REMARKS ABOUT CONNECTION AND DIRAC MATRICES

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ABSTRACT. The connection Laplacian L and the Dirac matrix D are both $n \times n$ matrices defined from a given finite abstract simplicial complex G with n sets. In both cases, there is interlacing of the eigenvalues for subcomplexes. This gives general upper bounds of the eigenvalues $\lambda_j \leq d_j$ both for L and D in terms of inclusion or intersection degrees. We conjecture that L always dominates both D and $g = L^{-1}$ in a weak Loewner sense. In a second part we look at dynamical systems (G, T), where T is a simplicial map on G. Both L and D generalize to dynamical versions L_T and D_T. L_T is still unimodular with an explicit Green function inverse g_T and $D_T = d_T + d_T^*$ still comes from an exterior derivative d_T. We also review the Lefschetz fixed point theorem $\sum_{x \in F} i_T(x) = \chi_T(G)$ for a simplicial map T on a finite abstract simplicial complex G. It implies the Brouwer fixed point theorem telling that simplicial map on a contractible finite abstract simplicial complex G always has at least one simplex fixed. This is all combinatorial as no geometric realizations are ever involved.

1. INTRODUCTION

1.1. If G is a **finite abstract simplicial complex** with n sets, both the **Dirac matrix** D as well as the **connection matrix** L are $n \times n$ matrices. The Dirac matrix $D = d + d^*$ uses the **exterior derivative** d is a matrix $d(x, y) = \text{sign}(x, y)$ if $|x|$ and $|y|$ differ by 1 and $y \subset x$ and $d(x, y) = 0$ else. The **connection matrix** is defined as $L(x, y) = 1$ if $x \cap y \neq \emptyset$ and $L(x, y) = 0$ else. Its inverse $g = L^{-1}$ defines **Green functions** $g(x, y)$. Both the **Hodge Laplacian** $H = D^2 = dd^* + d^*d$ and the connection Laplacian L^2 have non-negative spectrum. For 1-dimensional complexes, $L - L^{-1} = |D|^2$. [15].
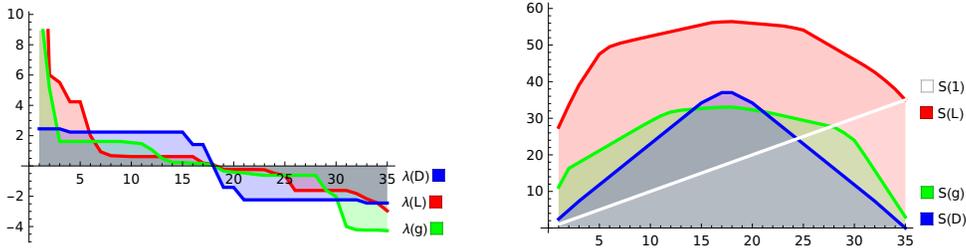

FIGURE 1. The figure show sthe spectra λ (left) and cumulative spectra S (right) of the connection matrix L, the Green matrix $g = L^{-1}$ and the Dirac matrix $D = d + d^*$ for a random complex G. L appears to weakly Loewner dominate both D and g.

1.2. The Dirac matrix uses "inclusion" $x \subset y$ while the connection matrix uses "intersection" $x \cap y \neq \emptyset$ for its definition. Both frame-works have spectral relations to the **Euler characteristic** $\chi(G) = \sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$. The kernels $\ker(H_k)$ of the Hodge

Laplacian blocks H_k are the **cohomology groups** of G and their dimensions $\dim(\ker(H_k))$ are the **Betti numbers** b_k. By Euler-Poincaré, they super-sum to the Euler characteristic $\sum_{k=0}^{q}(-1)^k b_k = \chi(G) = \sum_{x \in G} \omega(x)$. This identity can already be seen as an example of the Lefschetz fixed point formula, where the simplicial map is the identity and every $x \in G$ is a fixed point and the Euler characteristic is the Lefschetz number. Remarkably, the connection matrix L and so its square L^2 are both unimodular. The determinant is explicitly known as $\det(L) = \prod_{x \in G} \omega(x)$. The Euler characteristic is the total potential energy $\sum_{x,y \in G} g(x,y) = \chi(G)$. The number of positive eigenvalues minus the number of negative eigenvalues of L is equal to the Euler characteristic. For D the number of positive and negative eigenvalues is the same. If the spectra of D and L and the inverse $g = L^{-1}$ are plotted together then the graphs $j \to \lambda_j(D)$ and $j \to \lambda_j(L)$ and $j \to \lambda_j(g)$ cross close to each other. The cumulative spectra appear to be ordered: the cumulative sum $S_j(L)$ appears to dominate both $S_j(g)$ and $S_j(D)$.

1.3. The Hodge Laplacian $H = D^2$ is always a **reducible** matrix because it is block diagonal. It is also the sum of $D_k = d^*d$ and $D'_k = d^*d$ which are called a **combinatorical Laplacians**. The D_k plays a role for **analytic torsion** $A(G) = \prod_k \mathrm{Det}(H_k)^{k(-1)^{k+1}}$, where Det is the **pseudo determinant**, the product of the non-zero eigenvalues. Torsion $A(G)$ can be rewritten as a **super pseudo determinant** $\mathrm{SDet}(D) = \prod_k \mathrm{Det}(D_k)^{(-1)^k}$ of these Dirac blocks. Torsion satisfies $A(G) = |V|$ for contractible complexes, $A(G) = |V|/|V'|$ for even dimensional spheres and $|V||V'|$ for odd dimensional spheres, where $|V|$ is the number of zero-dimensional simplices and $|V'|$ is the number of facets, the highest dimensional simplices. Unlike in the continuum, where the subject is quite technical, analytic torsion is a topic for linear algebra in the finite [20].

1.4. In contrast to the Hodge Laplacian $H = D^2$, the connection matrix L and so its Laplacian L^2 are always irreducible non-negative $0-1$ matrices if G is connected. The reason is that L has only non-negative entries and that the connection graph of G has a finite diameter r implying that L^r has only positive matrix entries implying irreducibility. This implies that L (and so L^2) always has a unique **Perron-Frobenius eigenvalue**. As L is invertible with determinant 1 or -1, also its inverse $g = L^{-1}$ is an irreducible integer matrix. An interesting question which we can not answer yet is whether also $g = L^{-1}$ always has a unique maximal eigenvalue. The powers of g do not necessarily become positive now. But we see a unique maximal eigenvalue in the case in all examples considered so far. While the trace of L is n, the trace of g is the sum of the Euler characteristics $\sum_x \chi(U(x))$ of the smallest open sets $U(x) = \{y \in G, x \subset y\}$. This follows from the general Green-Star formula $g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$.

1.5. When looking at the **fusion inequality** $b(U) + b(K) \geq b(G)$ [22, 24] for an open closed pair $U \cup K = G$, where U is open and K is closed, we made use of the fact that the spectra move monotonically for the Hodge spectra. This had used the **Courant minimax principle**. But more is true: the spectra also interlace. As we will see, in both cases, we can get from L_G to L_K or from D_G to G_K by deleting some rows or columns belonging to maximal simplices, then use the Cauchy interlace theorem [9]: if μ_k are the eigenvalues of a principal $(n-1 \times n-1)$ sub-matrix of K, then $\lambda_k \leq \mu_k \leq \lambda_{k+1}$.

1.6. We suspect that $L \geq D$ in a **weak Loewner sense**, meaning that the eigenvalues λ_j of L and the eigenvalues μ_j of D satisfy $\sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j$ for all k. We also see that L weakly Loewner dominates its inverse g. These are open questions.

2

1.7. In the 1-dimensional case, there is the relation $|H| = L - L^{-1}$ between the **sign-less Hodge Laplacian** $|H| = |D|^2$ and the connection Laplacian. We called it the **hydrogen identity** [17]. For Kirchhoff spectra of graph, this can be useful for estimating the spectral radius of $|H|$ and so give upper bounds for the spectral radius of H in terms of the spectrum of L.

1.8. For **higher characteristics** $w_k(G)$ [21], which generalize Euler characteristic $w_1(G) = \chi(G)$, there are parallel stories. In each case, there are **connection Green functions** $g(x, y) = w(x)w(y)w_k(U(x) \cap U(y))$. It can be generalized to energized cases. The connection story is a 2-point Green function story and deals with matrices so that it is linear algebra. There are **k-point Green functions**. For higher characteristic, the matrix $g(x, y) = w(x)w(y)w_k(U(x) \cap U(y))$ is not unimodular any more in general.

2. THE CONNECTION LAPLACIAN

2.1. If G is a finite abstract simplicial complex containing n sets, the $n \times n$ connection matrix L can be defined as $L(x, y) = \chi(C(x) \cap C(y))$, where $C(x) = \{y \in G, y \subset x\}$ and where $\chi(A) = \sum_{y \in A} \omega(x)$ is the **Euler characteristic** with $\omega(x) = (-1)^{\dim(x)}$. This more general definition is useful for generalized energized versions, where $\omega(x)$ is an arbitrary non-zero function (which even can become division algebra-valued [18]). In the special case of the Euler characteristic, we have $L(x, y) = 1$ if x, y intersect and 0 else, because $C(x) \cap C(y)$ is the simplicial complex that is defined by the simplex $x \cap y$ and $\chi(C(x)) = 1$.

2.2. The matrix L is unimodular and has as inverse the integer matrix $g(x, y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$, where $U(x) = \{y \in G, x \subset y\}$ is the **star** of x. The determinant of L is known to be the **Fermi characteristic** $\prod_{x \in G} \omega(x)$, which is a multiplicative version of the Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x)$. We know that the number of positive eigenvalues of L minus the number of negative eigenvalues of L is the Euler characteristic of G. See [19].

2.3. We have seen already that if $K \subset G$ is a sub-complex of G, then $L(K)$ is a principal sub-matrix of $L(G)$ so that interlacing happens. This is related to the result (see [8]) that the non-zero eigenvalues of a graph G interlace the eigenvalues of the graph G in which an additional edge has been added. As $L(K)$ still is a connection matrix of a complex, it still is unimodular.

2.4. As for open subsets $U \subset G$, and if we take a vertex away, we still have an open set but we do not have a principal submatrix any more in general. The removal of a vertex also removed connections between higher dimensional simplices. Take the example of $U = \{\{1\}, \{1, 2\}\}$ in K_2. Then, $L(U)$ is a 2×2 matrix, where each entry is 1. This obviously is no more unimodular.

2.5. Define the **connection degree** of $x \in G$ as $d(x) = |\{y \in G, \chi(C(x) \cap C(y)) \neq 0\}|$, where $C(x) = \{y \in G, y \subset x\}$. In the 1-dimensional case, this is the vertex degree if x is a vertex. We also have $d((a, b)) = d(a) + d(b)$ if $x = (a, b)$ is an edge of the 1-dimensional complex. Let $d_1 \geq d_2 \cdots$ denote the ordered connection degrees of G and $\lambda_1 \geq \lambda_2 \cdots$ the ordered connection eigenvalues of L.

Theorem 1. *The connection Laplacian spectrum satisfies* $\lambda_j \leq d_j$.

Proof. Use induction with respect to the number of elements in G. Removing a maximal simplex x has the effect that the connection matrix is a principal sub-matrix. Let μ_j be an eigenvalues of the smaller reduced complex K. Induction tells $\mu_j(K) \leq d_j(K) \leq d_j(G)$. The spectral radius estimate $\lambda_1 \leq d_1$ follows from basic principles [9]: the spectrum L is majorized by the spectrum of $|L|$ which is bounded above by the maximal row or column sum which is d_1. Now $\lambda_2 \leq \mu_1 \leq d_1(K) \leq d_2(G)$ and $\lambda_3 \leq \mu_2 \leq d_2(K) \leq d_3(G)$ etc. $\square$

2.6. This does not give any information for negative eigenvalues because $d_j \geq 0$. An other inequality is the Schur-Horn inequality $\sum_{j=1}^{k} \lambda_j \leq k$ which follows because the diagonal entries of L are 1. This concludes the remark about the majorization of eigenvalues of the connection matrix in simplicial complexes. We turn now to the Dirac case.
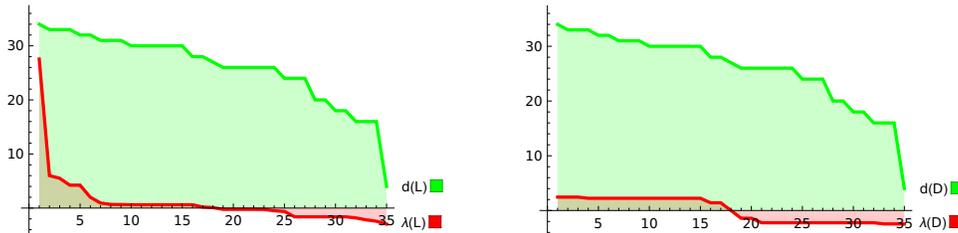

FIGURE 2. The estimates $\lambda_j(D) \leq d_j(D)$ and $\lambda_j(L) \leq d_j(L)$ where $d_j(D)$ and $d_j(L)$ are the ordered vertex degrees of D and L.

3. THE DIRAC MATRIX

3.1. The interlacing story has a parallel situation for the Dirac matrix E Let D_K denote the Dirac matrix of K if $D = D_G$ is the Dirac matrix of G. In the Dirac case, the interlacing not only works for closed sets, it also does for open sets:

Theorem 2. *If $K \subset G$ is a sub-complex of G, then the eigenvalues of D_K interlace the eigenvalues of D_G. The same holds if $U \subset G$ is an open set.*

Proof. The proof is the same and reflects the fact that going to a subcomplex can be achieved by removing successively maximal simplices in G that are not in K. As for open sets, removing a maximal simplex x only changes the matrix entries $D(x, y)$ of smaller dimensional simplices y contained in x. Unlike for L (where intersection matters) it does not affect other entries. $\square$

3.2. The estimate is very rough and far from optimal. Estimating the individual eigenvalues of $H = D^2$ is trickier. It was possible in the case of H_0, the Kirchhoff matrix [23]. By McKean-Singer, the non-zero eigenvalues of H restricted to even forms is the set set of non-zero eigenvalues of H restricted to odd forms. Eigenvalues therefore come in pairs. This can be encoded in short as $\text{str}(H^n) = 0$ for all $n > 0$. We also have $\text{str}(H^0) = \text{str}(1) = \chi(G)$. The Taylor expansion of the exponential function immediately gives the McKean-Singer symmetry $\text{str}(e^{-itH}) = \chi(G)$.

3.3. Define the **Dirac graph** of a simplicial complex has the graph in which the elements in the complex as vertices and where two vertices are connected if $D(x, y)$ is non-zero. Define the **Dirac graph degrees** as the vertex degrees of this graph. We can rewrite this as $d_x = \sum_k |D_{xy}|$. Again we assume that the eigenvalues are ordered in a descending manner and relabel

4

the sequence as $d_1 \geq d_2 \geq \cdots \geq d_n$ if $|G| = n$. Also the eigenvalues λ_j of D are ordered as $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$.

Theorem 3. *The ordered eigenvalues λ_j of the Dirac matrix of a complex satisfy $\lambda_j \leq d_j$, where d_j are the ordered Dirac graph vertex degrees. The same works for open sets in a complex.*

Proof. The proof is identical to the connection case and works with induction to the number n of simplices of G. In the same way one can estimate the eigenvalues of D first from above with the eigenvalues of $|D|$ in which all matrix entries are non-negative, then use the row sums $d_x = \sum_{y \in G} |D_{xy}|$ and have that the maximum of these sums is an upper bound for the spectral radius λ_1. $\square$

4. WAVE EQUATIONS

4.1. As D, L and $g = L^{-1}$ can have both positive and negative eigenvalues, it makes sense to compare them and see them on the same footing. Their squares D^2 and L^2 or g^2 produce then "Laplacians", self adjoint matrices that have non-negative eigenvalues only. They both can serve as generators of a **Dirac wave equation** $u_{tt} = -D^2 u$, as a **connection wave equation** $u_{tt} = -L^2 u$, or then the **Green connection wave equation** $u_{tt} = -g^2 u$. These equations are equivalent to Schrödinger evolution equations $u_t = \pm iDu$ in the Dirac case or $u_t = \pm iLu$ in the connection case or $u_t = \pm igu$ in the Green case.

4.2. In any of these situations, we have explicit solutions of the wave equations. The only thing which is needed for such a d'Alembert solution is the ability to see the Laplacian as a square of an other operator. Define first the **sinc function** $\text{sinc}(x) = \sin(x)/x$. It is an entire function as one can see by dividing the Taylor series of $\sin$ by x.

Theorem 4. *a) The Dirac wave equation is solved by $u(t) = \cos(Dt)u(0) + t \, \text{sinc}(Dt)u'(0)$. b) The connection wave equation is solved by $u(t) = \cos(Lt)u(0) + t \, \text{sinc}(Lt)u'(0)$. c) The Green wave equation is solved by $u(t) = \cos(gt)u(0) + t \, \text{sinc}(gt)u'(0)$.*

Proof. Both cos and sinc are entire functions. We obviously have for $t = 0$ the initial condition $u(0)$ right. Now note that $t\text{sinc}(Dt)$ has at $t = 0$ the derivative 1 so that also $u'(0)$ matches. $\square$

4.3. If $||Lt|| < \pi/2$ (rsp. $||gt|| < \pi/2$ or $||Dt|| < \pi/2$), the inverse problem of getting for any vector q and initial position $p = u(0)$, we can find an initial velocity $u'(0)$ such that $u(t) = q$. Scaling t to t/c obviously is equivalent to scaling D to Dc.

Corollary 1. *Assume cD as norm smaller than $\pi/2$, then the wave equation $u'' = -c^2 D^2 u$ has for a given initial end time boundary conditions $u(0)$ and $u(1)$ a unique solution. The same statement holds if D is replaced by L or by g.*

Proof. Use the solution formula $u(t) = \cos(cDt)u(0) + t\text{sinc}(cDt)u'(0)$ which gives $u'(0) = \text{sinc}(cD)^{-1}[u(1) - \cos(cD)u(0)]$. Because cD had norm smaller than $\pi/2$ the operator $\text{sinc}(cD)$ is invertible. $\square$

4.4. If $u(0)$ is localized on one simplex $x \in G$ and $u(1)$ is localized on a second simplex $y \in G$ and x, y are far remote. How come one can travel in time $t = 1$ from x to y? The paradox is resolved by noting that the initial velocity $u'(0)$ needed to go from x to y is not localized at x. It spreads over the entire stretch from x to y. We even can get from x to y if x and y are in different connectivity components of G: what will happen is that in one of the components we have an initial velocity $u'(0)$ and position $u(0) = e_x$ so that $u_1(x) = 0$ for all x in the first

component. In the other connectivity component, we have that $u_0 = 0$ and $u_1 = e_y$. There is an initial velocity which gets us there.

4.5. We should also note that for wave equations with continuum time $t \in \mathbb{R}$ and discrete space G, there is no finite propagation speed. There is no Huygens principle like in the continuum where one has a strong Huygens principle in odd dimensions or weak Huygens principle in even dimensions (see for example [6] page 73). If we write the solution to the wave equation $u_{tt} = -D^2 u$ in terms of eigenfunctions, then these eigenfunctions are global. Removing a simplex x at the end of the universe G impacts the eigenvalues and eigenfunctions of the Laplacian D^2 and so impacts the solution to the wave equation. The reason is that if G all solutions are entire functions in t; we even wrote down the solutions explicitly in terms of cos and sinc functions.

4.6. We have just seen that starting a wave so that $u(0)$ is localized at some simplex x and zero everywhere else. Then the solution of $u(t)$ involves every other simplex in the complex, even at the end of the "universe" G. However, by going to discrete time, we achieved to have a causal wave evolution. At time t, only simplices up to distance t matter. This **causality issue** can be important when making numerical experiments because we like to experiment with parts of the world G which are manageable and know that parts far away do not affect a given simplex for some time. In order to understand the spectral properties in an infinite lattice for example, we can explore the problem by looking at solutions of the Schrödinger equations as the spectral measures are directly related to dynamical properties See [11, 10]. To solve the **causality problem**, we need also to **discretize time**.

4.7. If L or D or g are of norm smaller than 1, we can now look at the **symplectic wave map** $(u, v) \to (2Lu - v, u)$ or $(u, v) \to (2Du - v, u)$ or $(u, v) \to (2gu - v, u)$ which are conjugated to a pair of unitary evolutions $(u, v) \to (Uu, U^*u)$. These are "cellular automata" maps for Laplacians H like D^2, L^2 or g^2 because in all these cases matrix entries $H(x, y)$ are zero if $U(x), U(y)$ do not intersect. The discretization of time comes with a change of the operator $L \to \tilde{L}$ but the spectral type of the operator does not change. Of course, if G is finite, there are just eigenvalues and there is no singular or absolutely continuous spectrum. The change to a discrete time however solves the causality issue and the wave needs time to get from x to y. If x, y are far away, then an initial wave for which the initial times u_0, u_1 are localized near x needs some time to reach y and this depends on the distance in the Dirac graph, or Connection graph or Green graph. It looks at first like a philosophical issue, but it is helpful in numerical simulations. It is also nice to see actual wave equations interpolated by cellular automata maps.

Theorem 5. *If $||cD|| < 1$, there is a symplectic matrix $(u, v) = (2Du - v, u)$ on $\mathbb{R}^{2n}$ that interpolates $(u_t, v_t) = i(c\tilde{D}, -c\tilde{D})$. The same statement holds if D is replaced by L or by g and the matrices are scaled such that $||cL|| < 1$ or that $||cg|| < 1$.*

Proof. In [10] we scale L to have norm smaller than 1, then took $\tilde{L} = \arccos L$ and use $L = \cos(\tilde{L})$ to get $U^n = \exp(in\tilde{L}) = \cos(n \arccos(L)) + i \sin(n \arccos(L)) = T_n(L) + iR_n(L)$ with **Chebychev polynomials** $T_n(x) = \cos(n \arccos(x)), R_n(x) = \sin(n \arccos(x))$. The unitary $U = L + i\sqrt{1 - L^2}$ solves $U + U^* = 2L$. The two evolutions

$$B = \begin{bmatrix} 2D & -1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix}$$

6

are conjugated via $B = S^{-1}AS$ using $S = \begin{bmatrix} U^* & U \\ 1 & 1 \end{bmatrix}$. While $u(n) = T_n(L)u(0) \pm R_n(L)u'(0)$ are not unitary, they are conjugated to unitary evolutions.

We also mentioned that $V = -iU$ solves $i(V - V*) = 2L$ if $U + U^* = 2L$ so that one can replace cos with sin which makes the similarity more clear. We have then compared three cases

$$\exp(i \arcsin(2hL)) = 1 - 2ihL - 2h^2L^2 - 2h^4L^4...$$

$$\exp(2ihL) = 1 - 2ihL - 2h^2L^2 + (4/3)ih^3L^3....$$

$$(hL - i)/(hL + i) = 1 - 2ihL - 2h^2L^2 + 2ih^3L^3....$$

where the later is the Cayley method. They all agree to second order $1 - 2ihL - 2h^2L^2$. $\qquad \square$

4.8. It is worth noting that the symplectic discretization of the wave equation is a time 1 step of an actual wave equation, just for a slightly deformed operator $\tilde{L} = \arcsin(L)$ or $\tilde{D} = \arcsin(D)$ or $\tilde{g} = \arcsin(g)$ and it is causal as every time step is of a cellular automaton type and so satisfies a weak Huygens principle (the strong Huygens principle is a miracle that happens only for odd dimensional manifolds). We therefore also have explicit solutions to these discrete time wave equations like $u(t) = \cos(t\tilde{D})u(0) + t\text{sinc}(t\tilde{D})u'(0)$. How come that the above causality does not work here? It would, if we look at the solution for general $t \in \mathbb{R}$. But we only look for integer t: indeed, the unitary evolution is for integer t given by Chebyshev polynomials for integer t. $e^{(\pm itarcsin(2hL))} = T_t(L) \pm iR_t(L)$. For $t = 1.5$ however the solution $u(t)$ would depend on all the space G and even on simplices at the other and of G.

5. WEAK LOEWNER ORDER

5.1. Assume A is a self-adjoint $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \cdots \lambda_n$ ordered in a decreasing manner. Define the **spectral sum** $S_k(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. The relation $S_k(A) \leq S_k(B)$ for all $1 \leq k \leq n$ defines a **partial order** on the set of self-adjoint matrices. (As for reflexivity, note that $S_1(A) \leq S_1(B)$ and $S_1(B) \leq S_1(A)$ implies $\lambda_1(A) = \lambda_1(B)$. Then use $S_2(A) \leq S_2(B)$ and $S_2(B) \leq S_1(A)$ that $\lambda_2(A) = \lambda_2(B)$ etc. As this order is weaker than the **Loewner order**, we call it the **weak Lowener order**.

5.2. The following was our starting point for this note as we aimed to prove that L spectral dominates D in a weak Loewner sense. The research then then meandered in the fall of 2025 to other topics. The entry point was was a **Brouwer conjecture** project from the summer 2025 [25]. It is a line of research, where the spectral sum $S_k(A)$ of a matrix is the focus of attention. The Brouwer conjecture [2] deals with the spectra of the **Kirchhoff matrix** $H_0(G)$ of a complex. The conjecture is $S_k \leq B_k$ for the spectral sum $S_k = \sum_{j=1}^k \lambda_j$ of the Kirchhoff matrix is bounded above by the **Brouwer bound** $B_k(G) = m + k(k+1)/2$, where $m = f_1(G)$ is the number of edges in G.

5.3. When looking at the spectra, it is natural to compare the spectra of L and D (as seen in Figure 1). The difference $L - D$ is not positive definite in general. We can ask a different question however: is $L - D$ is positive in a **weak Loewner order**? Experiments suggest that this is the case, but we can not prove it yet. Since the diagonal elements of L are 1, we know that from the Schur inequality that $S_k(L) \geq k$. Schur also gives $S_k(D) \geq 0$ which is obvious as the eigenvalues of D are symmetric: if λ is an eigenvalue, then $-\lambda$ is an eigenvalue. We see evidence however that L is larger than D in the weak Loewner sense:

Conjecture 1. *If L is the connection matrix and D the Dirac matrix, then $L - D$ satisfies $S_k(L) \geq S_k(D)$.*

5.4. Here are some attempts: Fan's theorem tells that $S_k(A + B) \leq S_k(A) + S_k(B)$ for all $1 \leq k \leq n$ in general. See [27, 7] and [5] call this spectral majorization. Fan's theorem also implies $S_k(D) \leq S_k(L) + S_k(D - L)$ but that gives only $S_k(L) \geq S_k(D) - S_k(D - L)$. A similar argument gives $S_k(L) \leq S_k(D) + S_k(L - D)$. we measure that $S_k(L + D)$ and $S_k(L - D)$ are very close.

5.5. The matrix $L - D$ has positive diagonal entries and is non-negative but that does not imply that it has no-negative eigenvalues. The matrix $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ for example has eigenvalues $7, -1$ even so it is positive and has all positive entries. Schur's theorem tells that if $B_{ij} >= 0, \forall i, j$, then $S_k(B) \geq 0$. For a Dirac matrix, we also have $S_k(-D) \geq 0$ simply because the spectra of D and $-D$ agree. The identity $k \leq S_k(L - D) \leq S_k(L) + S_k(D)$ shows that $S_k(L) \geq k - S_k(D)$.

6. Lefschetz fixed point Theorem

6.1. The **discrete Lefschetz fixed point theorem** [13] works for any **simplicial dynamical systems** (G, T), where T is a **simplicial map** and G is a simplicial complex. Unlike in the continuum, where one needs to add some assumptions like regularity or that T only has finitely many fixed points, there are no conditions whatsoever in the discrete. We review first the language of simplicial complexes as [13] formulated the theorem for Whitney complexes of graphs, which is almost no loss of generality as every simplicial complex defines a graph. The proof in [13] which was closer to Hopf's proof in the continuum. As mentioned in various expositions like [16] before, it can be proven by **heat deformation**. We give here a bit more details.

6.2. A simplicial complex G comes with the **Alexandrov topology**, in which the stars $U(x) = \{y \in G, x \subset y\}$ form a topological basis. A **simplicial map** is a continuous map that originates from a map on vertices. The Lefschetz formula generalizes the Euler-Poincaré formula and implies the Brouwer's theorem assuring that a simplicial map on a complex homotopic to K_1 has at least one fixed point. [1]

6.3. If a map T on a simplicial complex is continuous then it preserves the order structure. But a continuous map is not necessarily a simplicial map in general: a simplicial maps can not map smaller dimensional parts to higher dimensional ones. For example, a constant map $G \to G, x \to z$ which maps to a constant positive dimensional $z \in G$ is continuous, but it is not a simplicial map. The Lefschetz theorem needs to access orientation of a simplex in an other simplex and so needs that T comes from a simplicial map. It seems not impossible that there might be a generalization to continuous maps but we have not seen this to work yet.

6.4. Lets first define some parts which enter the Lefschetz fixed point theorem. If $T : G \to G$ is a simplicial map, a **fixed point** $x \in G$ is a simplex that satisfies $T(x) = x$. Denote with $F = F_T$ the set of fixed points. Define $\omega(x) = (-1)^{\dim(x)}$. If T is a simplicial map, the **index** of x is defined as $i_T(x) = \omega(x)\text{sign}(T|x)$. The map T induces the **Koopman matrix** $\mathcal{U}_T f = f(T(x))$, which as a linear map on $l^2(G)$ and so a $n \times n$ matrix.

---

[1]A.E. Brouwer from the Brouwer conjecture is not related to L.E.J. Brouwer from the Brouwer fixed point theorem. The later had no children of his own.

Lemma 1. $\text{str}(\mathcal{U}_T) = \sum_{x \in F} i_T(x)$

Proof. If a vertex x is a fixed point of T then $\mathcal{U}(e_x) = e_x$, meaning that $\mathcal{U}(x,x) = 1$. If $T(x) = y$ and $x \neq y$, then $\mathcal{U}(x,y) = e_x e_y = 0$. The analysis now reduces to a situation where G is a complete complex (the set of all non-empty subsets of a set with n elements). Then, every permutation $T : G \to G$ is a simplicial map and the fixed points are unions of cycles in the cycle decomposition of T. The index of a cycle y is 1, the index of a pair of cycles is -1, the index of a triplet of cycles is 1 etc. The sum $\sum_{x \in F} i_T(x)$ is in this case the Euler characteristic of a complete complex K_l, where l is the number of cycles of the permutation defined by T, which agrees with $\text{str}(\mathcal{U}_T)$. $\qquad\square$

6.5. In the special case, when T is the identity, then $\mathcal{U}$ is the identity matrix and $\text{str}(\mathcal{U}_T) = \chi(G)$ and $i_T(x) = \omega(x)$. This is a situation where the agreement follows from the definitions.

6.6. The **Lefschetz number** $\chi_T(G)$ is defined to be the super trace of the map T induced on the kernel of $H = D^2 = \oplus_{k=0}^q H_k$:

$$\chi_T(G) = \text{str}(\mathcal{U}_T|\text{ker}(H)) = \sum_{j=0}^q (-1)^q \text{tr}(\mathcal{U}_T|\text{ker}(H_k)) \ .$$

It generalizes the **Euler characteristic** $\chi(G)$, which is the situation when T is the identity. In that case, $\mathcal{U}_T u = u$ is the identity matrix and the super trace on the kernel of H_k is the Betti number b_k. The Lefschetz fixed point theorem then reduces to the Euler-Poincaré formula:

Theorem 6 (Lefschetz fixed point theorem). *$\sum_{x, T(x) = x} i_T(x) = \chi_T(G)$ for any simplicial map T on any simplicial complex G.*

6.7. To prove this, we first reduce the case of a simplicial map to an invertible simplicial map. While not all continuous maps are simplicial maps, every homeomorphism of a simplicial complex necessarily is a simplicial map. Proof: if v is a vertex then $T(v)$ must be a vertex too (as if it would be a positive dimensional simplex x, then $x = T(v)$ implied $T^{-1}(x) = v$). But then also $T^{-1}(w) = v$ for any vertex $w \subset x$, which would contradict invertibility. A simplicial map is not necessarily open as it can map G to a vertex $\{v\}$ and so map an open set (like G) to a closed set (like a singleton set $\{v\}$). Note that the following is not true for "simplicial complexes" as often considered in the literature as geometric realizations of finite abstract simplicial complexes. (Even [4], where finite abstract complexes first appeared also look at geometric realizations [4].)

Lemma 2. *If T is a simplicial map on a finite abstract simplicial complex G, then the attractor $K = \bigcap_{k \geq 1} T^k(G)$ of T is a simplicial complex.*

Proof. We only need to check that $T(G)$ of G is closed so that by induction $T^k(G)$ is closed implying that the intersection (it is a finite intersection) is closed. In order to see that $T(G)$ is closed, let $y = T(x)$ be in the image $T(G)$. By definition of a simplicial map, this is given as $y = \{T(v), v \in x\}$. Let $z \subset y$ be an arbitrary subset, then look at $u = \{v \in x, T(v) \in z\}$. This set is in G because G is a simplicial complex and so closed under the operation of taking non-empty subsets. Consequently, $z = T(u)$. We see that $z \in T(G)$. As every $z \subset y \in T(G)$ is also in $T(G)$, The set $T(G)$ is closed and so a simplicial complex. $\qquad\square$

6.8. For an arbitrary map $T : G \to G$ on a finite set G, we have:

Lemma 3. *The map restricted to the attractor $K = \bigcap_{k \geq 0} T^k(G)$ is a permutation.*

Proof. Use induction for $n = |G|$. It is clear for $n = 1$. The induction step uses that if $T : G \to G$ is given, then $T(G)$ is T invariant and either has the same number of elements or not. In the first case, it is a permutation, otherwise we can use induction. If T is a simplicial map on a simplicial complex G, then the attractor K of T is closed. The map T is now invertible on K and so a permutation. □

6.9. The proof of the Leschetz fixed point theorem can now can be reduced to homeomorphisms. This is an argument which is close to what Hopf's proof did:

Lemma 4 (Reduction to Homeomorphism)**.** *If T is a simplicial map on G and K is its attractor of T, then $\sum_{x \in G, T(x)=x} i_T(x) = \sum_{x \in K, T(x)=x} i_T(x)$ and $\chi_T(G) = \chi_T(K)$.*

Proof. If $T(x) = x$, then x obviously is in the attractor because $T^k(x) = x$ for every $k \geq 1$. As for the second statement, let x be a simplex that is in G but not in $T(G)$. Let $e_1, \ldots e_l$ be a basis in $\ker(H_k)$ and $\tilde{e}_1 \ldots \tilde{e}_l$ the vector for which $\tilde{e}_j(x) = 0$ and $\tilde{e}_j(y) = e_j(y)$ otherwise. Now, $\mathrm{tr}(\mathcal{U}_T | \ker(H_k)) = \sum_j e_j \cdot \mathcal{U}_T(e_j)$ as $\mathcal{U}_T e_j$ has a zero entry at position x we have $\sum_j \tilde{e}_j \cdot \mathcal{U}_T(e_j)$. This is $\sum_j \mathcal{U}_T(\tilde{e}_j) \cdot e_j$. Because K is T-invariant, also $\mathcal{U}_T(\tilde{e}_j)$ has a zero entry at x. The trace is now $\sum_j \mathcal{U}_T(\tilde{e}_j) \cdot \tilde{e}_j$, meaning that the simplex x has become irrelevant for the Lefschetz number. Doing so for every element $x \in G \setminus K$ proves the second claim $\chi_T(G) = \chi_T(H)$. □

6.10. The **McKean-Singer symmetry** [26] (see [12] for the discrete case) assures that $\mathrm{str}(e^{-tH}) = \chi(G)$ follows from $\mathrm{str}(H^n) = 0$ for $n > 0$:

Lemma 5 (McKean Singer)**.** $\mathrm{str}(H^n) = 0$ *for $n > 0$ so that $\mathrm{str}(e^{-tH}) = \chi(G)$ is independent of t.*

Proof. The Dirac matrix D produces a bijection between the non-zero spectrum of the even forms and the non-zero spectrum of the odd forms: if $Lu = \lambda u$ and $u \in l^2(\bigcup_{k,\mathrm{even}} G_k)$ (even forms), then $L(Du) = \lambda(Du)$ and Du is in $l^2(\bigcup_{k,\mathrm{odd}} G_k)$ (odd forms). (This argument does not work if λ is a zero eigenvalue because then, Du is the zero vector.) There is an isomorphism between the kernel on even forms and odd forms only if the Euler characteristic is zero. □

Lemma 6. *Assume T is homeomorphism of G and $H = D^2$ is the Laplacian:*
a) $\mathcal{U}$ commutes with H.
b) $\mathcal{U}$ commutes with the heat flow e^{-tH}.
c) $\mathcal{U}_t = e^{-tH}\mathcal{U}$ is block diagonal for all t with the same blocks than H.

Proof. If T is a homeomorphism, then T preserves the order structure and also the dimensions of the simplicies. It does not necessarily preserve the orders and so the sign of a subsimplex in a given simplex.
a) The relation $\mathcal{U}d = \pm d\mathcal{U}$ holds by definition. It implies that $\mathcal{U}D = \pm D\mathcal{U}$ and so $\mathcal{U}H = H\mathcal{U}$.
b) We can make a Taylor expansion in t of e^{-tH} and use that $\mathcal{U}$ commutes with every power H^n.
c) Both $\mathcal{U}$ and e^{-tH} preserve k-forms and so are block diagonal. □

10

6.11. The McKean symmetry now immediately works also in a dynamical setting:

Lemma 7. $\mathrm{str}(H^n \mathcal{U}) = 0$ *for $n > 0$ so that $\mathrm{str}(e^{-tH}\mathcal{U})$ is also independent of t.*

Proof. Diagonalize H and look at an eigenvector f on even forms with $Hf = \lambda f \neq 0$ then $H(Df) = DHf = D\lambda f = \lambda(Df)$ gives an eigenvectors Df on odd forms. Now look at $H^n U$. If f is an eigenvector on k-forms, then we especially know that Uf remains a vector on k-forms. $\mathcal{U}$ commutes with D on a non-zero eigenspace of $H\mathcal{U}$. If $H^n \mathcal{U}f = \lambda f$. Then $H^n \mathcal{U}(Df) = H^n D \mathcal{U}f = DH^n \mathcal{U}f = D\lambda f = \lambda(Df)$. □

6.12. We can now complete the proof of the Lefschetz fixed point theorem:

Proof. The deformed Koopman operators $\mathcal{U}_t f = e^{-tH}\mathcal{U}f$ is for $t = 0$ equal to $\mathcal{U}_0 = \mathcal{U}$. For $t \to \infty$, it is the induced map on harmonic forms. In the limit $t \to \infty$, where $P = \lim_{t \to \infty} e^{-tH}$ is the projection onto the kernel of H, it is the super trace of $\mathcal{U}$ on the kernel of H. This was defined to be the Lefschetz number. □

6.13. Remark: Finite topological spaces form a slightly more general category than the set of **finite pre-ordered sets** where **monotone maps** are morphisms. The later can be identified with the class of finite topological spaces that are T_0 spaces. For example: for the **Sierpinski space** $\mathcal{O} = \{\emptyset, \{1\}, \{1, 2\}\}$, the constant map $T(x) = 0$ is continuous but not open because $\{0\}$ is not open. Open sets are the upward-closed sets for the specialization pre-order $x \leq y$ if $x \in \overline{\{y\}}$. This means $x \leq y$ if and only if $y \in U(x)$ if and only if $x \subset y$. On a simplicial complex, the specialization pre-order is the subset partial order $x \subset y$. A theorem of Barmak-McCord assures that every finite T_0-spaces encodes the homotopy type of a finite abstract simplicial complex. The Kolmogorov quotient gets from a finite topological space, a T_0 space using the equivalence relation $x \sim y$ if $\overline{\{x\}} = \overline{\{y\}}$.

6.14. Examples:
1) if $G = K_n$ and T be a rotation map, there is one fixed point $x = G$ and $i_T(x) = 1$. **2)** Let $G = K_n$ and assume T has k cycles. Then there are k fixed points. Let y be a fixed simplex on which T is a rotation then $i_T(y) = \omega(y)$. Now $\sum_{y \in F} i_T(y) = \sum_{y \in F}(-1)^{\dim(y)} = \sum_{y \in F} \omega(y)$. This is by definition the super trace of U.

6.15. Remark: The theorem generalizes to correspondences $T(x) = \bigcup T_i(x)$ which have the property that the inverse S of T is a simplicial map. One just can look at the transformation S then. The ω limit set of S then is the α limit set of T. This models $T(x) = -x^2$ on the discrete circle C_n, which has trace 1 on constant functions and trace -1 on 1-harmonic forms, so that the Lefschetz number is 2. There are also always the two fixed points $0, -1$.

7. DYNAMICAL CONNECTION AND DIRAC MATRICES

7.1. Related to the Lefschetz fixed point theorem, one can look at **dynamical connection matrices** $L_T(x, y) = \chi(C(x) \cap C(T(y)))$ and dynamical Green function matrices $g_T(x, y) = \omega(x)\omega(y)\chi(U(x) \cap U(T(y)))$. These are no more symmetric matrices in general. A bit surprisingly, these still are unimodular matrices, even so the spectrum is now complex in general. We have the following generalization of the unimodularity theorem. We were pretty excited to see this to pan out in examples until we realized that the proof is easy:

Theorem 7. $L_T(x, y)^{-1} = g_T(x, y)^*$.

7.2. This is easier to see if one makes the problem harder and assume that T is a general permutation of the basis. This can be implemented as a permutation matrix P. The result follows from the very general formula in linear algebra:

Lemma 8. *If A, B are $n \times n$ matrices and P_T a permutation matrix defining a permutation of the basis. Define $A_T = AP_T$ and $B_T = BP_T$. Then $A_T B_T^*$ is independent of T.*

Proof. Just compute it $A_T B_T^* = AP^T (P^T)^* B = AB$. $\qquad\square$

7.3. This obviously directly produces the theorem above after noticing that $L_T = LP_T$ and $g_T = gP_T$. Also the **potential energy theorem** generalizes

Theorem 8. $\sum_{x,y} g_T(x,y) = \chi(G)$

7.4. Also this follows directly from the usual energy theorem $\sum_{x,y} g(x,y) = \chi(G)$, noting that

Lemma 9. *If A is an $n \times n$ matrix and P_T a permutation matrix and $A_T = AP_T$. Then $\sum_{x,y} A_T(x,y)$ is independent of T.*

Proof. The transformation T just reshuffles the sum. $\qquad\square$

7.5. One can also wonder about **dynamical Dirac matrices**. Define the **dynamically deformed exterior derivative** $d_T(x,y) = s(x, T(y))$. We have

Theorem 9. $H = (d_T + d_T^*)^2$ *is independent of T.*

Proof. Also this follows from the above simple lemma. The matrices $d_T d_T^*$ and $d_T^* d_T$ both do not depend on T so that D_T^2 does not depend on T. $\qquad\square$

8. Code

8.1. Here is the code that generated Figure 1 and Figure 2:

```
Generate[A_]:=If[A=={},A,Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]];        L=Length;        T=Transpose;
Connection[G_]:=Table[If[L[Intersection[G[[i]],G[[j]]]]>0,1,0],{i,L[G]},{j,L[G]}];
F[G_]:=Delete[BinCounts[Map[L,G]],1];      S[x_]:=Signature[x];
s[x_,y_]:=If[SubsetQ[x,y]&&(L[x]==L[y]+1),S[Prepend[y,Complement[x,y][[1]]]]*S[x],0];
Dirac[G_]:=Module[{f=F[G],d,n=L[G]},d=Table[s[G[[i]],G[[j]]],{i,n},{j,n}]; d+T[d]];
K=RandomGraph[{6,12}];G=Whitney[K];n=Length[G];
A=Connection[G]; B=Dirac[G];J=Inverse[A];
EA = Reverse[Sort[Eigenvalues[1.0*A]]]; EJ= Reverse[Sort[Eigenvalues[1.0*J]]];
EB = Reverse[Sort[Eigenvalues[1.0*B]]];
ea = Table[Sum[EA[[j]],{j,k}],{k,n}]; ej = Table[Sum[EJ[[j]],{j,k}],{k,n}];
eb = Table[Sum[EB[[j]],{j,k}],{k,n}]; di = Table[k, {k, n}];
DA = Reverse[Sort[Table[L[Select[G, L[Intersection[G[[k]],#]]>0 &]],{k,n}]]];
DB = Reverse[Sort[Table[L[Select[G, SubsetQ[G[[k]],#] || SubsetQ[#,G[[k]]] &]],{k,n}]]];
ListPlot[{EA,EJ,EB},Joined->True,Filling->Axis]
ListPlot[{ea,ej,eb},Joined->True,Filling->Axis]
ListPlot[{EA,DA},Joined->True,Filling->Axis]
ListPlot[{EB,DB},Joined->True,Filling->Axis]
```

8.2. And here is some illustration of the Lefschetz fixed point theorem.

```
Generate[A_]:=If[A=={},A,Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]];   L=Length;   s[x_]:=Signature[x];
s[x_,y_]:=If[SubsetQ[x,y]&&(L[x]==L[y]+1),s[Prepend[y,Complement[x,y][[1]]]]*s[x],0];
W[k_]:=Map[Sort,Select[G,L[#]==k &]];
Dirac[G_]:=Table[s[G[[i]],G[[j]]]+s[G[[j]],G[[i]]],{i,n},{j,n}];
Str[X_]:=Sum[X[[k,k]]*(-1)^(L[G[[k]]]-1),{k,n}];
g=CompleteGraph[{3,4,2}]; GraphJoin[CycleGraph[4], CycleGraph[5]];
G=Whitney[g]; n=L[G]; V=Flatten[Select[G,L[#]==1 &]]; DD=Dirac[G]; HH=DD.DD;
AutQ[T_]:=Module[{G1=W[2]},G1==Sort[Map[Sort,(G1 /. Table[V[[k]]->T[[k]],{k,L[V]}])]]];
S[T_]:=Table[V[[j]]->T[[j]],{j,L[V]}]; Q=Transpose[NullSpace[HH]];
P=Q.Inverse[Transpose[Q].Q].Transpose[Q]; Aut=Select[Permutations[V],AutQ[#] &];
U[T_]:=Table[x=G[[k]];y=G[[l]];z=y /.Table[V[[j]]->T[[j]],{j,L[V]}];
          If[x==Sort[z],s[z],0],{k,n},{l,n}];
f[T_]:=Module[{J=S[T]},Fix=Select[G,Sort[# /.J]==#&];ind[x_]:=(-1)^(L[x]-1)*s[x/.J];
   {If[Fix=={},0,Total[Map[ind,Fix]]], Str[P.U[T]]}];
T=RandomChoice[Aut]; J=S[T]; UU=U[T]; Fix=Select[G,Sort[# /. J]==# &];
{Total[Map[ind,Fix]], Str[UU], Str[P.UU]}
R=Map[f,Aut]; {Union[R],Total[R]/L[R]}
```

8.3. Finally we look at some code which illustrates the episode on dynamical matrices D_t, L_T, g_T.

```
Generate[A_]:=If[A=={},A,Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]]; L=Length; s[x_]:=Signature[x];
s[x_,y_]:=If[SubsetQ[x,y]&&(L[x]==L[y]+1),s[Prepend[y,Complement[x,y][[1]]]]*s[x],0];
F[G_,k_]:=Map[Sort,Select[G,L[#]==k&]]; Euler[G_]:=Total[Map[w,G]]; RP=RandomPermutation;
S[G_,T_]:=Module[{V=Flatten[F[G,1]]},Table[V[[j]]->T[[j]],{j,L[V]}]]; IS=Intersection;
FindAut[G_]:=Module[{V=Flatten[F[G,1]],H=F[G,2],T},T=Permute[Range[L[V]],RP[L[V]]];
    While[Not[H==Sort[Map[Sort,(H/.S[G,T])]]],T=Permute[Range[L[V]],RP[L[V]]]]; T];
d[G_,W_]:=Table[x=G[[i]];y=G[[j]]/.Q;s[x,y],{i,L[G]},{j,L[G]}]; Clear[g,K];
Dirac[G_,Q_]:=d[G,G]+Transpose[d[G,W]]; W[G_,k_]:=Map[Sort,Select[G,L[#]==k&]];
U[G_,x_]:= Select[G,SubsetQ[#,x] &];  w[x_]:=-(-1)^L[x];  w[x_,y_]:=w[x]*w[y];
g[G_,Q_]:=Table[x=G[[k]];y=G[[l]];z=y/.Q;w[x,y]*Euler[IS[U[G,x],U[G,z]]],{k,n},{l,n}];
K[G_,Q_]:=Table[If[L[IS[G[[i]],G[[j]]/.Q]]>0,s[G[[i]]]*s[G[[j]]],0],{i,L[G]},{j,L[G]}];

t=CompleteGraph[{2,3,4,1}]; G=Whitney[t]; n=L[G]; T=FindAut[G];   Q=S[G,T];
D1 = Dirac[G, Q]; D0 = Dirac[G,S[G,Range[Length[F[G,1]]]]];  gT=g[G,Q]; LT=K[G,Q];
D1.D1==D0.D0
Transpose[LT] == Inverse[gT]
Euler[G]==Total[Flatten[gT]]
```

9. Remarks

9.1. The Lefschetz fixed point theorem goes over to cohomologies of higher characteristics. Simplicial cohomology is just the first, linear cohomology. The next is quadratic cohomology, where one looks at the set $G^{(2)}$ of all simplex pairs (x, y) for which $x \cap y \neq \emptyset$. There is an exterior derivative d on functions on $G^{(2)}$ and so a Dirac matrix $D^{(2)} = d + d^*$, a Hodge Laplacian $H^{(2)}$ and cohomology in the form of the kernels of the blocks of this Laplacian, where the block $H_k^{(2)}$ belongs to functions on simplices for which $\dim((x, y)) = \dim(x) + \dim(y) = k$. A simplicial map $T : G \to G$ induces a map $T^{(2)} : G^{(2)} \to G^{(2)}$ given by $T^{(2)}((x, y)) = (T(x), T(y))$. A fixed point is a pair (x, y) of intersecting simplices such that $T^{(2)}((x, y)) = (x, y)$, its index is $i_{T^2}(x, y) = i_T(x)i_T(y)$, the Lefschetz number is the super trace of $T^{(2)}$ on cohomology as before. The proof of the higher Lefschetz theorems are exactly the same as in the simplicial cohomology case: the Dirac matrix produces a McKean-Singer symmetry. One then looks at the heat flow in the Koopman case $\mathcal{U}_{T^{(2)}} e^{-tH^{(2)}}$ which for $t = 0$ gives the sum $\sum_{(x,y) \in F} i_{T^{(2)}}((x, y))$ of indices of fixed points and for $t \to \infty$ the Lefschetz number $\chi_{T^{(2)}}(G)$ for quadratic cohomology.

9.2. The classical Birkhoff fixed point theorem from 1913 (revised a 100 years ago in 1926 [3]) deals with area-preserving homeormorphisms of the cylinder $X = \mathbb{T} \times [0, 1]$ that rotate the boundary in different directions. Since the classical Lefschetz number $\chi_T(X)$ in this case is zero, the Lefschetz theorem does not work and indeed, the area-preservation property as well as the twist conditions have to enter somewhere. Disregarding any of them allows to get examples without any fixed points, like rigid rotations or where one of the rotating invariant boundaries is an attractor. We contemplated in 2024 whether quadratic cohomology could work to prove this last theorem of Poincaré: quadratic cohomology has been proven to be interesting already to distinguish the cylinder X from the Möebius strip M as $b^{(2)}(X) = (0, 0, 1, 1, 0)$ and $b^{(2)}(M) = (0, 0, 0, 0, 0)$ [14]. This cohomology is more intersting already for a 2 dimensional complex, where the Hodge matrix $H^{(2)}$ has 5 blocks because the maximal dimension of a simplex pair (x, y) is $2 + 2 = 4$. We argued then that the Lefschetz number for an area preserving map twisting is 2. The area-preserving property should imply that the trace of the induced map on the $k = 2$ cohomology is 1. The twisting condition then assure that the trace on the induced map on the $k = 3$ cohomology is -1. The later deals with maps on pairs (x, y) where x is a triangle and y is an edge in x. If this argument holds, then the super trace is 2 and imply that there are at least two fixed points. One of the difficulties with such an approach is also that simplicial area preserving maps on a cylinder are rather rigid. We have difficulty however to build examples of simplicial maps on a cylinder that satisfy the area preservation and twist conditions.

9.3. Every simplicial complex G, defines a **Dirac zeta function** $\zeta(z) = \sum_j \lambda_j^{-s}(H)$, where λ_j are the non-zero eigenvalues of $H = D^2$. And then there is a **connection Zeta function** $\sum_j \lambda_j^{-s}(L^2)$, where λ_j are the eigenvalues of L^2, and where L is the connection Laplacian. One can also look at the **Green zeta function** $\sum_j \lambda_j^{-s}(g^2)$ For a one-dimensional complex G and the Hodge zeta function, there is symmetry about the imaginary axes: the functional equation is $\zeta(z) = \zeta(-\bar{z})$. A comparision of the zeta functions in the case of matrices D, L, g would be interesting.

9.4. If T is an automorphism of a complex G, the **dynamical zeta function** is defined as

$$\zeta_T(s) = \exp(\sum_{k=1}^{\infty} \chi_{T^k}(G)s^k/k) \ .$$

It is the generating function for all the **Lefschetz numbers** χ_{T^k} of the iterates of T. These are rational functions [13].

9.5. Given an isometry of a classical Riemannian manifold M, one can look at Vietris-Ribbs complexes G in M. It induces a continuous map on the complex if simplices are mapped into simplices. It is rarely a simplicial map however. We would like to have a frame work in which not only the cohomology of the complex is the cohomology of the manifold but also the Lefschetz number of the map is the same than the map in the finite. This needs to be explored more.

9.6. The nonlinear discrete sine-Gordon partial differential equations $Lu = c\sin(u)$ and $Hu = c\sin(u)$ have completely different behavior. In the first case, $u = 0$ is the only solution for small c while in the later case there are Harmonic forms $Hu = 0$ that solve the case $c = 0$. We can say that in the connection case, we have a **stability of the vacuum**. The fact that $u = 0$ is stable follows from the fact that L is invertible. It is an interesting phenomenon which persists when the simplicial complexes are infinite. The case of small c in the case $Hu = c\sin(u)$ is subtle in the infinite case, even for $\mathbb{Z}$. There is still some sort of stability but it is of KAM nature. For large c, one can use the standard implicit function theorem. The **anti-integrable limit** $c \to \infty$ gives many non-trivial solutions for these two non-linear equations. [1] We hope to investigate more what happens in the connection case. There is always a smallest c after which $Lu = c\sin(u)$ starts to have solutions different from the vacuum.

10. QUESTION

10.1. The first question is illustrated in Figure 1:

A) Is it true that $L \geq D$ or $L \geq g$ in the weak Loewner order?

10.2. We also do not see yet an argument assuring that the inverse of L has a maximal eigenvalue. It looks as if $g^n = L^{-n}$ is conjugated to a matrix for which all entries are positive.

B) Does $g = L^{-1}$ always have a unique maximal eigenvalue?

10.3. While D has spectrum that is symmetric about 0, this is not the case for L. The number of positive eigenvalues and the number of negative eigenvalues differ by the Euler characteristic. The spectral radius of g is interesting because $g(x, y)$ can be interpreted as a potential energy between x and y. The spectral radius also controls the **spectral gap** and so the stability of the vacuum $u = 0$.

C) Is the spectral radius of L always larger than the spectral radius of g?

10.4. We need to extend the Lefschetz fixed point theorem from simplicial maps to situations where it can model homeomorphisms of manifolds with or without boundary. The following is a guiding challenge.

D) Can the Birkhoff fixed point theorem be proven via quadratic cohomology?

REFERENCES

[1] S. Aubry and G.Abramovici. Chaotic trajectories in the Standard map. the concept of anti-integrability. *Physica D*, 43:199–219, 1990.

[2] A.E. Brouwer and W.H. Haemers. *Spectra of graphs*. Springer, 2012.

[3] M. Brown and W.D. Neumann. Proof of thePoincaré-Birkhoff fixed point theorem. *Mich. Math. J.*, 24:21–31, 1975.

[4] M. Dehn and P. Heegaard. Analysis situs. *Enzyklopaedie d. Math. Wiss*, III.1.1:153–220, 1907.

[5] A.M. Duval and V. Reiner. Shifted simplicial complexes are Laplacian integral. *Trans. Am. Math. Soc*, 354:5413–4344, 2002.

[6] L.C. Evans. *Partial Differential equations*, volume 19 of *Graduate Studies in Mathematics*. AMS, second edition, 2010.

[7] K. Fan. On a theorem of Weyl concerning eigenvalues of linear transformations i. *Proc. Nat. Acad. Sci. USA*, 35:652–655, 1949.

[8] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer Verlag, 2001.

[9] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, second edition edition, 2012.

[10] O. Knill. A remark on quantum dynamics. *Helvetica Physica Acta*, 71:233–241, 1998.

[11] O. Knill. Singular continuous spectrum and quantitative rates of weakly mixing. *Discrete and continuous dynamical systems*, 4:33–42, 1998.

[12] O. Knill. The McKean-Singer Formula in Graph Theory. http://arxiv.org/abs/1301.1408, 2012.

[13] O. Knill. A Brouwer fixed point theorem for graph endomorphisms. *Fixed Point Th. and Appl.*, 85, 2013.

[14] O. Knill. The cohomology for Wu characteristics. http://arxiv.org/abs/1803.06788, 2017.

[15] O. Knill. Hear the Euler characteristic of a simplicial complex. https://arxiv.org/abs/1711.09527, 2017.

[16] O. Knill. On Atiyah-Singer and Atiyah-Bott for finite abstract simplicial complexes. https://arxiv.org/abs/1708.06070, 2017.

[17] O. Knill. The Hydrogen identity for Laplacians. https://arxiv.org/abs/1803.01464, 2018.

[18] O. Knill. Division algebra valued energized simplicial complexes. https://arxiv.org/abs/2008.10176, 2020.

[19] O. Knill. The energy of a simplicial complex. *Linear Algebra and its Applications*, 600:96–129, 2020.

[20] O. Knill. Analytic torsion for graphs. https://arxiv.org/abs/2201.09412, 2022.

[21] O. Knill. Characteristic topological invariants. https://arxiv.org/abs/2302.02510, 2023.

[22] O. Knill. Cohomology of open sets, 2023. https://arxiv.org/abs/2305.12613.

[23] O. Knill. Eigenvalue bounds of the Kirchhoff Laplacian. *Lin. Alg. and its Appl.*, 701:1–21, 2024.

[24] O. Knill. Fusion inequality for quadratic cohomology, 2024.

[25] O. Knill. Remarks about the brouwer conjecture. https://arxiv.org/abs/2508.07550, 2025.

[26] H.P. McKean and I.M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.

[27] H. Weyl. Inequalities between the two kinds of eigenvalues of a linear transformation. *Proc. Nat. Acad. Sci. USA*, 35:408–411, 1949.

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