CHARACTERISTIC TOPOLOGICAL INVARIANTS

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Abstract. The higher characteristics $w_m(G)$ for a finite abstract simplicial complex G are topological invariants that satisfy k-point Green function identities and can be computed in terms of Euler characteristic in the case of closed manifolds, where we give a new proof of $w_m(G) = w_1(G)$. Also the sphere formula generalizes: for any simplicial complex, the total higher characteristics of unit spheres at even dimensional simplices is equal to the total higher characteristic of unit spheres at odd dimensional simplices.

1. Summary

1.1. We prove **k-point Green function formulas**

$$w_m(G) = \sum_{x_j \in G} g_m(x_1, \dots, x_k) = \sum_{x \in G^k} \prod_{j=1}^{k} w(x_j) w_m(\bigcap_{j=1}^{k} U(x_j))$$

for the **m'th characteristic** of $A \subset G$

$$w_m(A) = \sum_{x \in A^m, \bigcap_j x_j \in A} \prod_{j=1}^{m} w(x_j) \ ,$$

where $w(x) = (-1)^{\dim(x)}$ and where G is a finite abstract simplicial complex. The **stars** $U(x) = \{y, x \subset y\}$ define a topological base of a finite non-Hausdorff **topology** on G. For two arbitrary open sets U, V, the **valuation formula**

$$w_m(U \cup V) + w_m(U \cap V) = w_m(U) + w_m(V)$$

holds for all $m \geq 1$. This makes the higher characteristics w_m **topological invariants**: they agree for homeomorphic complexes. It also makes more general energized versions of w_m sheaf ready. For **manifolds**, $w_m(M) = w_1(M)$.

1.2. We use short hand $w_m(A) = \sum_{X \in A^m, \bigcap X \in A} w(X)$, summing over $X = (x_1, \dots, x_m) \in A^m$ with $\bigcap X \in A$, meaning $\bigcap_{j=1}^{m} x_j \in A$ and $w(X) = \prod_{j=1}^{m} (-1)^{\dim(x_j)}$. The **Euler characteristic** is $w_1(A)$ and the **Wu characteristic** is $w_2(A)$. **k-point Green function** maps a k-point configuration $X \in G^k$ to $g_m(X) = w(X) w_m(U(X))$ with $U(X) = \bigcap_{j=1}^{k} U(x_j))$. The **energy formula** $w_m(G) =$

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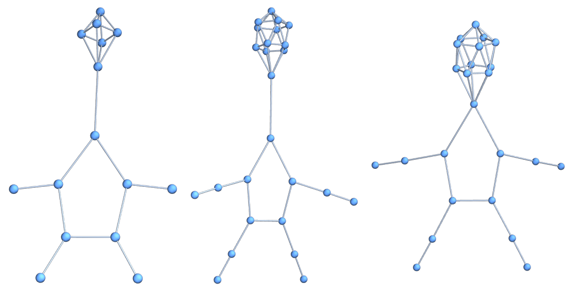

FIGURE 1. These complexes G, H, K are all homotopic to a **wedge sum** of a 2-sphere and a 1-sphere. The first two complexes G, H are homeomorphic. The complex K however is not homeomorphic to G. We have Euler $\omega_1(G) = \omega_1(H) = \omega_1(K) = 1$ and Betti vector $b_1(G) = b_1(H) = b_1(K) = (1,1,1)$. For Wu $\omega_2(G) = \omega_2(H) = 11, b_2(G) = b_2(H) = (0,0,11,1,1)$ and $\omega_2(K) = 9, b_2(K) = (0,0,11,3,1)$. Then $\omega_3(G) = \omega_3(H) = -29, b_3(G) = b_3(H) = (0,0,0,31,3,2,1)$ and $\omega_3(K) = -23, b_3(K) = (0,0,0,31,12,5,1)$.

$\sum_{X \in G^k} g_m(X)$ involves open sets in G and so is sheaf theoretic. Unlike in $w_m(G)$, there is no $\bigcap X \in G$ assumption in the Green part. The closure $B(x)$ of a **star** $U(x)$ is called the **unit ball**. Its boundary $S(x)$ is the **unit sphere** also known as **link**. The **local valuation formula**

$$w_m(B(x)) = w_m(U(x)) - (-1)^m w_m(S(x))$$

holds for all stars $U(x)$, but in general not for arbitrary open sets U with with $B(X) = \overline{U(X)}$ and $S(X) = B(X) \setminus U(X) = \delta U(X)$.

1.3. The special **one-particle case** $k = 1$ gives for all $m \geq 1$ a Gauss-Bonnet or Poincaré-Hopf type formula

$$w_m(G) = \sum_{x \in G} w(x) w_m(U(x)) \ ,$$

which reduces the computation of $w_m(G)$ to local expressions. We can think of $g_m(x)$ as a "curvature" or interpret it as an "index". Note that the Gauss-Bonnet type formula for open sets does not generalize if G is replaced by an open set U. It only works if G is replaced by a smaller simplicial complex. The unit balls $B(x) = \overline{U(x)}$ for example are by themselves again simplicial complexes and we also have $w_m(G) = \sum_{x \in G} w(x) w_m(B(x))$.

1.4. For all complexes G, the **sphere formula**

$$\sum_{x \in G} w(x) w_m(S(x)) = 0$$

holds. We have seen that in [7, 10] for $m = 1$. It follows from the local valuation formula and the **ball formula** $w_m(G) = \sum_{x \in G} w(x) w_m(B(x))$. This is can be useful for large complexes, where we can use the ball formula again for computing $w_m(B(x))$, especially if many $B(x)$ have the same type or are known. The sphere formula for example proves that if G is a complex for which the characteristic w_m of all spheres $S(x)$ is constant $c \neq 0$ then $w_m(G) = 0$. This applies for odd-dimensional manifolds or more generally for odd-dimensional **Dehn-Sommerville d-manifolds** $\mathcal{X}_d$ [4] recursively defined by the property $\chi(G) = 1 + (-1)^d$ and that all unit spheres satisfy $S(x) \in \mathcal{X}_{d-1}$. The induction starts with $\mathcal{X}_{-1} = \{0\}$, where the **void** $0 = \{\}$ is the empty complex, which is also the (-1)-sphere.

1.5. Unlike manifolds, Dehn-Sommerville spaces $\mathcal{X} = \bigcup_k \mathcal{X}_k$ define a **monoid** under the **join operation** $G + H = G \cup H \cup \{xy, x \in G, y \in H\}$. It contains the **sphere monoid** $\mathcal{S}$ of all spheres in which the (-1) sphere 0 is the zero element. The join of a p-spheres with a q-sphere is a $(p+q+1)$-sphere using induction and $S_{G+H}(x) = S_G(x) + H$ for $x \in G$ or $S_{G+H}(x) = G + H(x)$ for $x \in G$. It is natural to look at generalized d-manifolds which are complexes for which all unit spheres are in $\mathcal{X}_{d-1}$. This is a larger class than d-manifolds but which like manifolds is no sub-monoid of the monoid $(\mathcal{G}, +)$ of all complexes.

1.6. We have proven the energy formula in the case $m = 1, k = 2$ already as $\chi(G) = \sum_{x,y} g(x, y)$, where

$$g(x, y) = w(x)w(y)\chi(U(x) \cap U(y))$$

are the matrix entries of the inverse of the **connection matrix** $L(x, y) = 1$ if $x \cap y \neq \emptyset$ and $L(x, y) = 0$ else. In the case $m = 1, k = 1$, we have seen the super trace expression $\chi(G) = w_1(G) = \sum_{x \in G} w(x)\chi(U(x))$ involving $g(x, x) = \chi(U(x))$. We then generalized the energy formula to energized complexes, where $w(x)$ was replaced by an arbitrary ring-valued function $h(x)$. We had explored such **energized complexes** also in the case when h was division algebra valued.

1.7. The energy theorem can more generally be seen as a linear transformation which turns an **interaction energy function** $h_m : G^m \to R$ on intersecting points to a Green function $g_m : G^k \to R$ on k-tuples of **open stars**. The parameters m and k can be arbitrary positive integers. The energy

$$w_m(A) = \sum_{X \in A^m, \bigcap X \in A} h_m(X)$$

then defines k-**point potential values**

$$g_m(X) = w(X) w_m(U(X))$$

or $X \in G^k$. We then still have for any $m, k \geq 1$ the relation

$$w_m(G) = \sum_{X \in G^m, \bigcap X > 0} h_m(X) = \sum_{X \in G^k} g_m(X) \; .$$

Of course, for a general choice of the interaction energy h, there is no topological invariance. Indeed, already requiring the weaker combinatorial property of being invariant under Barycentric refinements forces h to be of the form $h(x) = w(x)$.

1.8. The more general picture with **energized local interaction** h_m explains, why a positive semi-definite h produces positive semi-definite g: if h_m is zero except for some fixed X_0, where $h_m(X_0) = 1$, then $g_m : X \in G^k \to g_m(X)$ is positive semi definite form. Convex combinations of positive semi definite forms h_m in m variables are mapped into convex combinations of positive semi definite forms g_m in k variables. In the case $m = 1, k = 2$ for example, where the Green function matrix $g(x, y)$ is the inverse of the connection matrix $L(x, y)$, we have a duality between positive definite quadratic forms. In this case, L and g are even isospectral. They gave isospectral positive definite integral quadratic forms. One can write $L(x, y) = w_m(V(x) \cap V(y))$ with $V(x) = \overline{\{x\}}$. For $|h(x)| = 1, k = 2, m = 1$ we had them inverses of each other. [5, 8, 6].

1.9. As for references, we should add that the subject has a large cultural background starting with Max Dehn and Poul Heegaard [2] who first defined abstract simplicial complexes in 1907. Together with finite topological spaces as first considered by Pavel Alexandrov [1] in 1937, this produces a powerful framework. We pointed out some reasons in [10, 9] why we do not want to look at geometric realizations. An example is that the geometric realization of the double suspension of a Poincaré homology sphere is homeomorphic to a 5-sphere even so the combinatorial complexes are obviously not homeomorphic in a discrete sense. To remain in the finite topology allows to avoid difficulties from the continuum.

1.10. The mathematics of Dehn,Heegard or Alexandrov is not only mathematically simpler as it is only combinatorics of finite sets of sets only, we can also branch off into area, where the continuum is awkward. For example, we can look at the **cohomology of open sets** in a simplicial complex. This is richer than the cohomology of closed sets. Since the Betti vector of a single set $\{x\}$ (which by definition is an open set $\{x\}$ with locally maximal x in a simplicial complex) is the basis vectors $e_{|x|}$, where $|x|$ is the cardinality of the set, we can realize any Betti vector with open sets. We would not know how to realize a given Betti vector with a simplicial complex. In the continuum, the cohomology of an open set is usually only considered in the sense of a limit of cohomologies of compact subsets. Even the homotopy is different. An open interval $\{\{1, 2\}\}$ for example has Euler characteristic -1 and is not homotopic to a single point $\{\{1\}\}$ which has Euler characteristic 1. These two spaces should not be considered homotopic, for any sensible definition of homotopy. But we can identify $A = \{\{1, 2\}\}$ for example with $B = \{\{2\}, \{1, 2\}, \{2, 3\}\}$ which is also an open interval. The open sets A, B are homeomorphic. Their boundaries $\delta A = \overline{A} \setminus A = \{\{1\}, \{2\}\}$ and $\delta B = \overline{A} \setminus B = \{\{1\}, \{2\}\}$ are both 0-spheres. Unlike for general finite set of sets (sometimes called multi-graphs), the cohomology and topology works well for both open and closed sets. Almost everything we describe here fails for general set of sets which are neither closed nor open.

2. Notations

2.1. A **finite abstract simplicial complex** G is a finite set of sets, closed under the operation of taking non-empty subsets. G carries a finite topology $\mathcal{O}$, in which the set of all **stars** $U(x) = \{y, x \subset y\}$ is the **topological base**. If $V = \bigcup_x x$ is the **vertex set**, the collection of vertex stars $U(x)$ with $x = \{v\}$ with $v \in V$ are a **topological subbase** which when closed under intersection produces the topological base and when closed under intersection and union produces the topology. The closed sets in the topology are the sub-simplicial complexes. If $x \subset y$, then $U(y) \subset U(x)$. A map $f : G \to H$ between simplicial complexes is **continuous** if $f^{-1}(U)$ is open in G if U is open in H. A **simplicial map** is a map from $V(G)$ to $V(H)$ that lifts to an order preserving map $G \to H$. Simplicial maps are continuous; however not all continuous maps are simplicial maps. An example is a constant map onto a simplex c of positive dimension.

2.2. The topology on G is **Kolmogorov** (T0) but neither **Fréchet** (T1), nor **Hausdorff** (T2). As any finite topology, it is **Alexandroff**, meaning that there are smallest neighborhoods $U(x)$ of every $x \in G$. These are the **stars** $U(x) = \{y \in G, x \subset y\}$. The topology is **Zariski type** because the closed sets in the topology agreeing with sub-simplicial complexes of G. In the case when G is the Whitney complex of a graph (V, E), formed by the vertex sets of complete subgraphs, then subgraphs define closed sets. Every G again defines a graph, where $\Gamma = (V, E) = \{(x, y), x \subset y \text{ , or , } y \subset x\}$. The Whitney complex G of this graph is the **Barycentric refinement** G_1 of G. The topology has the desired connectivity properties of G. The just described graph obtained from G has the same connectivity properties than the topology of G. The topology induced from the geodesic distance of the graph would produce the **discrete topology** on G and render G completely disconnected. The non-Hausdorff property is inevitable. In the case when G comes from a graph Γ, the graph Γ is the Čech nerve of the **topological subbase** of vertex stars. The Čech nerve of the **topological base** of all stars is Γ_1, the Barycentric refinement of Γ.

2.3. Elements in G also known as **simplices** or **faces** or simply called **sets**. G is is a set of sets of V and the topology is a set of sets in $\mathcal{O}$. The closure $\overline{A}$ of an arbitrary set $A \subset G$ is the smallest closed set in G containing A. We need to distinguish three different things: (i) the set $x \in G$ as an element or point of G, (ii) the subset $U(x) = \{x\} \subset G$ and (iii) its closure $K(x) = \overline{\{x\}} = \{y, y \subset x\} \subset G$, which is the simplicial complex generated by $\{x\}$. The set $A(x)$ is open only if x is a locally maximal simplex (meaning not contained in any larger simplex) and closed only if x has dimension 0. In general, the set $A(x) = \{x\} \subset G$ is neither open nor closed but x always defines two natural sets, the **open set** $U(x)$ and the **closed set** $K(x)$. The open set $U(x)$ is the smallest open set containing x, the closed set $K(x)$ is the smallest closed set containing x. $U(x)$ is the **star** and $K(x)$ the **core**. The closure $B(x)$ of $U(x)$ is the **unit ball**, and its boundary $S(x) = B(x) \setminus U(x)$ of $U(x)$ is the **unit sphere**. The **dimension** $\dim(x)$ of $x \in G$ is defined as $|x| - 1$, where $|x|$ is the **cardinality** of x. We write $X = (x_1, \dots, x_k) \in G^k$ to address a k**-tuple** of points $x_j \in G$, and define $w(x) = (-1)^{\dim(x)}$ and $w(X) = \prod_{j=1}^k w(x_j)$. We write shorthand $\bigcap X$ for $\bigcap_{j=1}^k x_j$. Similarly, we write $\bigcup X$ for $\bigcup_{j=1}^k x_j$. We usually do not care about the order of the elements in X but allow that the same element appears multiple times. The configuration $X = (x, x, x, \dots, x)$ for example is a k-point configuration in which all points are the same.

2.4. We often look at functions $h : G \to R$, where R is some algebraic object like a **ring**. It should have an additive structure that is commutative (with respect to addition). It could be $\mathbb{Z}$ or a finite Abelian group for example, it can also have more structure like a vector space over a field or an operator algebra. Here we look at rings like $R = \mathbb{Z}$ but the multiplicative structure of the ring does not really enter. It could be a division algebra like the quaternions for example, where the multiplication is not commutative. As we have expressions like $w(x)w_m(U(x))$ in our main result, a multiplication with 1 or -1. If R is an additive group, then $-r$ denotes the additive inverse of r in the group. As the theme is part of a finitist approach to mathematics, we prefer finite objects. Even in the case $R = \mathbb{Z}$, the **range** of a function $f : G \to R$ is always finite so that we still deal with finite objects, despite the fact that there is no a priori cap on the size of R. Having a ring rather than only an additive group has the advantage that on can also look at objects like determinants (or in the non-commutative ring case **Dieudonné determinants**) which can play a role in the vicinity of what we do here. Having a ring is a familiar frame work in other parts of mathematics.

2.5. Rather than have a fixed ring R it is possible to attach a ring $R(x)$ to every open set $R(U)$. The ring $R(x)$ is called the **ring of sections**. To fix the relations, one needs **restriction maps**. Because the open sets $U(x)$ are minimal, the ring $R(x)$ can be called the **stalk** at x. Its elements are the **germs** at x. Since $x \subset y$ implies $U(y) \subset U(x)$, a **sheaf** is determined by giving **restriction maps** $r(x, y) : U(x) \to U(y)$ satisfying **pre-sheaf properties** $r(x, x) = Id$ and $r(y, z) \circ r(x, y) =$

$r(x,z)$ if $x \subset y \subset z$. Having all the germs $R(x)$ and the transition maps fixed, one already has the existence (called **gluing**) and uniqueness (called **locality**) which are necessary to have a **sheaf** and not only a pre-sheaf. An example is $w(x) = (-1)^{\dim(x)}$ with restriction maps $r(x,y) = w(x)w(y)$ for $x \subset y$. There is a unique extension of w to all open sets $w(U) = \prod_{x \in U} w(x)$. In analogy to $\chi(U) = w_1(U) = \sum_{x \in U} w(x)$, we have called this the **Fermi characteristic** $\phi(U)$ of U [7]. It is equal to $\det(L)$ with the connection matrix $L(x,y) = \chi(K(x) \cap K(y))$, the inverse of the Green function matrix $g(x,y) = w(x)w(y)\chi(U(x) \cap U(y))$.

2.6. For a general function $h : G^m \to R$, where we write $h(X) = h(x_1, \cdots, x_m)$, we can define $w_m(G) = \sum_{X \in G^m, \bigcap X \in G} h(X)$ and more generally,

$$w_m(A) = \sum_{X \in A^m, \bigcap X \in A} h(X)$$

for $A \subset G$ for sets A that are not necessarily simplicial complexes. No symmetry like that $h(X) = h(Y)$ if X and Y are permutations is assumed. We mostly take $h(X) = \prod_{j=1}^m (-1)^{\dim(x)}$ because this assures that $w_m(G)$ are **combinatorial invariants**, meaning invariant under Barycentric refinements. For k arbitrary stars $U(x_1), \dots, U(x_m)$, define $U(X) = \bigcap_{j=1}^k U(x_j)$ and $\omega_m(U) = \sum_{X \in G^m, \bigcap X \in G} w(U(X))$. The first characteristic $w_1(G) = \sum_{x \in G} w(x)$ is the **Euler characteristic** of G, the second characteristic $w_2(G) = \sum_{x,y,x \cap y \in G} w(x)w(y)$ is the **Wu characteristic** and the third characteristic is

$$w_3(G) = \sum_{x,y,z,x \cap y \cap z \in G} w(x)w(y)w(z) \ .$$

While for any subsets A, B, we have for $m = 1$ the property $w_1(A \cup B) = w_1(A) + w_1(B) - w_1(A \cap B)$, this valuation formula fails for $m > 1$ for general A, B, even for closed A, B in general. But $w_m(G)$ is a **multi-linear valuation** if extended to $w_m(G_1, \dots, G_m)$ with $G_j \subset G$ arbitrary subsets of G. Now $G_j \to w_m(G_1, \dots, G_m)$ with all other sets $G_i, i \neq j$ fixed, satisfies linearity $\omega(A) + \omega(B) = \omega(A \cup B) + \omega(A \cap B)$.

2.7. The 1-point complex is defined to be contractible. If G is a complex and both $G \setminus U(x)$ and $S(x)$ are contractible, then G is called **contractible**. Two complexes G, H which can morphed into each other by homotopy reductions and extensions are called **homotopic**. This is an equivalence relation on the space of all complexes. Homotopy does not honor dimension and so is not topological. Homotopy preserves $w_1(G)$ but not $w_m(G)$ with $m > 1$. The higher characteristics are topological invariants in the sense that they are preserved under homeomorphisms, an other equivalence relation: H is called a **continuous image** of G if there exists a Barycentric refinement G_m and a continuous map $f : G_m \to H$ (of course using the finite topology defined above), such that $f^{-1}(S(x))$ is homeomorphic to $S(x)$ (inductively defined as the maximal dimension of $S(x)$ is smaller than the maximal dimension of G) and such that for every locally maximal $x \in G$ of dimension d, the complex $f^{-1}(B(x))$ is a d-ball. G and H are homeomorphic, if G is a continuous image of H and H is a continuous image of G. A d-ball is a complex of the form $G - U(x)$, where G is a d-sphere. A d-sphere is a d-manifold G such that $G - U(x)$ is contractible. A **d-manifold** G is a complex such that for all x, the unit sphere $S(x)$ is a $(d-1)$-sphere. The empty complex is the (-1)-sphere.

3. Energy theorem

3.1. We assume $m, k \geq 1$ are integers. Define the m'th order **potential energy** of the **k-point configuration** $X \in G^k$ as $g_m(X) = w(X)w_m(U(X))$. By design, it is zero if $U(X) = \bigcap U(x_j) = \emptyset$ which is the case if at least one of the points is out of reach of the others. Even for non-intersecting x, y it can happen that $U(x) \cap U(y)$ is non-empty like if x, y are zero-dimensional parts of a higher dimensional simplex z, where z is in $U(x) \cap U(y)$. This is a case where x, y can not be separated by open sets. Our goal is to have for any $k \geq 1$ and any $m \geq 1$, the m'th characteristic can be expressed using **k-point Green function entries** $g_m(x_1, \dots, x_k) = \prod_{j=1}^k w(x_j) w_m(\bigcap_{j=1}^k U(x_j))$. We think of this as the **m-th potential energy** of the k-point configuration $X = (x_1, \dots, x_k)$.

3.2. Our main theorem tells that the m'th characteristic is the **total energy** over all possible k'point configurations.

Theorem 1 (Energy)**.** $w_m(G) = \sum_{X \in G^k} g_m(X)$.

Proof. The theorem is simpler to prove if formulated more generally like if

$$h_m(x_1, \dots, x_m) = \prod_{j=1}^{m} w(x_j)$$

is replaced by a general function $h_m(x_1, \dots, x_m)$ of m variables. The reason is that the map $h_m \to g_k$ is linear so that we only need to verify the statement in the simplest possible case where h is 1 only for a single configuration $Z = (z_1, \dots, z_m)$ and 0 else. The left hand side is then 1. On the right hand side, we have to look at all $X = (x_1, \dots, x_k)$ for which $Z \in U(X) = U(x_1) \cap U(x_2) \cap \cdots \cap U(x_m)$. As we will see however below, that condition will assure that $\bigcup Z \subset U(x_j)$ for all $j = 1, \dots, m$. Therefore, $\sum_{X \in G^k} w(X) w_m(U(X)) = \prod_{j=1}^k \sum_{x_j \in G} w(x_j) w_m(U(x_j)) = 1$. $\square$

3.3. A second major point is the **sphere formula** for $S(X) = \delta U(X) = B(x) \setminus U(x)$.

Theorem 2 (Sphere formula)**.** $0 = \sum_{X \in G^k} w(X) w_m(S(X))$.

This means that the total energy of all unit spheres of even configurations is the same than the total energy of all unit spheres of odd configurations.

Proof. The energy theorem also works of $U(X)$ is replaced by $B(X)$, which is the closure of $U(X)$. The two equations

$$0 = \sum_{X \in G^k} w(X) w_m(U(X))$$

$$0 = \sum_{X \in G^k} w(X) w_m(B(X))$$

and the local valuation formula

$$w_m(U(X)) - (-1)^m w_m(S(X)) = w_m(B(X))$$

prove the theorem. □

3.4. Let us add already a remark which however leads to an other story to which we hope to be able to write more about in the future: the theorem also works for the **dual spheres** $\hat{X} = S(x_1) \cap \cdots \cap S(x_k)$ which is different from $S(X) = \delta(U(x_1) \cap \cdots \cap U(x_k))$.

$$0 = \sum_{X \in G^k} w(X) w_m(\hat{X}) \; .$$

3.5. The dual spheres $\hat{X}$ play an important role in other places like in **graph coloring**. In the case when G is a d-manifold and if all points in X are adjacent in the metric in which two points $x, y \in S(x)$ have distance 1, then $\hat{X}$ is always a $(d-k)$-sphere. If $k = 1$ then $\hat{X} = S(x)$, which is in the manifold case a $(d-1)$-sphere. For $k = 2$, and $x, y \in S(x)$ we have $\widehat{(x, y)} = S(x) \cap S(y)$ is a $(d-2)$-sphere. We have looked at this earlier in the context of graphs, where $X = (v_1, \dots, v_k)$ are the vertices of a complete graph K_k and so is associated to a vertex x in the Barycentric refined graph. If G is a d-manifold, then by definition $\hat{X} = \hat{x}$ is a $(d-k)$-sphere. We have seen this as a **duality** because $\bigcap_{v \in x} S(v) = \hat{x}$ and $\bigcap_{v \in \hat{x}} S(v) = x$. This can be seen as a duality between $(k-1)$-spheres (the boundary sphere of a simplex) and $(d-k)$-spheres. Shifting k, there is a duality between k-spheres and $(d+1-k)$-spheres.

3.6. Still dwelling on that, moving cohomology from simplices to spheres could allow then to see **Poincaré duality** more elegantly, avoiding concepts like CW complexes which are necessary already in elementary setups like that the cube is the dual of the octahedron. When the cube is seen as a 2-sphere, it by definition is not a simplicial complex but only a more general CW complex if one wants to understand it as dual to the octahedron. So, if one wants to avoid the continuum (as we do), one usually goes to CW complexes. Poincaré himself already struggled quite a bit with the difficulty that the dual of a simplicial complex is not a simplicial complex. We will see the duality in the context of δ-sets which generalize simplicial complexes too but more naturally than CW complexes. δ sets are more general than simplicial sets, a construct which has a bit more structure than δ sets. But simplicial sets as a subclass of δ sets less accessible. Entire articles have been written just to explain the definition.

3.7. Here is an other remark, maybe more for the mathematical physics minded: motivated by the **Fock picture** in which G^k are considered as k-particle configurations, we could sum up the theorem over k and get for example

$$w_m(G) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{X \in G^k} g_m(X) \; .$$

If we would replace $w(x)$ with $h(x) = (-1)^{\dim(x)}/2$ and set $h(X) = \prod_j h(x_j)$ and still use $U(X) = \bigcap_j U(x_j)$, if $X = (x_1, \dots, x_k)$ and setting $g_m(X) = h(X) w_m(U(X))$, this would allow to write $w_m(G) = \sum_X g_m(X)$ where X runs over all particle configurations ranging from single particles $k = 1$ to pair interactions $k = 2$, triple interactions $k = 3$ etc. The total **energy of space** is then the sum over all interaction energies overall possible particle configurations X. If we think of $g_m(X)$ as the **potential energy** of the particle configuration X, then $w_m(G)$ is the sum over all possible potential energies which particle configurations which can be realized in G.

3.8. Here are special cases, written out in more detail. First of all, lets write down the definitions of **Euler characteristic**

$$w_1(G) = \sum_{x \in G} w(x)$$

and **Wu characteristic**

$$w_2(G) = \sum_{x,y \in G^2, x \cap y \neq \emptyset} w(x) w(y) \; .$$

i) For $m = 1$, we have expressions for the Euler characteristic:
$w_1(G) = \sum_x w(x) w_1(U(x)) = \sum_x g_1(x)$
$w_1(G) = \sum_{x,y} w(x) w(y) w_1(U(x) \cap U(y)) = \sum_{x,y} g_1(x,y)$
$w_1(G) = \sum_{x,y,z} w(x) w(y) w(z) w_1(U(x) \cap U(y) \cap U(z)) = \sum_{x,y,z} g_1(x,y,z)$
$w_1(G) = \sum_{x,y,z,w} g_1(x,y,z,w)$
The first two expressions for $k = 1$ and $k = 2$ have appeared in [7]. The inverse of the matrix $g_1(x,y)$ is $L(x,y) = 1$ if $x = y$ and $L(x,y) = 0$ if $x \neq y$.
ii) Next come expressions for the Wu characteristic:
$w_2(G) = \sum_x w(x) w_2(U(x)) = \sum_x g_2(x)$
$w_2(G) = \sum_{x,y \in G} w(x) w(y) w_2(U(x) \cap U(y)) = \sum_{x,y} g_2(x,y)$
$w_2(G) = \sum_{x,y,z \in G} w(x) w(y) w(z) w_2(U(x) \cap U(y) \cap U(z)) = \sum_{x,y} g_2(x,y,z)$

$w_2(G) = \sum_{x,y,z,w} g_2(x,y,z,w)$
iii) And here are expressions for the third characteristic
$w_3(G) = \sum_x w(x) w_3(U(x)) = \sum_x g_3(x)$
$w_3(G) = \sum_{x,y} w(x)w(y)w_3(U(x) \cap U(y)) = \sum_{x,y} g_3(x,y)$
$w_3(G) = \sum_{x,y,z} w(x)w(y)w(z)w_3(U(x) \cap U(y)) \cap U(z)) = \sum_{x,y,z} g_3(x,y,z)$
$w_3(G) = \sum_{x,y,z,w} g_3(x,y,z,w)$

Remarks: **1)** The theorem works if $h(X) = w(X)$ is replaced by any R-valued interaction function $h : G^k \to R$. We would for example set $w_2(A) = \sum_{x,y,x \cap y \in A} h(x,y)$ and get this equal to $\sum_{x,y \in G^2} g_2(x,y)$ with

$$g_2(x,y) = w(x)w(y)w_2(U(x) \cap U(y))$$

or equal to $\sum_{x \in G} g_1(x)$ with $g_1(x) = w(x)w_2(U(x))$.
2) The energy theorems in the case of $k = 1$ are of Gauss-Bonnet type because we attach a fixed curvature $g_m(x)$ to a point. We can actually interpret this also as a Poincaré-Hopf theorem. There is some duality here as $U(x)$ can be seen dual to $K(x)$.
3) The energy formula reduces the time for the computation of the characteristic substantially. Especially for $k = 1$, where we have only to compute the m'th characteristic of the $n = |G|$ stars $U(x)$.
4) Instead of self-interactions of all G^j, we could take open sets $G_j \subset G$ and get expressions

$$w_m(G_1, \dots, G_k) = \sum_{X, x_j \in G_j} g_m(X) \; .$$

For example, $w_m(A,B,C) = \sum_{x \in A, y \in B, z \in C} g_m(x,y,z)$, where

$$g_m(x,y,z) = w(x)w(y)w(z)w_m(U(x) \cap U(y) \cap U(z)) \; .$$

Think of this as the **total m'th characteristic energy** of the three sets A, B, C.
5) We have seen that in order to prove the theorems, it is helpful to energize more generally $h(x_1, \dots, x_m)$ and set $w_m(G) = \sum_{X \in G^m, X > 0} h(X)$. The Green function procedure which maps functions $h : G^k \to R$ to functions $g : G^m \to R$ with $g_m(X) = w(X)w_m(U(X))$ is linear. It is therefore enough to verify the claim for basis elements, where h(x) is zero except at $h(x_1, \dots, x_k) = 1$.

4. Valuation Lemma

4.1. The following theorem is a main reason, why higher characteristics are topological invariants. The following valuation formula does not work for closed sets in general already if $m \geq 2$.

Lemma 1 (Valuation). *$w_m(U \cup V) = w_m(U) + w_m(V) - w_m(U \cap V)$ for all open U, V and all $m \geq 1$.*

4.2. The key why this works is:

Lemma 2 (Patching)**.** *Given $X \in G^k$ and U, V are open, define $z = \bigcap X$. If $z \in U \cap V$, then both x, y are simultaneously in U, V and so $x \in U \cap V$ and $y \in U \cap U$.*

Patching. If $z = \bigcap X$ is in $U \cap V$, then every simplex w which contains z is both in U and V and so in $U \cap V$ so that w is in U as well as in V and $U \cap V$. Because this works for any w, it works for any of the points x_j. Since all $x_j$are in $U, V, U \cap V$ also the union $z = \bigcup X$ is in all three sets. ☐

Valuation. Counting X in $U \cup V$ is the sum of the counts in U and the counts in V with a double count if X is in the intersection. ☐

Remark:
1) For $m > 1$, this in general does not hold for closed sets nor for a mixture of closed and open sets if $m > 1$. We will see for general m that for an open set $A = U(x)$, a closed disjoint set $B = S(x)$ and closed union $B(x)$ and empty intersection we have $\omega_m(U(x)) - \omega_m(S(x)) = \omega_m(B(x))$ but for odd m that $\omega_m(U(x)) + \omega_m(S(x)) = \omega_m(B(x))$.
a) For $G = \{(1), (2), (3), (12), (23)\}$, the two closed sets $A = \{(1), (2), (12)\}, B = \{(2), (3), (23)\}$ intersect in $C = \{(2)\}$. Now $w_2(A) = w_2(B) = -1$ now $\omega_2(A) + \omega_2(B) = w_2(G) - \omega_2(S)$.
b) If G is the octahedron complex which is the join $C_4 \oplus \{a, b\}$ and $A = G \setminus U(a), B = G \setminus U(b)$ which are both balls intersecting in the circle $S = \{x_1, x_2, x_3, x_4, (x_1x_2), (x_2x_3), (x_3x_4), (x_4x_1)\}$ which is also closed. The pair $(ax_1) \subset A$ intersects with the pair $(bx_1) \subset B$ but these two pairs are both not in S. Still $w_1(A) + w_2(B) = w_2(G) + w_2(S)$.
Example a) shows that looking for more general relations is rather pointless as it depends on m and the dimension.
2) In the special case $m = 1$, where we deal with Euler characteristic, the valuation formula holds for all sets because there is no interaction between points. This is the reason that for $m = 1$, we have a homotopy invariant while for $m > 1$ we have topological invariants.

4.3. The local valuation formulas link two closed and an open set:

Lemma 3 (Local valuation)**.**

$$w_m(B(x)) = w_m(U(x)) - (-1)^m w_m(S(x))$$

This works more generally for the open sets $U(X) = \bigcap U(x_j)$ and $B(X) = \overline{U(X)}$ and $S(X) = B(X) \setminus U(X)$, the boundary. The reason is that $U(X)$ is a disjoint union of disjoint open $U(x)$.

5. The case of manifolds

5.1. For $x \in G$, the **unit ball** $B(x) = \overline{U(x)}$ and the **unit sphere** $S(x) = B(x) \setminus U(x)$ are both closed sets and so simplicial complexes. A complex G is called a d**-manifold** if every $S(x)$ is a $(d-1)$-sphere. A d-manifold is a d**-sphere**, if there exists x such that $G \setminus U(x)$ is contractible. In the case when G is a d-sphere, $G \setminus U(x))$ is declared to be a d**-ball**. A complex G is called **contractible** if there exists $x \in G$ such that both $S(x)$ and $G \setminus U(x)$ are contractible. These definitions are recursive with respect to the dimension d. The foundation is laid by assuming the **void** $0 = \emptyset$ is the (-1)-sphere and that the one-point complex called **unit** $1 = \{1\}$ is the smallest contractible complex.

5.2. A d**-manifold with boundary** is a complex G such that every $S(x)$ is either a $(d-1)$-sphere or a $(d-1)$ ball. The **boundary** $\delta G = \{x \in G, S(x) \text{ ball}\}$ is a closed subset of G. (Just check that if $x \in \delta G$ and $z \subset x$ then $S(z)$ is a ball). It is a $(d-1)$ manifold because for $z \in \delta G$ the set $S(z) \cap \delta G$ is a sphere because $B(z) \cap \delta G$ is a $(d-1)$ unit ball. We have to distinguish here between "closed= no boundary" and "closed=topologically closed". We also want to deal with the case of open manifolds without boundary like an open disk or topologically closed manifolds with boundary which are not closed as manifolds. Open manifolds model also infinite manifolds like in the continuum, where for example the contangent correspondence allows to see the space $(-1,1)$ to be naturally homeomorphic to $\mathbb{R}$. Let us use the notation U for an open manifold without boundary, G for the closure and S the boundary. So, $S = \delta G$ is the topological boundary.

5.3. We can verify $w_m(B(x)) = w_m(U(x)) - (-1)^m w_m(S(x))$ in the case of manifolds without boundary.

Lemma 4 (Local data in manifold case). *$w_m(U(x)) = (-1)^{dm}$, $w_m(S(x)) = 1 + (-1)^{d-1}$ and $w_m(B(x)) = (-1)^{d(m+1)}$.*

Theorem 3 (Manifolds M). *(i) For even-dimensional manifolds with or without boundary $w_m(M) = w_1(M)$. (ii) For odd-dimensional manifolds with boundary $w_m(M) = w_1(M) - (-1)^m w_1(S)$.*

Proof. For even dimensional manifolds with boundary S, we have $w_m(U(x)) = 1$ in the interior and $w_m(U(x)) = 0$ at the boundary, so that the formula $w_m(M) = \sum_x \omega(x) w_m(U)$ immediately shows that $w_m(M) = \sum_x w(x) = w_1(M)$. This also works for odd-dimensional manifolds without boundary. For odd-dimensional manifolds with boundary S, we have $w_m(B(x)) = 1$ in the interior and $w_m(B(x)) = -(-1)^m$ at the boundary. So $w_m(M) = w_1(M) - (-1)^m w_1(S)$ with or without boundary.

In the special case if the manifold has no boundary, we also can use the energy theorem $w_m(G) = \sum_x w(x)w_m(U(x))$, The previous lemma $w_m(U(x)) = w(x)^m = (-1)^{dm}$ shows that if d is even, then $w_m(U(x)) = 1$ and so $w_m(G) = \sum_x w(x) = w_1(G)$. If d is odd, then $w_1(G) = \sum_x w(x) = 0$ and this does not change when multiplying with the constant $w_m(x) = (-1)^{dm}$. □

6. Topological invariance

6.1. A simplex x is called **locally maximal** if it is not contained in any larger simplex. This means that the star $U(x) = \{x\}$ of a locally maximal point therefore has only one point x in G. A map $f : G \to H$ between simplicial complexes is called **continuous** if $f^{-1}(U)$ is open in G if U is open in H. A complex H is a **topological image** of G if there exists a continuous $f : G \to H$ such that $f^{-1}(S(x))$ is homeomorphic to $S(x)$ for all x and $f^{-1}(B(x))$ is a $dim(x)$-ball for every unit ball $B(x)$ with locally maximal x in H. If G and H are both topological images of each other, the two complexes are called **homeomorphic**. Also this definition is recursive. It relies on homeomorphism in dimension $(d-1)$. Quantities which are constant on homeomorphism classes are called **topological invariants**.

6.2. In comparison, a quantity is a **combinatorial invariant** if it stays the same under **Barycentric refinements**. We have in this context say what we mean with the Barycentric refinement of an open set. We can not just take the Whitney complex of the incidence graph as in the case of closed sets because the Whitney complex is a simplicial complex. With that definition, the Barycentric refinement of an open set would be a closed set. What we can do is to see an open set U of a complex G and define U_1 as the complement of the Barycentric refinement $(G \setminus U)_1$ in the Barycentric refinement G_1 of G. Bott defined **combinatorial invariant** as a quantity that does not change when making Barycentric refinements. For all $m \geq 1$ we have:

Theorem 4 (Invariance)**.** *w_m are topological invariants. w_m are combinatorial invariants.*

Proof. First check that w_m is the same on all d-balls B. This follows from the general manifold formula because B is a manifold with boundary. For odd m we have $w_m(B) = w_1(B)$. For even m, we have $w_m(B) = w_1(B) - w_1(S)$, where S is the boundary. The fact that w_m is invariant under Barycentric refinement is treated in the next section about the topological product. Given two general G, H which are homeomorphic. By definition, there is a continuous map from G_n to H. Now use the Ball formula $w_m(H) = \sum_x w(x)w_m(B(x))$. We claim that this is the same than $w_m(G) = \sum_{f^{-1}(B(x))} w(x)w_m(f^{-1}(B(x)))$ □

6.3. The property of being a **Dehn-Sommerville space** (as defined above again) is topological. If G, H are homeomorphic and G is Dehn-Sommerville, then H is Dehn-Sommerville. The proof can be done by induction. Having verified it for dimension up to d, then we have it for dimension $d+1$ because there is a topological correspondence of unit spheres making sure that all unit spheres are Dehn-Sommerville.

6.4. As an other question we should mention how to define homotopy for open sets. One idea two call two open sets U, V to be homotopic in G, if $U^c = G \setminus U$ and V^c are homotopic closed sets. But we would like to have that if U, V are homotopic, then the Betti vectors $b(U)$ and $b(V)$ should be the same. An open 3-ball for example has Betti vector $(0,0,0,1)$ and a closed 3-ball has Betti vector $(1,0,0,0)$. They should already not be considered homotopic because the Euler characteristic does not match. All this will hopefully be explored more in a future paper on the matter.

7. Topological product

7.1. The **topological product** $G \cdot H$ of two simplicial complexes G, H is the Whitney complex of the graph in which the Cartesian product $G \times H$ are the vertices and where (a,b) and (c,d) are connected either $a \subset b, c \subset d$ or if $b \subset a, d \subset c$. It is a set of subsets of $V = G \times H$ and again a simplicial complex.

7.2. Unlike other products like the **Shannon product**, the topological product is of a topological nature. It preserves manifolds and as we will see is compatible with all higher characteristics.

Lemma 5. *If G and H are manifolds, then $G \cdot H$ is a manifold.*

Proof. Lets outline again the argument: we have to look at the unit sphere $S((x,y))$ of a point (x,y) and show that it is a $d-1$-sphere. $S((x,y))$ is the union of two cylinders $S(x) \cdot B(y)$ and $B(x) \cdot S(y)$ which are glued together along $S(x) \cdot S(y)$. In the case of a product of a 1-manifold G and a 1-manifold H for example, the unit spheres $S(x,y)$ is a "square" which is the union of $S(x) \times B(y)$ (left right parts) $B(x) \times S(y)$ (top bottom part) intersecting in 4 points $S(x) \times S(y)$. To see that this is a sphere, collapse $B(y)$ to a point to see that $S(x,y)$ is homotopic to the join of $S(y)$ and $S(x)$ which is a sphere. One then checks case by case that each point in $S(x,y)$ has a $d-2$ sphere as unit sphere. For interior points in $B(y)$ or $B(x)$ this follows by induction. For points in $S(x) \times S(y)$ the unit sphere is a copy of two $(d-2)$ balls glued along a $(d-3)$ sphere and so a $(d-2)$ sphere. ∎

7.3. Similarly, one has by analyzing additionally the unit spheres of boundary points and knowing that the join of a ball with a sphere is a ball:

Lemma 6. *If G, H are manifolds with boundary then $G \cdot H$ is a manifold with boundary.*

For example, the product of two closed intervals is a square, the product of an interval with a circle is a closed cylinder.

7.4. The topological product is not associative. The complex $G \cdot 1$ is the Barycentric refinement G_1 of G so that $G \cdot (1 \cdot 1) = G_1$ but $(G \cdot 1) \cdot 1 = G_2$ is the second Barycentric refinement.

7.5. There is an algebraic **Stanley-Reisner** description (which is seen implemented in the code section). If the elements in $V = \bigcup_{x \in G} x$ are labeled with variables $t_1, \dots, t_q$, then every $x \in G$ can be written as a monomial $t(x) = t_{j_0} t_{j_2} \dots t_{j_{\dim(x)}}$ and $x \subset y$ is algebraically encoded by $t(x)|t(y)$. The complex G is a polynomial $\sum_{x \in G} t(x)$. In the same way, the complex H is given by the polynomial $\sum_{y \in H} t(y)$. The product $G \cdot H$ is now represented by the product of these two polynomials. The vertex set of $G \cdot H$ has $|G||H|$ elements.

7.6. For example, if $G = \{\{1\}, \{2\}, \{1,2\}\}$ and $H = \{\{3\}, \{4\}, \{3,4\}\}$. The product complex $G \cdot H$ is a complex for which the vertex set has 9 elements.

Theorem 5 (Product)**.** *$w_m(G \cdot H) = w_m(G) w_m(H)$.*

Proof. We have $w_m(G) = \sum_{x_1, \dots, x_m, \bigcap X > 0} h(x_1) \cdots h(x_m) = \sum_X h(X)$. Similarly, $w_m(H) = \sum_{y_1, \dots, y_m, \bigcap Y > 0} h(y_1) \cdots h(y_m) = \sum_Y h(Y)$. Now,

$$\begin{aligned} w_m(G \cdot H) &= \sum_{X,Y \bigcap X > 0, \bigcap Y > 0} h(x_1) \cdots h(x_m) h(y_1) \cdots h(y_m) \\ &= \sum_{X,Y} h(X) h(Y) \\ &= \sum_X h(X) \sum_Y h(Y) = w_m(X) w_m(Y) \; . \end{aligned}$$

□

7.7. Remarks:

1) A special case is if $H = 1 = \{\{1\}\}$, where $G_1 = G \cdot 1$ is the Barycentric refinement. And $w_m(G_1) = w_m(G)$ is a special important case.

2) As in the case $m = 1$, if G is the Whitney complex of a finite simple graph, we would like to know about the behavior of the curvature when taking products. In the case of the Shannon product and $m = 1$, we have seen that the curvatures multiply.

8. Code

8.1. Here is some code. As usual, one can copy-paste it from the ArXiv's LaTex source. The procedures should be pretty self-explanatory, given that the Wolfram language allows to write mathematical procedures in a form resembling pseudo code. As for the topological product, we repeat the algebraic frame work as we have used it [3], (before even knowing about Stanley-Reisner rings).

```
Closure[A_]:=If[A=={},{},Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]];
Boundary[A_]:=Complement[Closure[A],A];
Whitney[s_]:=If[Length[EdgeList[s]]==0,Map[{#}&,VertexList[s]],
   Map[Sort,Sort[Closure[FindClique[s,Infinity,All]]]]]];
UU[G_,x_]:=Module[{U={}},
     Do[If[SubsetQ[G[[k]],x],U=Append[U,G[[k]]]],{k,Length[G]}];U];
VV[G_,x_]:=Module[{U={}},
     Do[If[SubsetQ[x,G[[k]]],U=Append[U,G[[k]]]],{k,Length[G]}];U];
Basis[G_]:=Table[UU[G,G[[k]]],{k,Length[G]}];  Stars=Basis;
Cores[G_]:=Table[VV[G,G[[k]]],{k,Length[G]}];
SubBasis[G_]:=Module[{V=Union[Flatten[G]]},Table[UU[G,{V[[k]]}],{k,Length[V]}]];
UnitSpheres[G_]:=Module[{B=Basis[G]},
    Table[Complement[Closure[B[[k]]],B[[k]]],{k,Length[B]}]];
UnitBalls[G_]:=Map[Closure,Basis[G]];
Cl[U_,A_]:=Module[{V=U},Do[V=Union[Append[V,
    Union[V[[k]],A[[l]]]]],{k,Length[V]},{l,Length[A]}];V];
Topology[G_]:=Module[{V=B=Basis[G]},
    Do[V=Cl[V,B],{Length[Union[Flatten[G]]]}];Append[V,{}]];
GraphBasis[s_]:=Basis[Whitney[s]];
Nullity[Q_]:=Length[NullSpace[Q]];    sig[x_]:=Signature[x];
sig[x_,y_]:=If[SubsetQ[x,y]&&(Length[x]==Length[y]+1),
   sig[Prepend[y,Complement[x,y][[1]]]]*sig[x],0];
Fvector[G_]:=If[Length[G]==0,{},Delete[BinCounts[Map[Length,G]],1]];
Ffunction[G_,t_]:=Module[{f=Fvector[G],n},Clear[t];  n=Length[f];
   If[Length[VertexList[s]]==0,1,1+Sum[f[[k]]*t^k,{k,n}]]];
BarycentricGraph[s_]:=ToGraph[Whitney[s]];
BarycentricComplex[G_]:=Whitney[ToGraph[G]];
dim[x_]:=Length[x]-1;  w[x_]:=(-1)^dim[x];
Wu1[A_]:=Total[Map[w,A]];  Chi=Wu1;                          EulerChi=Wu1;
Wu2[A_]:=Module[{a=Length[A]},Sum[x=A[[k]];  Sum[y=A[[l]];
    If[MemberQ[A,Intersection[x,y]],1,0]*w[x]*w[y],{l,a}],{k,a}]];Wu=Wu2;
Wu3[A_]:=Module[{a=Length[A]},Sum[x=A[[k]];Sum[y=A[[l]];Sum[z=A[[o]];
    If[MemberQ[A,Intersection[x,y,z]],1,0]*
    w[x]*w[y]*w[z],{o,a}],{l,a}],{k,a}]];
RingFromComplex[G_,a_]:=Module[{V=Union[Flatten[G]],n,T,U},
      n=Length[V]; Quiet[T=Table[V[[k]]->a[[k]],{k,n}]];
      Quiet[U=G /.T];  Sum[Product[U[[k,l]],
      {l,Length[U[[k]]]}],{k,Length[U]}]];
ComplexFromRing[f_]:=Module[{s,ff},s={};  ff=Expand[f];
      Do[Do[If[Denominator[ff[[k]]/ff[[l]]]==1 && k!=l,
         s=Append[s,k->l]],  {k,Length[ff]}],{l,1,Length[ff]}];
      Whitney[UndirectedGraph[Graph[Range[Length[ff]],s]]]];
TopologicalProduct[G_,H_]:=Module[{},
      f=RingFromComplex[G,"a"];
      g=RingFromComplex[H,"b"];  F=Expand[f*g];  ComplexFromRing[F]];
```

8.2. The following lines illustrate some of the identities for random complexes.

```
G=Whitney[RandomGraph[{9,15}]]; G = Sort[G]; G = Map[Sort, G]; n = Length[G];
U1=Basis[G]; U2=Table[Intersection[U1[[k]],U1[[l]]],{k,n},{l,n}];
U3=Table[Intersection[U1[[k]],U2[[l,m]]],{k,n},{l,n},{m,n}];
V1=Cores[G]; V2=Table[Intersection[V1[[k]],V1[[l]]],{k,n},{l,n}];
V3=Table[Intersection[V1[[k]],V2[[l,m]]],{k,n},{l,n},{m,n}];
S1=Map[Boundary,U1]; S2=Table[Boundary[U2[[k,l]]],{k,n},{l,n}];
S3=Table[Boundary[U3[[k,l,m]]],{k,n},{l,n},{m,n}];
B1=Map[Closure,U1];  B2=Table[Closure[U2[[k,l]]],{k,n},{l,n}];
B3=Table[Closure[U3[[k,l,m]]],{k,n},{l,n},{m,n}];

Print[" Check energy formulas   "];
Wu1[G]==Total[Table[w[G[[k]]]*Wu1[U1[[k]]],{k,n}]];
Wu2[G]==Total[Table[w[G[[k]]]*Wu2[U1[[k]]],{k,n}]];
Wu1[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu1[U2[[k,l]]],{k,n},{l,n}]]]
Wu2[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu2[U2[[k,l]]],{k,n},{l,n}]]]
Wu1[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*w[G[[m]]]*Wu1[U3[[k,l,m]]],
   {k,n},{l,n},{m,n}]]]
Wu2[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*w[G[[m]]]*Wu2[U3[[k,l,m]]],
   {k,n},{l,n},{m,n}]]]

Print[" Check energy ball formulas  "];
Wu1[G]==Total[Table[w[G[[k]]]*Wu1[B1[[k]]],{k,n}]]
Wu2[G]==Total[Table[w[G[[k]]]*Wu2[B1[[k]]],{k,n}]]
Wu1[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu1[B2[[k,l]]],{k,n},{l,n}]]]
Wu2[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu2[B2[[k,l]]],{k,n},{l,n}]]]
Wu1[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*w[G[[m]]]*Wu1[B3[[k,l,m]]],
   {k,n},{l,n},{m,n}]]]
Wu2[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*w[G[[m]]]*Wu2[B3[[k,l,m]]],
   {k,n},{l,n},{m,n}]]]

Print[" Check sphere formulas          "];
0==Total[Table[w[G[[k]]]*Wu1[S1[[k]]],  {k,  n}]]
0==Total[Table[w[G[[k]]]*Wu2[S1[[k]]],  {k,  n}]]
0==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu1[S2[[k,l]]],{k,n},{l,n}]]]
0==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu2[S2[[k,l]]],{k,n},{l,n}]]]
0==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*w[G[[m]]]*Wu1[S3[[k,l,m]]],
    {k,n},{l,n},{m,n}]]]
0==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*w[G[[m]]]*Wu2[S3[[k,l,m]]],
    {k,n},{l,n},{m,n}]]]

Print[" Check local valuation formula  "];
Map[Wu1,U1]==Map[Wu1,B1]-Map[Wu1,S1]
Union[Flatten[Table[Wu1[U2[[k,l]]]+Wu1[S2[[k,l]]]-Wu1[B2[[k,l]]],{k,n},{l,n}]]]
Map[Wu2,U1]==Map[Wu2,B1]+Map[Wu2,S1]
Union[Flatten[Table[Wu2[U2[[k,l]]]-Wu2[S2[[k,l]]]-Wu2[B2[[k,l]]],{k,n},{l,n}]]]

Print[" Check product  "];
G=Whitney[RandomGraph[{6,10}]]; H=Whitney[StarGraph[5]];
GH=TopologicalProduct[G,H];
{Wu1[G],Wu1[H],Wu1[GH]}
{Wu2[G],Wu2[H],Wu2[GH]}

(* A bit more time consuming to compute *)
Wu3[G]==Total[Table[w[G[[k]]]*Wu3[B1[[k]]],{k,n}]]
Wu3[G]==Total[Flatten[Table[w[G[[k]]]*w[G[[l]]]*Wu3[B2[[k,l]]],{k,n},{l,n}]]]
```

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