Zykov join** [57]: the join $A \oplus B$ of two finite simple graphs is the disjoint union of the graphs augmented with edges connecting any vertex of A with any vertex of B . The monoid of graphs can be seen as a submonoid of the monoid of simplicial

complexes because every **finite simple graph** $\Gamma = (V, E)$ defines a simplicial complex G in which the vertex sets of complete subgraphs K_n with $n \geq 1$ of Γ form the sets of G . G is called the **Whitney complex** or **flag complex** or **order complex** of Γ . Since any simplicial complex G defines in turn a graph $\Gamma = (G, \{(x, y), x \subset y \text{ or } y \subset x\})$ whose simplicial complex G_1 is the Barycentric refinement of G , there is hardly any difference between the two frame works ‘simplicial complex’ or ‘graph’. It is mostly a change language. But there are complexes that are not Whitney complexes. But every **Barycentric refinement** G_1 of any complex G is by definition the Whitney complex of a graph.

12.8. Let us just repeat the definition in a graph setting: one says a subgraph $(W, F) \subset (V, E)$ **is induced by its vertex set** $W \subset V$ if $F = \{(a, b) \in E, a \in W, b \in W\}$. The **unit sphere** of a vertex $v \in V$ is the graph induced by all neighbor vertices $\{w \in V, (v, w) \in E\}$. The definition of **Dehn-Sommerville q -manifold graphs** proceeds in the same way as for simplicial complexes: the empty graph $0 = \{\emptyset, \emptyset\}$ is the (-1) -sphere graph. A graph is called a **Dehn-Sommerville q -manifold graph** if for every $v \in V$, the unit sphere $S(v)$ is a Dehn-Sommerville $(q - 1)$ -sphere graph for all $v \in V$. A graph is a **Dehn-Sommerville q -sphere graph** if it is a Dehn-Sommerville q -manifold graph and $\chi((V, E)) = \chi(G) = 1 + (-1)^q$.

12.9. Given a simplicial complex G we have a graph (G, E) in which the sets in G are vertices and $E = \{(a, b), a, b \in G, a \subset b \text{ or } b \subset a\}$. Given a vertex $v \in G$, then the Barycentric refinement of unit sphere $S(v)$ in the graph is the unit sphere $S(v)$ in G . For a k -simplex $x \in G$, the unit sphere $S(x)$ is the join of a $(k - 1)$ sphere $S^-(x)$ with a $q - k - 1$ sphere $S^+(x)$ which can be seen as the intersection $\bigcap_{w \subset x, w \in V} S(w)$. In principle, we can understand the geometry of G in terms of the graph geometry of the graph (G, E) . A Dehn-Sommerville q -manifold G defines a Dehn-Sommerville q -manifold graph (G, E) . On the other hand, a Dehn-Sommerville q -manifold graph defines a Dehn-Sommerville q -manifold G_1 , the Whitney complex of the graph.

APPENDIX: CODE

```
(* Basics *)
CleanGraph [s_]:=AdjacencyGraph[AdjacencyMatrix[s]];
CleanComplex [G_]:=Union[Sort[Map[Sort,G]]];
Generate [A_]:=If[A=={ },A,CleanComplex[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Whitney [s_]:=Generate[FindClique[s,Infinity,All]]; Closure=Generate;
FVector [G_]:=Delete[BinCounts[Map[Length,G]],1];
Euler [G_]:=Sum[(-1)^Length[G[[k]]],{k,Length[G]}];
OpenStar [G_,x_]:=Select[G,SubsetQ[#,x] &];
Basis [G_]:=Table[OpenStar[G,G[[k]]],{k,Length[G]}];
UnitSphere [G_,x_]:=Module[{U=OpenStar[G,x]},Complement[Closure[U],U]];
UnitSpheres [G_]:=Table[UnitSphere[G,G[[k]]],{k,Length[G]}];
```

```
(* Construction *)
RingFromComplex [G_,a_]:=Module[{V=Union[Flatten[G]],n,T,U},
n=Length[V]; Quiet[T=Table[V[[k]]->a[[k]],{k,n}]];
Quiet[U=G /.T]; Sum[Product[U[[k,1]],{1,Length[U[[k]]]}],{k,Length[U]}];
ComplexFromRing [f_]:=Module[{s,ff},s={}; ff=Expand[f];
Do[Do[If[Denominator[ff[[k]]/ff[[1]]]==1 && k!=1,
s=Append[s,k->1]],{k,Length[ff]},{1,1,Length[ff]}];
Whitney[UndirectedGraph[Graph[Range[Length[ff]],s]]];
GeometricProduct [G_,H_]:=Module[{f,g,F},
f=RingFromComplex[G,"a"];
g=RingFromComplex[H,"b"]; F=Expand[f*g]; ComplexFromRing[F];
TopologicalProduct=GeometricProduct;
```

```

ConeExtension [G_-]:=Module[{q=Max[Flatten[G]]+1,n=Length[G]},
  Generate[Table[Append[G[[k]],q],{k,n}]];
Suspension [G_-]:=Module[{q=Max[Flatten[G]]+1,n=Length[G]},
  Closure[Union[Table[Append[G[[k]],q],{k,n}],
    Table[Append[G[[k]],q+1],{k,n}]]];
DoubleSuspension [G_-]:=Suspension[Suspension[G]];
Addition [A_ ,B_-]:=Module[{q=Max[Flatten[A]],Q},
  Q=Table[B[[k]]+q,{k,Length[B]}];Sort[Union[A,Q]];
JoinAddition [G_ ,H_-]:=Union[G,H+Max[G]+1,
  Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1},0]]]];

AddTwoSphere [s_-]:=Module[{e,x,m,v=VertexList[s]},x=First[v]; e=EdgeList[s]; m=Length[v];
  UndirectedGraph[Graph[Union[e,{x->m+1,m+1->m+2,m+2->m+3,m+3->x,
    m+1->m+4,m+4->m+3,m+3->m+5, m+5->m+1,x->m+4,m+4->m+2,m+2->m+5,m+5->x}]]];
Bouquet [n_-]:=Nest[AddTwoSphere,CompleteGraph[{2,2}],n];

Shannon [A_ ,B_-]:=Module[{q=Max[Flatten[A]],Q,G={},Q=Table[B[[k]]+q,{k,Length[B]}];
  Do[G=Append[G,Sort[Union[A[[a]],Q[[b]]]],{a,Length[A]},{b,Length[Q]}];
  If[A=={},G={}; If[B=={},G={}; Sort[G]];

```

(* Level sets *)

```

R[G_ ,k_-]:=Module[{{},R[x_-]:=x->RandomChoice[Range[k]]; Map[R,Union[Flatten[G]]];
Surface [G_ ,g_-]:=Select[G,SubsetQ[#,/g,Union[Flatten[G]/.g]] &];
S[s_ ,v_-]:=VertexDelete[NeighborhoodGraph[s,v],v]; Sf[s_ ,v_-]:=FVector[Whitney[S[s,v]]];
Curvature [s_ ,v_-]:=Module[{f=Sf[s,v]},1+f.Table[(-1)^k/(k+1),{k,Length[f]}]];
Curvatures [s_-]:=Module[{V=VertexList[s]},Table[Curvature[s,V[[k]]],{k,Length[V]}];
ToGraph [G_-]:=UndirectedGraph[n=Length[G];Graph[Range[n],
  Select[Flatten[Table[k->l,{k,n},{l,k+1,n}],1],(SubsetQ[G[[#[[2]]],G[[#[[1]]]])&]];
Barycentric [s_-]:=ToGraph[Whitney[s]];

```

(* Cohomology *)

```

Coho [G_-]:=Module[{n,q,f,d,Dirac,U,H},
  n=Length[G]; q=Map[Length,G]-1;f=Delete[BinCounts[q],1];
  Orient[a_ ,b_-]:=Module[{z,c,k=Length[a],l=Length[b]},If[SubsetQ[a,b] &&
    (k==l+1),z=Complement[a,b][[1]];c=Prepend[b,z];Signature[a]*Signature[c],0];
  d=Table[0,{n},{n}]; d=Table[Orient[G[[i]],G[[j]],{i,n},{j,n}];
  Dirac=d+Transpose[d]; H=Dirac.Dirac; f=Prepend[f,0]; m=Length[f]-1;
  U=Table[v=f[[k+1]];Table[u=Sum[f[[1]],{l,k}];H[[u+i,u+j]],{i,v},{j,v}],{k,m}];
  Map[NullSpace,U];
Betti [G_-]:=Map[Length,Coho[G]];

Coho2 [G_ ,H_-]:=Module[{n=Length[G],m=Length[H],U={},d1,d2},len[x_-]:=Total[Map[Length,x]];
  Do[If[Length[Intersection[G[[i]],H[[j]]]]>0,U=Append[U,{G[[i]],H[[j]]}],{i,n},{j,m}];
  U=Sort[U,len[#1]<len[#2] &];u=Length[U];l=Map[len,U];w=Union[1];
  b=Prepend[Table[Max[Flatten[Position[1,w[[k]]]],{k,Length[w]}],0]; h=Length[b]-1;
  deriv1[{x_ ,y_-]:=Table[{Sort[Delete[x,k]],y},{k,Length[x]}];
  deriv2[{x_ ,y_-]:=Table[{x,Sort[Delete[y,k]]},{k,Length[y]}];
  d1=Table[0,{u},{u}]; Do[v=deriv1[U[[m]]]; If[Length[v]>0,
    Do[r=Position[U,v[[k]]]; If[r!={},d1[[m,r[[1,1]]]]=(-1)^k,{k,Length[v]}],{m,u}];
  d2=Table[0,{u},{u}]; Do[v=deriv2[U[[m]]]; If[Length[v]>0,
    Do[r=Position[U,v[[k]]]; If[r!={},d2[[m,r[[1,1]]]]=(-1)^(Length[U[[m,1]]]+k),
    {k,Length[v]}],{m,u}]; d=d1+d2; Dirac=d+Transpose[d]; L=Dirac.Dirac; Map[NullSpace,
  Table[Table[L[[b[[k]]+i,b[[k]]+j]],{i,b[[k+1]]-b[[k]]},{j,b[[k+1]]-b[[k]]}],{k,h}]];
Betti2 [G_ ,H_-]:=Map[Length,Coho2[G,H]]; Coho2 [G_-]:=Coho2[G,G]; Betti2 [G_-]:=Betti2 [G,G];

```

(* Invariants *)

```

X[k_ ,q_-]:=Module[{f},f[i_-]:=Table[If[i==j-1,1,0],{j,q+1}];
  Sum[(-1)^(j+q) Binomial[j+1,k+1] f[j],{j,k,q}]-f[k];
BarycentricOperator [q_-]:=Table[StirlingS2[j,i]i!,{i,q+1},{j,q+1}];
Y[k_ ,q_-]:=Reverse[Eigenvectors[Transpose[BarycentricOperator[q]]]][[k+1]];

```

```

Ffunction[G_]:=Module[{f=FVector[G],n},Clear[s];1+Sum[f[[k]]*s^k,{k,Length[f]}];
DehnSommervilleQ[G_]:=Module[{f},Clear[s];f=Ffunction[G];
  Simplify[f]==Simplify[(f/.s->-1-s)];

```

(* Connection Stuff *)

```

ConnectionLaplacian[G_]:=Table[If[MemberQ[G,Sort[Intersection[G[[i]],G[[j]]]],1,0],
  {i,Length[G]},{j,Length[G]}];
ConnectionGreenFunction[G_]:=Inverse[ConnectionLaplacian[G]];

```

```

w[x_-]:=-(-1)^Length[x];dim[x_-]:=Length[x]-1;
Wu1[A_-]:=Total[Map[w,A]];
Wu2[A_-]:=Module[{a=Length[A]},Sum[x=A[[k]],Sum[y=A[[1]]];
  If[MemberQ[A,Intersection[x,y]],1,0]*w[x]*w[y],{1,a},{k,a}];Wu=Wu2;
Wu3[A_-]:=Module[{a=Length[A]},Sum[x=A[[k]],Sum[y=A[[1]]];Sum[z=A[[o]]];
  If[MemberQ[A,Intersection[x,y,z]],1,0]*w[x]*w[y]*w[z],{o,a},{1,a},{k,a}];

```

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