CARTAN'S MAGIC FORMULA FOR SIMPLICIAL COMPLEXES

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Abstract. Élie Cartan's magic formula $L_X = i_X d + d i_X = (d + i_X)^2 = D_X^2$ relates the exterior derivative d, an interior derivative i_X and its Lie derivative L_X. We use this formula to define a finite dimensional vector space $\mathcal{X}$ of vector fields X on a finite abstract simplicial complex G. The space $\mathcal{X}$ has a Lie algebra structure satisfying $L_{[X,Y]} = L_X L_Y - L_Y L_X$ as in the continuum. Any such vector field X defines a coordinate change on the finite dimensional vector space $l^2(G)$ which play the role of translations along the vector field. If $i_X^2 = 0$, the relation $L_X = D_X^2$ with $D_X = i_X + d$ mirrors the Hodge factorization $L = D^2$, where $D = d + d^*$ we can see $f_t = -L_X f$ defining the flow of X as the analogue of the heat equation $f_t = -Lf$ and view the Newton type equations $f_{tt} = -L_X f$ as the analogue of the wave equation $f_{tt} = -Lf$. Similarly as the wave equation is solved by $\psi(t) = e^{iDt}\psi(0)$ with $\psi(t) = f(t) - iD^{-1} f_t(t)$, also any second order differential equation $f_{tt} = -L_X f$ is solved by $\psi(t) = e^{iD_X t}\psi(0)$ in $l^2(G, \mathcal{C})$. If X is supported on odd forms, the factorization property $L_X = D_X^2$ extends to the Lie algebra and $i_{[X,Y]}$ remains an inner derivative. If the kernel of $L_X : \Lambda^p \to \Lambda^p$ has dimension $b_p(X)$, then the general Euler-Poincaré formula $\chi(G) = \sum_k (-1)^k b_k(X)$ holds for every parameter field X. Extreme cases are $i_X = d^*$, where b_k are the usual Betti numbers and $X = 0$, where $b_k = f_k(G)$ are the components of the f-vector of the simplicial complex G. We also note that the McKean-Singer super-symmetry extends from L to Lie derivatives. (It also holds for L_X on Riemannian manifolds but appears to have been unnoticed there so far): the non-zero spectrum of L_X on even forms is the same than the non-zero spectrum of L_X on odd forms. We also can make a deformation $D'_X = [B_X, D_X]$ of $D_X = d + i_X + b_X$, $B_X = d_X - d_X^* + i b_X$ which produces a in general non-isospectral deformation of the exterior derivative d governed by the vector field X featuring inflationary initial size decay for d typical for such systems, leading to an expansion of space.

1. Introduction

1.1. When formulating physics in a discrete geometric frame work, one is challenged by the absence of a continuous diffeomorphism group. What is the analogue of a vector field on a finite abstract simplicial complex G? Discrete theory approaches

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like [7, 16, 19] have put forward some notions. What about a combinatorial discrete frame work [13]?

1.2. In order to get a continuum motion, one always can just look at paths in the unitary group on the Hilbert space $l^2(G)$ like for example isospectral Lax deformations $D' = [B, D]$ of the exterior derivative d defining $D = d + d^*$ [11, 10]. But we would like to have a definition of vector field which is formally identical to the classical case and which agrees with the classical case if the differential complex comes from a Riemannian manifold.

1.3. As in the continuum, like on a Riemannian manifold [1, 8], we would like to see vector fields related to 1-forms, possibly moderated by a Riemannian metric, The presence of an exterior derivative d then produces potential fields $F = dV$ which then can be used to generate dynamics like $x'' = -\nabla V(x)$. In that case, the Riemannian structure allows to transfer the 1-form dV into a vector field ∇V. In this note, we look at a vector field notion in the discrete which works on any finite abstract simplicial complex G, a finite set of non-empty sets x invariant under the process of taking non-empty finite subsets.

1.4. We see that there is a finite dimensional Lie algebra of vector fields X for which the Lie derivative L_X defines a coordinate change which commutes with exterior differentiation d. The coordinate changes allow to have a basic general covariance principle. The motion can be extended to a Hamiltonian frame work so that we could look also at analogues the Kepler problem, where a mass point moves in a central field given by a potential associated to the geometry. Given a vector field X, then the solution $\psi(t) = e^{iD_X t}\psi(0)$ of the second order and so a Newton type equation $f_{tt} = -L_X f$ with $L_X = D_X^2$ and $\psi(0) = f(0) - iD_X^{-1} f_t(0)$ resembles then the solution $e^{iDt}\psi(0)$ of the wave equation $f_{tt} = -Lf$ with $L = D^2$. It is the Cartan magic formula [5], which is now part of any differential topology book like [1]) which produces the analogy between the wave equation with Hodge Laplacian $L = dd^* + d^*d = (d+d^*)^2 = D^2$ and the Lie derivative $L_X = di_X + i_X d = (d+i_X)^2 = D_X^2$. The **Cartan formula** can therefore be seen as a key to port notions of ordinary differential equations on manifolds to discrete spaces like simplicial complexes.

1.5. The Cartan formula has been used in the past in discrete frame works (it appears in [17]). Usually, in the Newton case as well as in the wave case, one does not write the dynamics using complex coordinates. Here in the discrete, it is convenient as the equations become just Schrödinger equations giving paths in the unitary group. The wave equation case with Laplacian L is the most symmetric case, where the propagation of information happens in all directions or (if the momentum

f_t is chosen accordingly) allows to force the propagation into a specific direction. The analogue of ordinary differential equations are obtained when replacing L with L_X, in which X is a deterministic field, which assigns to a simplex, just one smaller dimensional simplex.

1.6. The note draws also from insight gained in [18, 12] and belongs to the theme of looking for finite dimensional analogues of partial differential equations in the continuum. The set-up for [18] builds on work like [3] and is an **advection model** for a **directed graph** G is $u' = -Lu$, where $L = \text{div}(V(\text{grad}_0(u)))$ is the directed Laplacian on the graph. As in the usual (scalar) Laplacian $L = d^*d$ for undirected graphs which leads to the **heat equation** $u' = -Lu$, the **advection Laplacian** uses difference operators: the modification of the gradient grad_0 which is the maximum of grad and 0. The divergence $\text{div} = d^*$ as well as $\text{grad} = d$ are defined by the usual exterior derivative d on the graph. The **consensus model** is the situation, where the graph G is replaced by its **reverse graph** G^T, where all directions are reversed. One can therefore focus on advection. A central part of [18] relates this to Markov chains. If $L = D - A$, then $M = AD^{-1}$ is a **left stochastic matrix**, a Markov operator, which maps probability vectors to probability vectors. The kernel of L is related to the fixed points of M. Assume $LD^{-1}u = 0$, then $(D - A)D^{-1}u = 0$ and $u = AD^{-1}u = Mu$. Perron-Frobenius allows to study the structure of the equilibria which are given by the kernel of L. This concludes the diversion into advection.

1.7. What is the connection? While the topic is related, we look here at differential equations on all differential forms and not only on 0-forms. Also, we don't yet really study the dynamics much and just establish the linear algebra set-up showing that there is an elementary way to define a Lie algebra of vector fields in a discrete set-up. The affinity to the continuum is that the formalism is not only similar but identical to the continuum. Whatever is done works both for Riemannian manifolds or finite abstract simplicial complexes or more generally for a differential complex. There are many open questions as seen at the end of this note: integer-valued deterministic X often produce integer eigenvalues of L_X for example. We would like to know when this is case appears.

1.8. An other angle emerged while teaching the multi-variable Taylor theorem in [14]. Already the single variable Taylor theorem can be seen as the solution $f(t,x) = e^{Dt}f(0)$ of the **transport equation** $f_t = Df$ with $D = d/dx$, a partial differential equation. Because also $f(t,x) = f(x+t)$ solves this partial differential equation, we have $f(x+t) = e^{Dt}f(0)$ which becomes so the Taylor theorem, provided the initial function $f(0)$ is real analytic. For a multivariate function we can replace $L = D^2$ with a Lie derivative $L_X = D_X^2$ and in the case of a constant field $X = v$, a Taylor expansion $f(x+tv) = f(x) + df(x)tv + d^2f(x)(tv)^2/2 + ...$ (The multivariable Taylor theorem can be formulated conveniently using directional (Fréchet) derivatives which is how textbooks like [6, 2] treats the subject in higher dimensions, avoiding tensor calculus.) As both L and L_X can be written as a square $L = D^2, L_X = D_X^2$, the analogy between the wave and Newton equation has appeared. The frame work shows that a"diffeomorphism Lie group" exists in any geometric structure with an exterior derivative; Taylor links the vector field Lie algebra with translation.

2. From Cartan to d'Alembert

2.1. Given a smooth compact manifold M or a simplicial complex G with **exterior derivative** $d : \Lambda^p \to \Lambda^{p+1}$, then every vector field X defines an **interior derivative** $i_X : \Lambda^p \to \Lambda^{p-1}$. The **Cartan magic formula** writes the **Lie derivative** L_X as $L_X = di_X + i_X d$. From the identities $d^2 = 0$ and $i_X^2 = 0$ follows that L_X commutes with d. We know already from the continuum, that without the di_X part, the naive directional derivative $i_X d$ alone would not work, as it would be coordinate dependent. A L_X commutes with d it leads to a chain homotopy between the complexes before and after the coordinate transformation. Like the **Hodge Laplacian** $L = dd^* + d^*d$, **we can write** L_X **as a square**: define $D_X = d + i_X$ and $D = d + d^*$. Then, $L_X = D_X^2$ and $L = D^2$. The **directional Dirac operator** D_X has also an adjoint D_X^* but it is different from D_X in general. Despite the notation used, the directional Dirac operator is not the directional derivative used in calculus. The operators d, i_X, D_X and L_X work on the linear space of all differential forms.

2.2. If $L = D^2$ is the Hodge Laplacian with Dirac operator $D = d + d^*$, then the **wave equation** is $f_{tt} = -Lf$. The **directional wave equation** is the formal analogue $f_{tt} = -L_X f$. Written in the d'Alembert form, it is $(\partial_{tt} + L_X)f = 0$. As L and L_X are both squares of simpler operators D and D_X, we can factor $(\partial_t + iD_X)(\partial_t - iD_t)f = 0$ or $(\partial_t + iD)(\partial_t - iD)f = 0$. The solutions $e^{\pm iD_X}$ which with Euler's formula $e^{iDt} = \cos(Dt) + i\sin(Dt)$ leads to the explicit solution $f(t) = \cos(Dt)f(0) + \sin(Dt)D^{-1}f_t(0)$, where D^{-1} is the pseudo-inverse is defined as $D^{-1}f_t^{\perp}(0)$ if $f_t^{\perp}(0)$ is in the orthogonal complement of the kernel of D.

2.3. So far, the solutions of the equations were real-valued functions in the Hilbert space $H = l^2(G, \mathbb{R})$. If G is a finite abstract simplicial complex, the Hilbert space is finite dimensional, and the frame work is part of linear algebra. It is convenient to build the **complex valued wave** $\psi(0) = f(0) - iD^{-1}f_t(0)$, (where again D^{-1} is the pseudo inverse) and get $\psi(t) = e^{iDt}\psi(0)$. The dynamics is now the solution to the **Schrödinger equation** $i\psi_t = -D\psi$, where $\psi(0)$ encodes the initial position $f(0)$ in its real part and the initial velocity $f_t(0)$ in its imaginary part. This works in the same way for D_X. In both cases, the wave equation for the real wave is equivalent to the Schrödinger equation for a complex wave. In the case of 0-forms, we have $L = d^*d$ and $L_X = i_X d$ is the **directional derivative** in the direction X. We summarize:

Given a geometric space with an exterior derivative d, the second order real wave equation $f_{tt} = -Lf$ is equivalent to the first order complex Schrödinger equation $\psi' = iD\psi$, leading to a d'Alembert solution $\psi(t) = e^{iDt}\psi(0)$ which can then be computed using a Taylor expansion and for which the real part of ψ gives the wave solution $f(t)$.

3. The Lie algebra

3.1. Let us assume now that we are in a finite dimensional geometric space with an exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$ leading to a differential complex on a graded vector space $\Lambda = \oplus_{p=0}^{\dim(G)} \Lambda^p$. A **vector field** is defined by a linear operator i_X on Λ which maps Λ^p to Λ^{p-1} and has the property that $i_X^2 = 0$. We can define a Lie algebra multiplication $Z = [X, Y]$ by first forming $L_X = i_X d + d i_X$ which is a map from Λ^p to Λ^p and then defining Z through the inner derivative

$$i_Z = i_{[X,Y]} = [L_X, i_Y] = L_X i_Y - i_Y L_X \ .$$

3.2. The field Z can be read of from i_Z. We also have the **Lie algebra relation**

$$\begin{aligned} L_Z &= i_Z d + d i_Z = L_X i_Y d - i_Y L_X d + d L_X i_Y - d i_Y L_X \\ &= L_X(L_Y - d i_Y) - i_Y L_X d + L_X d i_Y - (L_Y - i_Y d) L_X \\ &= L_X L_Y - L_Y L_X \ . \end{aligned}$$

3.3. Now, if $i_X i_Y = i_Y i_X = 0$, then $L_X i_Y - i_Y L_X = i_X L_Y - L_Y i_X$ because inter changing X and Y produces a change of sign of L_Z. As $L_Z = i_Z d + d i_Z$, also i_Z changes sign meaning Z changes sign. So, $i_Z = L_X i_Y - i_Y L_X = -(L_Y i_X - i_X L_Y)$.

3.4. These elementary matrix identities prove the following proposition which applies to any so derived Lie algebra of fields X with $i_X^2 = 0$.

Proposition 1. *Every vector field X defines an operator $D_X = d + i_X$ which has as a square a Lie derivative $L_X = D_X^2$. The set of L_X define a Lie algebra with $L_{[X,Y]} = L_X L_Y - L_Y L_X$. If X is in supported on odd forms, then $i_X^2 = 0$ and $L_X = D_X^2$ holds in the entire Lie algebra.*

3.5. Even so L_X is not self-adjoint in general, it plays the role of a Laplacian. In the case when L_X is not diagonalizable, we can not form the pseudo inverse when writing the dynamics in the complex but we can assume that the initial velocity is in the image of D_X. With this definition, also the adjoint operator d^* defines a vector field. We can still see $d^* = i_X$ for some field X. It belongs to the class of vector fields X for which the eigenvalues of D_X are real.

3.6. We could look at a subclass of "deterministic fields", which have the property that for a p-form f, the $(p-1)$-form $i_X f$ is supported on a single sub-simplex y of x, if f is supported on a single simplex x. This would correspond to classical vector fields close to the discrete Morse theory frame-work [7]. If X is supported on one-dimensional simplices, this is close to the discrete Morse theory set-up. Unlike in the continuum, these "deterministic fields" are not invariant under addition, nor under the Lie algebra multiplication $[X, Y]$. When taking the commutator of L_X and L_Y for such fields, then the corresponding inner derivative $i_{[X,Y]}$ connects simplices which are not directly connected.

3.7. If the simplicial complex is one-dimensional, or if X is restrained to p-forms with odd p (or even if we like) then $i_Z = i_{[X,Y]}$ satisfies again $i_Z^2 = 0$ and the factorization $L_Z = (i_Z + d)^2 = D_Z^2$ holds in the entire Lie algebra. In general, the new interior derivative is only nilpotent, because $i_Z^{1+\dim(G)} = 0$. The Mathematica procedures below allow to support X onto any subset of forms but we mostly use the case when X is supported on odd-dimensional forms.

3.8. Let us briefly look at the 1-dimensional (single-variable) classical case $M = \mathbb{R}$, where $L_X = i_X d$ is what we understand to be the usual derivative d/dx. Technically, the exterior derivative d produces from a 0-form $f \in \Lambda^0$ a 1-form $df dx \in \Lambda^1$ which is in a different vector space than f. But for the constant vector field $X = 1$, the combination $i_X d$ produces again an element in Λ^0. Since $i_X^* = 0$ on 0-forms, we have $i_X d = i_X d + d i_X = L_X$. Now $e^{L_X t} f = \sum_{n=0}^{\infty} (d/dx)^n f(x) X^n t^n / n!$ is by the **Taylor formula** equal to $f(x + tX)$, illustrating that the derivative d/dx is the generator of the translation. The flow $\phi_X f = f(x + tX)$ is a solution of a **transport equation** and not the wave equation. To get an analogue of the later, we need a second derivative in time and so a symplectic or complex structure. It is first a bit puzzling to see the Lie derivative L_X as a second order operator. But the Cartan formula shows that also in one dimensions, $L_X = i_X d = (d + i_X)^2 = D_X^2$ is second order. In calculus, we usually think of the derivative d as a map on a space of **scalar functions** and not as a map from 0-forms to 1-forms. The inner derivative i_X which brings us back to 0-forms is silently assumed in calculus. This identification of 0-forms and 1-forms can not be done in the discrete because the dimension $v_1 = |E|$ of 1-forms is different from the dimension $v_0 = |V|$ of 0-forms on a graph $G = (V, E)$. Still, the general frame work applies and the wave equation $f_{tt} = -L_X f$ can be written as $(\partial_t - iD_X)(\partial_t + iD_X) = 0$ with $D_X = d + i_X$.

3.9. We would like to point out that the McKean-Singer symmetry [15] which holds for simplicial complexes and the operator $L = D^2$ [9] remains valid also for $L_X = D_X^2$:

Proposition 2 (McKean-Singer symmetry). *The non-zero spectrum of L_X on even differential forms Λ^{even} is the same than the non-zero spectrum of L_X on odd differential forms Λ^{odd}.*

Proof. The proof is the same than in the continuum [4] or used in [9]: the operator D_X which exchanges Λ^{even} and Λ^{odd} gives a translation between the eigenvectors belonging to non-zero eigenvalues. The discrepancy between the kernels on odd and even forms is by definition the Euler characteristic. □

3.10. Also the nonlinear **Lax type deformation** [11, 10] of the Dirac operator generalizes from D to D_X. These Lax equations are

$$D_X' = [B_X, D_X]$$

where $D_X = d + i_X + b_X, B_X = d_X - d_X^* + ib_X$. Unlike for $i_X = d^*$, where $L = L_X$ is the Hodge Laplacian, the deformation is now not isospectral in general and therefor not expected to be integrable. The expansion rate in different part of space or differential forms happens differently. Still, these systems remain interesting non-linear differential equations and the corresponding $d(t)$ still satisfies $d(t)^2 = 0$ producing an exterior derivative after deformation. As in the case of the wave equation, the deformed $d(t)$ keeps the same cohomology.

4. Physics

4.1. Any mathematical theory with some quantum gravitational ambitions should be able to be powerful enough to solve the Kepler problem effectively in any scale: in the large, it should lead to the classical Kepler problem, in the very large to relativistic motion in a Schwarzschild metric and in the very small to the quantum dynamics in the Hydrogen atom. No current theory passes this Kepler test: no theory can yet describe a point in the influence of a central field classically, relativistically and quantum mechanically, not just in principle or a perturbative patch work but in an elegant manner, leading to quantitative results which match experiments in all three scales. It should be able to describe the motion of satellites or planets, also relativistically, predict the emission patters of gravitational waves emitted by a binary system or the structure of the Mendeleev table in the small.

4.2. The general covariance principle in physics states that physical laws are independent of the coordinate system. This means that the laws should not only be invariant under a finite dimensional symmetry group like Euclidean or Lorentz symmetry but they should be invariant under the diffeomorphism group of the manifold. Additionally, in case of fibre bundles, additional gauge symmetries might apply, but this is also part of the general covariance principle. An example are the Maxwell equations $dF = 0, d^*F = j$ leading in the Coulomb gauge $d^*A = 0$ to the Poisson equation $LA = j$. If we move into a new coordinate system, then the transported equations look the same. An other example are the Einstein field equations $G = eT$, relating the geometric Einstein tensor with the energy tensor T using a proportionality factor e, the Einstein constant. The covariance there is there the statement that G and T are tensors. How can one port the general covariance principle to the discrete? A naive request would be to look at laws only which are invariant after applying a deformation through a vector field. Since L_X and d commute, any law which only involves the exterior derivative does this. Examples are the wave, the heat or the Schrödinger equation.

5. Examples

5.1. The simplest case with a non-trivial Lie algebra is when $G = \{\{1\},\{2\},\{1,2\}\}$ is the Whitney complex of the complete graph K_2. In that case,

$$d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, d^* = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The general inner derivative (vector field) has the form

$$i_X = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}.$$

The general operator $D_X = d + i_X$ and $L_X = D_X^2 = di_X + i_X d$ then is

$$D_X = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -1 & 1 & 0 \end{bmatrix}, L_X = \begin{bmatrix} -a & a & 0 \\ -b & b & 0 \\ 0 & 0 & b-a \end{bmatrix}.$$

The eigenvalues of L_X are $\{0, b-a, b-a\}$. Given an other vector field

$$i_Y = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix},$$

one can form $i_Z = L_X i_Y - i_Y L_X = i_X L_Y - L_Y i_X$ which is

$$i_Z = \begin{bmatrix} 0 & 0 & av-bu \\ 0 & 0 & av-bu \\ 0 & 0 & 0 \end{bmatrix},$$

leading to

$$D_Z = \begin{bmatrix} 0 & 0 & av-bu \\ 0 & 0 & av-bu \\ -1 & 1 & 0 \end{bmatrix}, L_Z = \begin{bmatrix} bu-av & av-bu & 0 \\ bu-av & av-bu & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This satisfies $L_Z = L_X L_Y - L_Y L_X$. The eigenvalues of L_Z are all zero. Indeed $L_Z^2 = 0$.

5.2. Here is the general case if $G = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\}\}$ is the Whitney complex of a linear graph of length 2.

$$d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}, d^* = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With a vector fields

$$i_X = \begin{bmatrix} 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D_X = \begin{bmatrix} 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & a_6 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

This leads to

$$L_X = \begin{bmatrix} -a_1-a_2 & a_1 & a_2 & 0 & 0 \\ -a_3 & a_3 & 0 & 0 & 0 \\ -a_6 & 0 & a_6 & 0 & 0 \\ 0 & 0 & 0 & a_3-a_1 & -a_2 \\ 0 & 0 & 0 & -a_1 & a_6-a_2 \end{bmatrix}.$$

Given an other vector field

$$i_Y = \begin{bmatrix} 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D_Y = \begin{bmatrix} 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & b_6 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

we get

$$i_Z = \begin{bmatrix} 0 & 0 & 0 & -a_2b_1 - a_3b_1 + a_1(b_2 + b_3) & a_2(b_1 + b_6) - (a_1 + a_6)b_2 \\ 0 & 0 & 0 & a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 \\ 0 & 0 & 0 & a_1b_6 - a_6b_1 & a_2b_6 - a_6b_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of L_X contain 0 as well as the following two eigenvalues, each with multiplicity two:
$\left(\pm\sqrt{a_1^2 + 2a_1(a_2 - a_3 + a_6) + (a_2 + a_3 - a_6)^2} - a_1 - a_2 + a_3 + a_6\right)/2$. We see that real eigenvalues are quite common but that imaginary eigenvalues of L_X can occur in the 4-dimensional space of vector fields. The operator L_Z does in general not have zero eigenvalues. They can even become complex. We also see that the commutator L_Z now tunnels between places which were not directly connected in G.

5.3. For the complete complex $G = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$, we can look at the general

$$D_X = d + iX = \begin{bmatrix} 0 & 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e & f & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & g \\ -1 & 0 & 1 & 0 & 0 & 0 & h \\ 0 & -1 & 1 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

Now, given two general D_X, D_Y, we have $L_X = D_X^2, L_Y = D_Y^2$. If $g = h = i = 0$ then, $L_X i_Y - i_Y L_X = i_X L_Y - L_Y i_X$ and $i_Z = L_X i_Y - i_Y L_X$ has the property that $i_Z^2 = 0$.

5.4. Let $G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$ be the complex of the cyclic graph C_4. The exterior derivative d and an example of an interior derivative i_X are:

$$d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, i_X = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This leads to the Dirac operator $D = d + d^*$ and the directional Dirac operator $D_X = d + i_X$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, D_X = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Hodge Laplacian $L = D^2$ and Lie derivative $L_X = D_X^2$ are

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}, L_X = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

The eigenvalues of D are $\{-2, 2, -\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}, 0, 0\}$, the eigenvalues of D_X are $\{-\sqrt{2}, -\sqrt{2}, -1, -1, 1, 1, 0, 0\}$. The eigenvalues of L_X are $\{4, 4, 2, 2, 2, 2, 0, 0\}$.

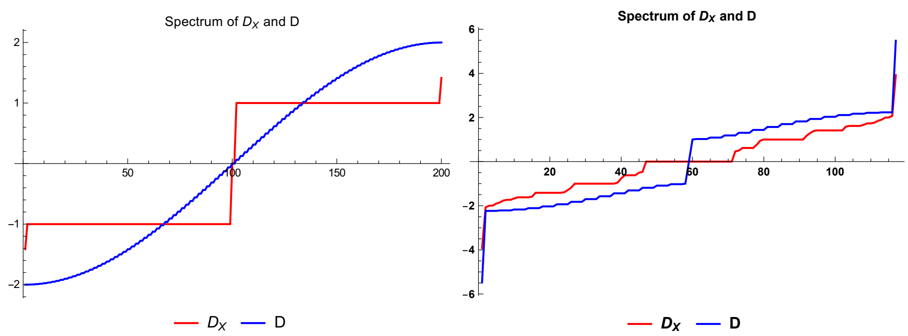

FIGURE 1. The figures show the spectrum of D and D_X for a cycle graph spectrum and a wheel graph spectrum. In both cases, we start with $i_X = d^*$, then take away each entry 1 or -1 with probability 1/2. D_X still has the $\sigma(D_X) = -\sigma(D_X)$ symmetry from the Dirac operator D but the energies of the particles are smaller as it is more difficult to travel.

6. Mathematica procedures

6.1. Here is the code which computes the Dirac operator D and the vector field analogue D_X for any simplicial complex G. The code can be copy pasted when accessing the LaTeX source of this document on the ArXiv. The first part computes the matrices given in the above example. For the vector field, we chose for i_X just to take the first non-zero entry of d^*:

```
G={{1},{2},{3},{4},{1,2},{2,3},{3,4},{4,1}};
n=Length[G]; Dim=Map[Length,G]-1; f=Delete[BinCounts[Dim],1];
Orient[a_,b_]:=Module[{z,c,k=Length[a],l=Length[b]},
  If[SubsetQ[a,b] && (k==l+1),z=Complement[a,b][[1]];
  c=Prepend[b,z]; Signature[a]*Signature[c],0]];
d=Table[0,{n},{n}];d=Table[Orient[G[[i]],G[[j]]],{i,n},{j,n}];
dt=Transpose[d]; DD=d+dt; LL=DD.DD;
HX[x_]:=Block[{u=Flatten[Position[Abs[x],1]]},If[u=={},0,First[u]]];
iX=Table[0,{n},{n}];
Do[l=HX[dt[[k]]]; If[l>0,iX[[k,l]]=dt[[k,l]]],{k,f[[1]]}];
DX=iX+d; LX = DX.DX;
```

6.2. Here is the code to generate D_X and L_X for a generate random finite abstract simplicial complex G:

```
(* Generate a random simplicial complex                    *)
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]
R[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}]; Generate[A]];
G=Sort[R[10,20]];
```

```
(* Computation of exterior derivative                                        *)
n=Length[G]; Dim=Map[Length,G]-1; f=Delete[BinCounts[Dim],1];
Orient[a_,b_]:=Module[{z,c,k=Length[a],l=Length[b]},
  If[SubsetQ[a,b] && (k==l+1),z=Complement[a,b][[1]];
  c=Prepend[b,z]; Signature[a]*Signature[c],0]];
d=Table[0,{n},{n}];d=Table[Orient[G[[i]],G[[j]]],{i,n},{j,n}];
dt=Transpose[d]; DD=d+dt; LL=DD.DD;

(* Build interior derivatives iX  and iY                                     *)
UseIntegers=False;
e={}; Do[If[Length[G[[k]]]==2,e=Append[e,k]],{k,n}];
BuildField[P_]:=Module[{X,ee,iX=Table[0,{n},{n}]},
X=Table[If[UseIntegers,Random[Integer,1],Random[]],{l,Length[e]}];
Do[ee=G[[e[[l]]]]; Do[If[SubsetQ[G[[k]],ee],
  m=Position[G,Sort[Complement[G[[k]],Delete[ee,2]]]][[1,1]];
  iX[[m,k]]= If[MemberQ[P,Length[G[[m]]]],X[[l]],0]*
              Orient[G[[k]],G[[m]]]],{k,n}],{l,Length[e]}]; iX];

(* Build Laplacians LX,LY,LZ, plot spectrum of D and DX and matrices  *)
iX=BuildField[{1,3,5,7,9}]; iY=BuildField[{1,3,5,7,9}];
DX=iX+d; LX=DX.DX; DY=iY+d; LY=DY.DY; iZ1=LX.iY-iY.LX; iZ2=iX.LY-LY.iX;
Print[iZ1==iZ2]; iZ=iZ1; LZ=Chop[iZ.d+d.iZ]; DZ=iZ+d;
dx="\!\(\*SubscriptBox[\(D\), \(X\)]\)";
lx="\!\(\*SubscriptBox[\(L\), \(X\)]\)"; pl=PlotLabel;
GraphicsGrid[{{MatrixPlot[DX,pl->dx], MatrixPlot[LX,pl->lx]},
             {MatrixPlot[DD,pl->"D"], MatrixPlot[LL,pl->"L"]}}];
u1 = Sort[Chop[Eigenvalues[1.0 DX]]]; u2 = Sort[Eigenvalues[1.0 DD]];
u1=N[Round[u1*10^6]/10^6];              (* clear tiny imaginary parts  *)
S=ListPlot[{u1, u2},Joined ->True,PlotLegends ->Placed[{dx,"D"},Below],
 PlotRange -> All, PlotStyle -> {Red, Blue},
 PlotLabel -> "Spectrum of \!\(\*SubscriptBox[\(D\), \(X\)]\) and D"];

(* Compute Betti numbers, compare bosonic and fermionic part              *)
chi=Sum[-f[[k]](-1)^k,{k,Length[f]}]; f=Prepend[f,0]; m=Length[f]-1;
U=Table[v=f[[k+1]];
   Table[u=Sum[f[[l]],{l,k}];LL[[u+i,u+j]],{i,v},{j,v}],{k,m}];
Cohomology = Map[NullSpace, U]; Betti = Map[Length, Cohomology]
chi1=Sum[-Betti[[k]](-1)^k,{k,Length[Betti]}];
EV=Map[Eigenvalues,U];
EVFermi=Table[EV[[2k]],{k,Floor[Length[EV]/2]}];
EVBoson=Table[EV[[2k-1]],{k,Floor[(Length[EV]+1)/2]}];
extract[u_]:=Module[{v=Flatten[u],w={}},
   Do[If[Abs[v[[k]]]>10^(-8),w=Append[w,v[[k]]]],{k,Length[v]}];Sort[w]];
extract[EVFermi]==extract[EVBoson]

(* Now the same for LX                                                      *)
U=Table[v=f[[k+1]];
   Table[u=Sum[f[[l]],{l,k}];LX[[u+i,u+j]],{i,v},{j,v}],{k,m}];
EV=Map[Eigenvalues,U];
EVFermi=Table[EV[[2k]],{k,Floor[Length[EV]/2]}];
EVBoson=Table[EV[[2k-1]],{k,Floor[(Length[EV]+1)/2]}];
extract[EVFermi]==extract[EVBoson]
{EVFermi,EVBoson}
```

```
Total[Abs[N[extract[EVFermi] - extract[EVBoson]]]]

Cohomology = Map[NullSpace, U]; Betti = Map[Length, Cohomology];
chi2=Sum[-Betti[[k]](-1)^k,{k,Length[Betti]}];
{chi,chi1,chi2}
```

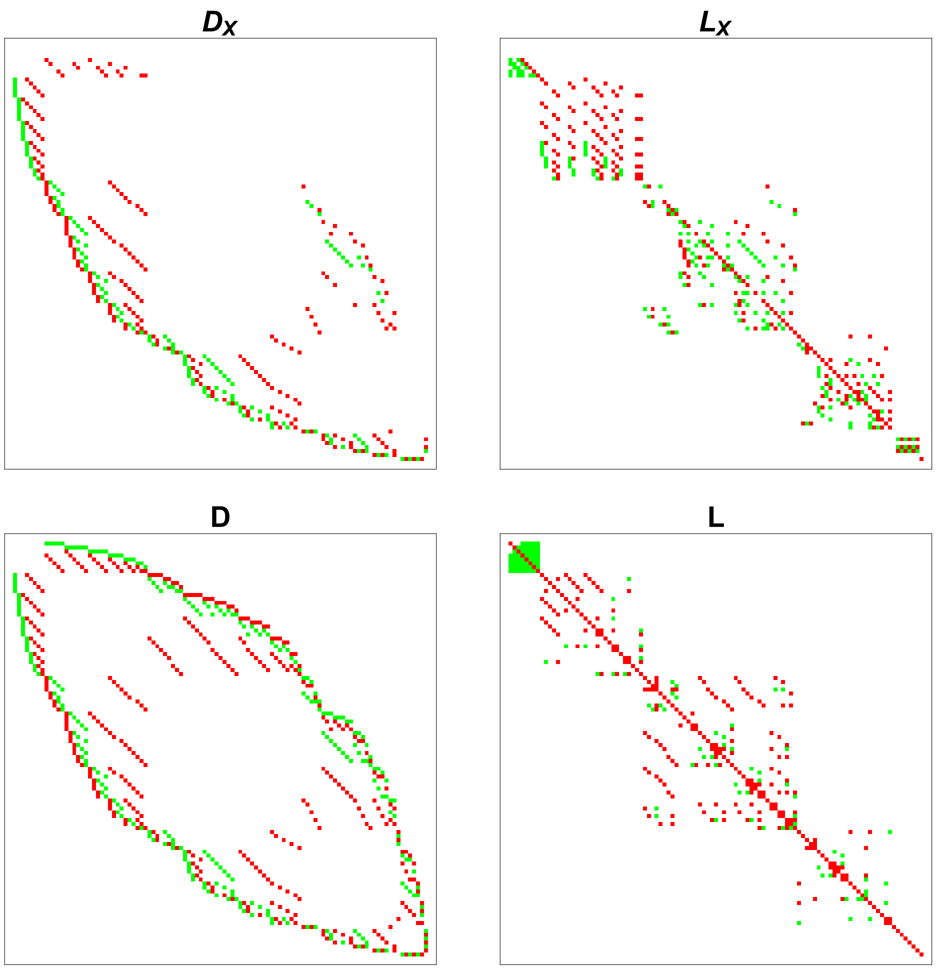

FIGURE 2. The matrices D_X, L_X, D, L in the case of a random complex. This was produced with the code above.

6.3. And finally, here is the self-contained procedure which does the isospectral deformation of the exterior derivative by deforming $D'_X = [B_X, D_X]$. In the case $i_X = d^*$, this is the standard **Lax isospectral deformation** we have seen before

[11, 10]. In an other extreme case, when $X = 0$, then d is not deformed at all. We still have the inflationary initial decay of d typical for that type of integrable dynamical system. The decay of d means by the Connes formula that there is an expansion of space because if the derivative operator become small, then the distances grow.

```
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]
R[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}];Generate[A]];G=Sort[R[5,8]];
n=Length[G];  fv=Delete[BinCounts[Map[Length,G]],1];
cn=Length[fv];br={0};Do[br=Append[br,Last[br]+fv[[k]]],{k,cn}];

Orient[a_,b_]:=Module[{z,c,k=Length[a],l=Length[b]},
  If[SubsetQ[a,b] && (k==l+1),z=Complement[a,b][[1]];
  c=Prepend[b,z];  Signature[a]*Signature[c],0]];
d=Table[0,{n},{n}];d=Table[Orient[G[[i]],G[[j]]],{i,n},{j,n}];
dt=Transpose[d];  DD=d+dt;  LL=DD.DD;

UseIntegers=False;e={};  Do[If[Length[G[[k]]]==2,e=Append[e,k]],{k,n}];
BuildField[P_]:=Module[{X,ee,iX=Table[0,{n},{n}]},
X=Table[If[UseIntegers,Random[Integer,1],Random[]],{l,Length[e]}];
Do[ee=G[[e[[l]]]];  Do[If[SubsetQ[G[[k]],ee],
  m=Position[G,Sort[Complement[G[[k]],Delete[ee,2]]]][[1,1]];
  iX[[m,k]]=  If[MemberQ[P,Length[G[[m]]]],X[[l]],0]*
              Orient[G[[k]],G[[m]]]],{k,n}],{l,Length[e]}];  iX];
iX=BuildField[{1,3,5,7,9}];  iY=BuildField[{1,3,5,7,9}];
DX=iX+d;  LX=DX.DX;  DY=iY+d;  LY=DY.DY;  iZ=LX.iY-iY.LX;  DZ=iZ+d;LZ=DZ.DZ;

T[A_]:=Module[{n=Length[A]},Table[If[i<=j,0,A[[i,j]]],{i,n},{j,n}]];
UT[{DD_,br_}]:=Module[{D1=T[DD]},       (* Lower triangular block   *)
Do[Do[Do[D1[[br[[k]]+i,br[[k]]+j]]=0,{i,br[[k+1]]-br[[k]]}],
  {j,br[[k+1]]-br[[k]]}],{k,Length[br]-1}];D1];
RuKu[f_,x_,s_]:=Module[{a,b,c,u,v,w,q},u=s*f[x];  (* Runge Kutta   *)
a=x+u/2;v=s*f[a];b=x+v/2;w=s*f[b];c=x+w;  q=s*f[c];x+(u+2v+2w+q)/6];

DD=DX;  d0=UT[{DD,br}];  e0=Conjugate[Transpose[d0]];
M=1000;  delta=2/M;  u={};   (* Deformation with Runge Kutta        *)
Do[d=UT[{DD,br}];e=Conjugate[Transpose[d]];
BB=d-e;  CC=d+e;  MM=CC.CC;  b=DD-CC;  VV=b.b;
B=BB+1.0*I*b;  f[x_]:=B.x-x.B;  DD=RuKu[f,1.0 DD,delta];
u=Append[u,Total[Abs[Flatten[Chop[d]]]]],{m,M}];
DDX=DD;  LLX=DDX.DDX;

{Total[Abs[Flatten[Chop[d.d]]]],  Total[Abs[Flatten[Chop[e.e]]]]}

F[x_]:=If[x==0,0,-Log[Abs[x]]];   (* Plot the size of d       *)
v=M*Table[F[u[[k+1]]]-F[u[[k]]],{k,Length[u]-1}];
ListPlot[v,PlotRange->All]
```

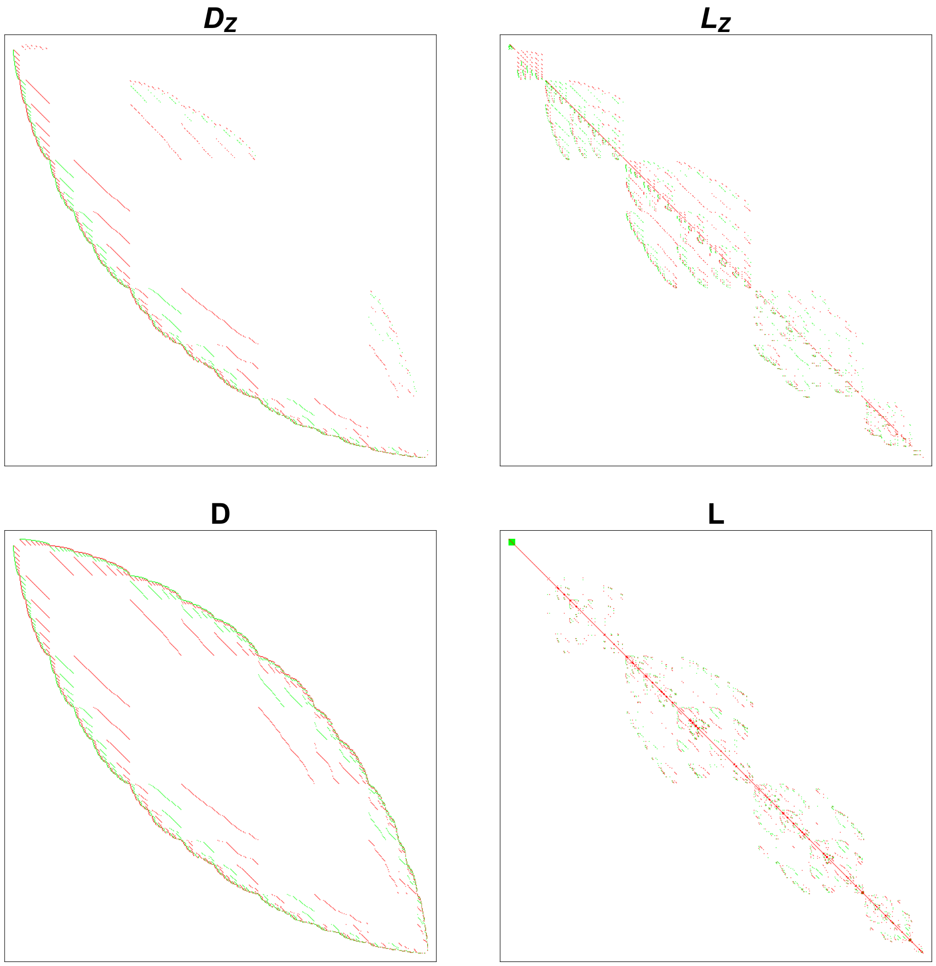

Figure 3. The matrices D_Z, L_Z, D, L in the case of a random complex with f-vector $(10, 45, 120, 192, 165, 73, 15, 1)$. Also this was produced with the code above, where $Z = [X, Y]$ is the commutator of two random vector fields.

7. Questions

7.1. The operators D_X and L_X are not symmetric in general so that complex eigenvalues can appear. In that case, the solution $\psi(t) = e^{iD_X t}\psi(0)$ can grow exponentially. D_X still often has real eigenvalues, leading to quasi-periodic solutions as the orbits $e^{iD_X t}\psi(0)$ form a subgroup of a finite dimensional torus, if the graph is finite. Actually, if $i_X(k, l) \neq 0$ only for one l, then we implement a deterministic vector field. In that case the eigenvalues of L_X often non-negative integers taking values in $\{0, 1, \dots, \dim(G)+1\}$. We would like to understand the spectrum.

7.2. For $D = i_X + i_X^* + d + d^*$ we have $D^2 = L_X + L_X^* + L$. Now, if we average that over all possible vector fields using a measure which is homogeneous, we expect the L_X to average out and get the wave equation governed by L. Can one make this more precise and see the wave equation $f_{tt} = -L$ as an average of deterministic flows $f_{tt} = -L_X$?

7.3. We often integer eigenvalues of L_X if i_X has integer values. In small dimensional examples, we can compute general formulas for the eigenvalues but integer eigenvalues also often appear for large random simplicial complexes. Under which conditions does L_X have integer eigenvalues?

7.4. The eigenvalues of L_X are most of the time real if the entries of I_X are non-negative multiplies of d^*. They can become imaginary in general, if the signs are changed. Can we find conditions which assures a real spectrum?

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