REMARKS ON THE BROUWER CONJECTURE

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ABSTRACT. The Brouwer conjecture (BC) in spectral graph theory claims that the sum of the largest k Kirchhoff eigenvalues of a graph are bounded above by the number m of edges plus k(k+1)/2. We show that (BC) holds for all graphs with n vertices if n is larger or equal than 4 times the square of the maximal vertex degree. We also note that (BC) for graphs implies (BC) for quivers.

1. IN A NUTSHELL

1.1. The **Brouwer conjecture** (BC) is the statement in spectral graph theory that $S_k = \sum_{j=1}^{k} \lambda_j \leq m + k(k+1)/2 = B_k$ for all $1 \leq k \leq n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ are the non-increasing ordered eigenvalues of the Kirchhoff matrix of the graph and m is the number of edges. The now 20 year old conjecture is still open despite considerable effort like [6, 20, 14, 29, 27, 7, 24].

1.2. We add here some remarks related to the conjecture. Some involve the sum $D_k = \sum_{j=1}^{k} d_j$ of the largest k vertex degrees. In the case of quivers, graphs with possible loops and r additional multiple connections, the constants have to be adapted to $B_k = m + r + k(k+1)/2$. Define also $H_k = m + r + k^2$ which recently appeared in [22] for finite simple graphs (where $r = 0$).

1.3. Here are the five major points we cover here. "Quiver" can always be replaced with "finite simple graph" as the later is the special case of a quiver without loops and multiple conditions.

- $D_k \leq S_k \leq 2D_k$ holds for all quivers.
- $D_k \leq B_{k-1}$ holds for all quivers.
- (BC) holds for all quivers G with $n(G) > 4d_1(G)^2$ vertices.
- If (BC) holds for simple graphs, it holds for quivers.
- $S_k \leq H_k$ holds for all quivers.

1.4. We also mention that if we would know the inequality $S_k \leq B_k$ for all $k \leq s$ and for all graphs G, then (BC) would hold for all graphs satisfying $\lambda_1 \leq s$. As of now, this is only known for $s = 2$ and is not very useful yet. We do not even know whether there is an example of a graph with $\lambda_1 + \lambda_2 + \lambda_3 > m + 6$.

2. THE BROUWER CONJECTURE

2.1. Let $G = (V, E)$ be a **finite graph** with n **vertices** V and m **edges** E. Self loops and multiple connections both count as edges. The **Kirchhoff matrix** K of such a quiver G is defined as $K = F^T F$, where the $m \times n$ matrix F is the **quiver gradient** while its transpose, the $n \times m$ matrix F^T is the **quiver divergence** [19] and $K = F^T F = \text{divgrad}$ is the **quiver Laplacian**. The matrix K can be written as $D - A$ with diagonal **vertex degree matrix** D (in which every loop counts as 1) and **adjacency matrix** A in which $-A_{ij}$ counts the number

of edges between node i and node j. Let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ denote the **eigenvalues** of K, ordered in a non-increasing manner. The identity $K = D - A$ shows that eigenvalues of G do not depend on the choice of the orientations on E, despite the fact that the orientation was used to define the gradient matrix F. We remind in an appendix a bit of more notation about quiver calculus as the literature is not uniform in terminology. As in [19] we also add computer algebra code for quivers.

2.2. Assume first that G is a **finite simple graph**, an undirected quiver without multiple connections and no loops. Let $S_k(G) = \sum_{j=1}^{k} \lambda_j$ be the **cumulative descending spectral sum**. The **Brouwer bound** is defined as $B_k(G) = m(E) + k(k+1)/2$, where $m(G)$ is the **number of edges** of G. The **conjecture of Brouwer** (BC) (see [4], page 53) has first been formulated if G is a finite simple graphs, meaning that no loops nor multiple connections are allowed.

Conjecture 1 (BC). *If G is a finite simple graph, then $S_k \leq B_k$ for all $1 \leq k \leq n$.*

2.3. The conjecture is open. In all our experiments so far, we also see that also the **sign-less Brouwer conjecture** (BC+) holds, in which the Kirchhoff Laplacian K is replaced by $|K|$. This is related to the **connection Laplacian** L which is a $n \times m$ unimodular matrix, has the property that $L - L^{-1}$ is block diagonal with $K = |F^T F|$ and $K_1 = |FF^T|$ as blocks. There has been considerable interest in the conjecture during the last couple of years [6, 20, 14, 10, 29, 27, 7, 24]. The signless Brouwer conjecture (BC+) has appeared first in [10], where the cases $k = 1, 2$ are covered. We see that if (BC) or (BC+) holds for finite simple graphs, it extends to all quivers when suitable adapted in the case of multiple connections.

3. A SANDWICH

3.1. Let us first look at relations with **vertex degree sequences**. Define $D_k(G) = \sum_{j=1}^{k} d_j$ if $d_1 \geq d_2 \geq \cdots \geq d_n$ are the sorted vertex degrees of the quiver. These are also the diagonal elements of K when sorted in a non-increasing manner. The **Schur inequality**, which relates diagonal entries and eigenvalues for any symmetric matrix, immediately gives $S_k \geq D_k$. We proved in [19], the general upper bound $\lambda_k \leq d_k + d_{k+1}$ for all quivers and all eigenvalues λ_k. This especially implies:

Theorem 1 (Sandwich). *For any quiver we have $D_k \leq S_k \leq 2D_k$.*

3.2. Especially for larger n, we see in experiments that the spectral sum S_k is sandwiched between D_k and cD_k with c close to 1 if $k \geq 2$. The sequences D_k, S_k entering the inequalities $D_k \leq S_k \leq 2D_k$ are concave down sequences, while the Brouwer bound B_k is concave-up functions in k. Elementary is the following estimate. We suspect that it could have been one on of the motivations to (BC) even this is so not explicitly stated as such in the literature.

Proposition 1. *$D_k \leq B_{k-1}$ for all quivers.*

Proof. (i) Assume first that G has no loops. Look at the sub-graph which is the union of the k star graphs centered at the k vertices with largest vertex degree. Such a graph has at least $D_k - k(k-1)/2$ edges as maximally $k(k-1)/2$ can be double counted. This shows that $D_k - k(k-1)/2 \leq m$.
(ii) If we add a loop to a vertex in the graph, then B_k increase by 1 because m increases by 1. The D_k can only increase by either 0 or 1 depending on whether the loop was at one of the k largest degree vertices.

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(iii) If we add an additional multiple connection, then D_k can only grow by 2 or less, while $B_{k-1} = m + r + k(k-1)/2$ grows by 2 as both m and r (the redundancy) grow by 1. We will define the redundancy more precisely r in the next section. $\qquad\square$

3.3. We would like to know the largest $c > 1$ such that the concave down sequence cD_k is "tangent" to the concave up sequence B_k. We know that there exists a constant $1 < b < 2$ defined as the smallest b such that $S_k \leq bD_k$. We often see that $D_k \leq S_k \leq cD_k \leq B_k$ which would hold if $b \leq c$. The difficulty of the Brouwer conjecture can be illustrated in reminding that even for $k = 3$ the statement $S_3 \leq B_3$ is not yet known in general. Written out, this case $k = 3$ means that the sum of the 3 largest eigenvalues and the number m of edges satisfy $\lambda_1 + \lambda_2 + \lambda_3 \leq m + 6$. Even that case is open.

4. Extension to quivers

4.1. The total number r of **redundant edges** in a quiver is defined as the minimal number of edges which when removed produces a graph without multiple connections. This **redundancy** $r = r(G)$ can also be read off from the Kirchhoff matrix K as $r = \sum_{i<j} \text{sign}(K_{ij}) - K_{ij}$. In this more general setting, again set $S_k = \sum_{j=1}^{k} \lambda_j$ and define $B_k = m + r + k(k+1)/2$, if m still counts the total number of edges, including loops and multiple connections and where r is the redundancy of G. For graphs without multiple connections, this regresses to $B_k = m + k(k+1)/2$ which is the usual Brouwer bound. Note that adding a multiple connection increases B_k by 2 as both m and r get increased.

4.2. We will just see that if G is a (BC) graph, then adding a loop or a multiple connection keeps the extended graph in (BC).

Theorem 2. *If (BC) holds all finite simple graphs G, then (BC) holds for all quivers G.*

Proof. (i) Let v be the vertex which belongs to the x'th row in the matrix K. The **Hadamard perturbation formula** $\lambda_i' = |v_x|^2$ for a unit eigenvector v shows after integration from $t = 0$ to $t = 1$ that if a loop is added, the eigenvalues can only grow. Comparing the trace shows that S_k can maximally grow by 1 while the Brouwer sum B_k grows by exactly 1.
(ii) If we add an additional edge (x, y), the Hadamard perturbation formula for the deformation gives $\lambda_k' = (v_x - v_y)^2 \geq 0$. Looking at the trace shows that the spectral sum S_k grows maximally by 2. The Brouwer sum $B_k = m + r + n(n+1)/2$ grows for each k by exactly 2 during the deformation, because m is increased by 1 and r is increased by 1. $\qquad\square$

4.3. Remarks.
a) The same same upgrade is possible for the sign-less conjecture (BC+). We therefore would only need to show the sign-less conjecture for finite simple graphs and get the sign-less conjecture for general quivers.
b) Instead of referring to Hadamard deformation, one could also invoke general principles of symmetric operators. If $K \leq L$ are two symmetric matrices, then $\lambda_j(K) \leq \lambda_j(L)$. The perturbation adds a projection $e_j \cdot e_j^T$ in the first case and a multiple of a projection $(e_i - e_j) \cdot (e_i \cdot e_j)^T$ in the second case.

4.4. Extending the conjecture to quivers can be motivated by modeling **Schrödinger cases** (in the sense of providing discrete "tight binding approximation" versions of $-\Delta + V$, where the loops model a **potential values** $V(v)$ at a vertex v. It also allows for extended investigations. We can for example look at the least amount l of loops which are additionally needed at each vertex such that (BC) can be proven.

Proposition 2. *If sufficiently many loops are attached at every vertex, then (BC) hold.*

Proof. If l loops are attached to each vertex, the Kirchhoff matrix becomes $K + lI_n$ so that $S_k(G + L) = S_k(G) + l * k$ and $B_k(G + L) = B_k(G) + l * n$. If l is large enough, then $S_k(G) + lk \leq B_k(G) + ln$ holds for all $k < n$. But for $k = n$, we have $S_n(G) \leq B_n(G)$ which is known. $\qquad\square$

4.5. This motivates to stratify the conjecture and to ask for example, whether it is possible to prove the conjecture for all graphs with a generous number of $l = 1000$ loops attached at each vertex.

5. Large graphs

5.1. Illustrated by the fact that we do not know whether $S_3 \leq B_3$ in general, there appears to be a difficulty to establish the bound for small k. We have also sharp situations for very dense graphs, as the case complete graphs K_n illustrate, where $k = n - 1$ is sharp. This suggests that we should take large enough graphs, but keep the edge degree bounded in order to be able to make a statement.

5.2. Let us start with an interlacing result. If we remove an edge from the graph, the number of vertices does not change, so that applying the Cauchy-interlace theorem appears not possible. Removing an edge however produces a principal submatrix of the 1-form Laplacian $K' = FF^T$ which is essentially isospectral to $K = F^T F$ and which is a $m \times m$ matrix if there are m edges. Let $\lambda_1 \geq \lambda_2 \cdots \lambda_n$ be the eigenvalues of K. If $H = G - e$ is a subgraph in which one of the edges has been removed, we get the eigenvalues $\mu_1 \geq \mu_2 \cdots \mu_n$. We say that μ_l list is **interlaced** with the λ_l list, if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n = \lambda_n = 0$.

Lemma 1 (See [11] theorem 13.6.2). *The eigenvalues of $K(G - e)$ interlace the eigenvalues of $K(G)$.*

Proof. Here is a new "supersymmetry proof" Because $K = F^T F$ and $K' = FF^T$ are essentially isospectral, it is enough to show the interlacing for K'. But $K'(G)$ has the row and column by e deleted, so that by the **Cauchy interlace theorem**, the spectra of $K'(G)$ and $K'(G - e)$ are interlaced. $\qquad\square$

5.3. The general statement that the eigenvalues $\lambda_k(H) \leq \lambda_k(G)$ if H is a subgraph of G follows also from the **Courant variational principle**. But more is true. The spectrum of a subgraph always interlaces the spectrum of the graph. One can ask whether the result $\lambda_j \leq d_j + d_{j+1}$ (decreasing eigenvalue and vertex degree assumption), proven in [19] can also be proven by induction while removing edges. The answer is "no" because removing an edge reduces some vertex degrees rendering induction impossible. The extension of the result to quivers was necessary.

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5.4. The next theorem is a contribution to the Brouwer conjecture that adds confidence that (BC) holds for all graphs. Recall that n is the number of vertices of the graph, while m is the number of edges of the graph and d_1 is the **maximal vertex degree**.

Theorem 3. *If G is a connected graph for which $n \geq 4d_1^2$, then (BC) holds for G.*

Proof. We can first assume that G is a finite simple graph, then use that we can upgrade the result from graphs to quivers. Take a spanning tree $H_0 \subset G$. This requires that G is connected. We know (BC) holds for all trees [14]. We will now produce a sequence of graphs $H_0 \subset H_1 \subset \cdots \subset H_{m-n+1} = G$, where going from $A = H_l$ to $B = H_{l+1}$ is done by adding an edge of G. We show that if $A = H_L$ satisfies (BC), then $B = H_{l+1}$ satisfies (BC). Now use the above lemma which assures that if A has eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and B has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots \geq \lambda_n = \mu_n = 0$. Especially $\mu_j \geq \lambda_{j+1}$.

First case (i) $k \geq 2d_1$: this implies $k \geq \lambda_1$.

$$
\begin{aligned}
S_k(B) &= \lambda_1 + (\lambda_2 + \lambda_3 + \cdots + \lambda_k) \\
&\leq \lambda_1 + (\mu_1 + \mu_2 + \cdots \mu_{k-1}) \\
&\leq \lambda_1 + m(A) + k(k-1)/2 \\
&= \lambda_1 - k + m(A) + k(k+1)/2 \\
&\leq m(A) + k(k+1)/2 \\
&< m(B) + k(k+1)/2 = B_k(B) \ .
\end{aligned}
$$

Second case (ii) $k \leq 2d_1$: Then $k\lambda_1 \leq (2d_1)^2$ and

$$
\begin{aligned}
S_k(B) &= \lambda_1 + \lambda_2 + \cdots + \lambda_k \\
&\leq k\lambda_1 \\
&\leq (2d_1)\lambda_1 \\
&\leq (2d_1)(2d_1) \\
&= 4d_1^2 \\
&\leq n \\
&\leq m+1 \\
&\leq m + k(k+1)/2 = B_k(B) \ .
\end{aligned}
$$

So, also $B = H_{l+1}$ satisfies (BC). We continue as such until $H_{m-n+1} = G$ is reached and see that G satisfies (BC). $\qquad\square$

5.5. Examples.
a) In the case of a **cyclic graph** C_n, we have $m = n$ and $d_j = 2$ and the result works for $n \geq 4 * 2^2 = 16$. Cyclic graphs are well known to be in (BC): the case $n = 2, n = 3$ is covered with complete graphs. For $n \geq 4$, we can use that $1+2k \leq k(k+1)/2$. Now, $\lambda_j \leq 4\sin(\pi j/(2n))^2$ has the explicit sum $S_k = 1 + 2k - \csc(2n/\pi)\sin((1+2k)\pi/(2n)) \leq 1 + 2k \leq k(k+1)/2 \leq n + k(k+1)/2 = B_k$.
b) For a general **triangle free graph**, the Barycentric refinement doubles m but keeps the maximal vertex degree constant. So, after doing more than $\log_2(4d_1^2/m)$ refinement steps to a triangle free graph, we are in (BC).
c) For a vertex **regular graph** of degree d, we have $nd = 2m$. Now $4d^2 = 4(2m)^2/n^2 = 16m^2/n^2 \leq n$ if $16m^2 \leq n^3$.

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d) For a random graph in the Erdoes-Renyi space $E(n, p)$, the vertex degree expectation is about $d = pn$ so that we need $4p^2n^2 < n$ or $p < 1/\sqrt{(2n)}$ in order that the theorem can be applied for large n. There are results about random graphs in [26, 7]. Rocha showed that (BC) holds asymptotically almost surely.

e) Two discrete manifolds can be connected via a **connected sum**. In the special case when doing a connected sum with a spheres we get a notion of homeomorphism. If we do connected sums using manifolds with a global upper bound on the vertex degree, then taking a sufficiently large manifold of this type will get us into the class (BC).

f) For a **discrete 2-manifold** we can make a soft Barycentric refinement By taking the union of triangles and vertices as new vertices and connect two if one is contained in the other or if the intersection is an edge. This operation does not increase the maximal vertex degree if the maximal vertex degree is already 6 or more.

6. The snap reduction

6.1. We have just seen that edge removal produces interlacing eigenvalues. There is a snap construction within the class of quivers, which has the effect of removing a vertex and adding loops. This operation was used in [19]. It is somehow dual to the edge removal because the number of edges does not change, while the number of vertices decreases by 1.

6.2. Assume G is a finite simple graph or more generally a quiver without multiple connections. If we remove a vertex from G and snap the edges to the nearest connections so that edges morph to loops, we end up with a quiver $H = G - v$ with $n - 1$ vertices for which the total edge sum m of edges remains the same if no loop was present at v and where the total edge sum m has decreased to $m - l$ if l loops had been present at v. We call $G - v$ the **snap reduction** of G.

6.3. An observation in [19] was that the snap reduction has the effect that the Kirchhoff matrix $K(G-v)$ is a principal sub-matrix of the Kirchhoff matrix $K(G)$ of G. The **Cauchy interlace theorem** then relates the eigenvalues of $H = G - v$ and G. In the Brouwer we can use the snap reduction if k is larger than the spectral radius. If k is smaller than the spectral radius, we have to bound k terms each smaller or equal than the spectral radius.

Lemma 2 (Snap lemma). *Let G be a quiver without multiple connections and let $H = G - v$ be a snap reduction of G. Then $B_k(H) \leq B_k(G)$ and $S_k(G) \leq B_k(G) + (\lambda_1 - k)$ and especially $S_k(G) \leq B_k(G)$ for all $k \geq \lambda_1(G)$. Also $\lambda_1(H) \leq \lambda_1(G)$.*

Proof. If we remove a simple connection from v and snap it to a loop at w, then the edge number m either stays the same or decreases by the number of loops at m. Therefore, also $B_k = m + k(k+1)/2$ stays the same or becomes smaller. The Kirchhoff matrix of the snapped graph is a **principal sub-matrix** of the Kirchhoff matrix of G. This was true also if the vertex v has loops. Let μ_j denote the eigenvalues of the snapped graph $H = G - v$ Then, using $B_k(H) = B_{k-1}(H) + k$ and $B_k(H) \leq B_k(G)$ (following $m(H) \leq m(G)$), we have by induction and using $k \geq \lambda_1$:

$$S_k(G) = \sum_{j=1}^{n} \lambda_j \leq \lambda_1 + S_{k-1}(H) \leq \lambda_1 + B_{k-1}(H) = \lambda_1 - k + B_k(H) \leq \lambda_1 - k + B_k(G) \leq B_k(G) .$$

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[1]Alan Lew informed me that Brouwer's conjecture is known to hold for regular graphs, by work of Mayank [25] and Berndsen [**?**] in their M.Sc. Theses.

□

6.4. Define $\mathcal{G}_\sigma$ as the set of quivers with spectral radius $\leq \sigma$.

Lemma 3. *The class $\mathcal{G}_\sigma$ is invariant under snap reduction and edge removal.*

Proof. In both cases, the spectrum of the reduced graph is interlaced, the spectral radius of $H = G - v$ is smaller or equal than the spectral radius of G. So, if $G \in \mathcal{G}_\sigma$ then $H \in \mathcal{G}_\sigma$.   □

6.5. Here is a way to enlarge graphs for which (BC) holds: if we take a graph H satisfying (BC) and which has enough edges, then we can extend as long as we do not increase the maximal vertex degree. The reason is that we keep an upper bound on the spectral radius.

Proposition 3. *If (BC) holds for a connected graph H with $m(H) > 4d_1(H)^2$ and H is a subgraph of G with $d_1(G) \leq d_1(H)$, then (BC) also holds for G.*

Proof. If G is a quiver with m vertices and spectral radius λ_1. If $m \geq \lambda_1^2$, then G satisfies (BC) if H satisfies BC. (i) If $k \leq \sigma(G) = \lambda_1$ then $D_k = \lambda_1 + \cdots + \lambda_k \leq k\sigma(G) \leq \sigma(G)^2 \leq m \leq m + k(k+1)/2 = B_k$.
(ii) In the case $k \geq \sigma(G) = \lambda_1$, we use the snap lemma. Let G be a quiver with $n + 1$ vertices. Take any vertex v in G and snap it to get a graph $H = G - v$. The new graph H has a Kirchhoff matrix with interlaced spectrum so that the induction step works also in the case $k \geq \sigma = \lambda_1$. The problem is that the snapped graph has in general less edges so that $m(H) \geq \lambda_1(H)^2$ does not hold.   □

6.6. Here is a question. A **2-manifold** is a finite simple graph such that every unit sphere is a circular graph with 4 or more elements. A **2-manifold with boundary** is a finite simple graph for which every unit sphere is either a circular graph with 4 or more elements or then a path graph of length 2 or more. An example of a 2-manifold with boundary is to take a boundary-less graph and remove a vertex. It would be nice to know whether all q-manifolds are in (BC). We so far only know that it holds for large enough manifolds provided that the curvature stays bounded below.

7. BROUWER THRESHOLD

7.1. Define the **Brouwer threshold** s as the largest number s such that the Brouwer estimate $S_k(G) \leq B_k(G)$ holds for all $k \leq s$ for all finite simple graphs. This threshold is a **global number** that only depends on the progress of research. The Brouwer conjecture is equivalent to the statement that $s = \infty$. As for now (2025), we only know $s = 2$. That the Brouwer threshold s can be pushed up in the future is a reasonable expectation. Also the next statement is a motivation to increase the threshold. It does not invoke any condition in the number of edges.

Corollary 1. *If s is the Brouwer threshold. For all quivers with $\lambda_1 \leq s$, the Brouwer conjecture holds.*

Proof. This also immediately follows from combining the two lemmas and using induction in the class $\mathcal{G}_\sigma$ using that once we have established Brouwer for all finite simple graphs of order n, then it holds for all quivers of order n. As for the induction assumption, the claim holds for $n = 1$, where it holds unconditionally.   □

8. SIGNLESS LAPLACIAN

8.1. The **signless Laplacian** $|K|$ is defined as $|K| = D+A$ or $|K|_{ij} = |K_{ij}|$. If $K = F^T F$, then $|K| = |F|^T |F|$. The 1-form version is $K' = |F||F^T|$. If we place F into the lower sub-diagonal of a $(n+m) \times (n+m)$ matrix which is zero everywhere else, we call it the **exterior derivative** d. Now define the **Dirac matrix** $D = d + d^*$ which is a symmetric $(n+m) \times (n+m)$ matrix. Since $d^2 = 0$ the matrix $H = D^2 = (d+d^*)^2 = dd^* + d^*d$ is block diagonal and has in the first block a $n \times n$ Kirchhoff matrix $K = F^T F$. The second block is the $m \times m$ matrix $K_1 = FF^T$. The Hodge Laplacian is $H = K \oplus K_1$. The two blocks are essentially isospectral, meaning that their non-zero eigenvalues agree.

8.2. If we replace d with $|d|$ and d^* with $|d^*|$ the same analysis works but now $|H| = |K| \oplus |K_1|$ and the two matrices $|K|$ and $|K_1|$ are still essentially isospectral. (Note that the sign-less Laplacian in higher dimensions is no more block diagonal as $d^2 = 0$ does not hold any more then.) It is known that the spectral radius of $|K|$ is larger or equal than the spectral radius of K, but there is no definite inequality between the smaller eigenvalues. The eigenvalues of H and $|H|$ are very close in general. Because $|H| = |D|^2$ also $|K|$ has only non-negative eigenvalues. One can now formulate the Brouwer conjecture BC^+ for sign-less Laplacians. See [7].

8.3. The vertices and edges of a graph form a one-dimensional simplicial complex with $n + m$ simplices x. We can look at the **connection matrix** L which is defined as $L_{xy} = 1$ if and only if the two simplices x, y intersect. In [17] was noted that the 0-1 matrix L is **unimodular** meaning that its determinant is either 1 or -1. More precisely, the inverse matrix $g = L^{-1}$ has entries $w(x)w(y)\chi(U(x) \cap U(y))$, where $w(x) = (-1)^{\dim(x)}$ and $U(x) = \{y, x \subset y\}$ is the **star** of x. For example, if $x = y$ is a vertex, then the diagonal entry $g_{xx} = 1 - \deg(x)$ is related to the vertex degree $\deg(x)$. We also proved $\sum_{x,y} g(x,y) = \chi(G)$ and that the number of positive eigenvalues of L minus the number of negative eigenvalues is also $\chi(G)$. All this could be generalized to energized complexes [18].

8.4. The connection Laplacian and the Hodge Laplacians are defined in any dimension, for any finite abstract simplicial complex. In one dimensions, we have the **hydrogen identity** $L - L^{-1} = |H|$. [16]. The eigenvalues λ_j of $|H|$ are now related to the eigenvalues μ_j of L by $\lambda_j = \mu_j - 1/\mu_j$. If the eigenvalues of $|H|$ are ordered $\lambda_1 \geq \lambda_2 \geq ...$ then the eigenvalues of L are ordered $\mu_1 \geq \mu_2 \geq ...$ as long as the $\lambda_j > 1$.

8.5. We made use of the hydrogen identity to estimate the spectral radius of $|H|$ and so the spectral radius of K from above. We have $\lambda_1(K) \leq max(\lambda_1(L), \lambda_1(g))$ because $\lambda_1(K) \leq \lambda_1(|K|) = \lambda_1(|H|) = \lambda_1(L) - \lambda_1(L)^{-1}$. If H be a subgraph of G in which one of the edges has been deleted. Let $L(H)$ be the connection Laplacian of H and $L(G)$ the connection Laplacian of G.

Lemma 4. *Let H be a subgraph of G, then $L(H)$ is a principle submatrix of $L(G)$. The eigenvalues of $L(H)$ are therefore interlaced with the eigenvalues of $L(G)$.*

Proof. Any edge can only interact with itself or with vertices. $\qquad\square$

8.6. This generalizes to higher dimensional simplicial complexes in that if H is a subcomplex of G, there is interlacing. It follows that the connection spectral radius of a subcomplex H of G is smaller or equal than the connection spectral radius of G. We see so far in experiments that the spectral radius of $g = L^{-1}$ is smaller or equal than the spectral radius of L. We do not know whether this is true in general.

9. Brouwer-Haemers

9.1. For quivers without multiple connections, we had proven in [19] a Brouwer-Haemers lower bound [3] (written there when the eigenvalues were ordered in increasing order) $\lambda_j \geq d_j - (n-j)$ which implies in the decreasing order $\lambda_j \geq d_j - j + 1$. See also [4] Proposition 3.10.2, where the decreasing order with the stronger $d_j - j + 2$ is used, but which required some condition on the graph.) The work of [4, 3, 12, 23, 13], had an increased lower bound, which needed however cases like $G = K_n$. The slightly weaker $\lambda_j \geq d_j - j + 1$ was proven with the snap induction proof and is true unconditionally and holds also in the quiver case without multiple connections.

9.2. In our earlier stage of the work we called $m + k(k+1)$ the **Brouwer-Haemers upper bound**. For quivers with multiple connections, the adaptation is $m + r + k(k+1)$. Alan Lew noticed that our proof of $S_k \leq m + k(k+1)$ was incorrect. He also informed us of [22] with the stronger $S_k \leq H_k$ with $H_k = m + k^2$ after an earlier $S_k \leq m + k^2 + 15k \log(k) + 65k$ [21]. We call the H_k and its adaptation for quivers with redundancy $H_k = m + r + k^2$ now the **Lew bound**.

9.3. When looking at general graphs with loops but without multiple connections, the edge degree bound was

$$\lambda_j \leq d_j + d_{j+1} \leq 2d_j .$$

This implies $S_k \leq 2D_k$ given in [19]. Experiments show that this in general competes with the Brouwer estimate. There is in general a middle interval of k values, where one of the two estimates is better. In [19] we had opted to arrange the eigenvalues in increasing order as in Riemannian geometry. In the context of the Brouwer conjecture most of the literature take a decreasing order.

9.4. If $\overline{\lambda}_j$ are the eigenvalues of the graph complement $\overline{G}$, then $\lambda_k + \overline{\lambda}_{n-k+1} = n$ and $\sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{k} \overline{\lambda}_{n-j+1} = nk$ so that the Brouwer estimate also gives a bound on the smallest k eigenvalues of the graph complement. See [6].

9.5. In general, the vertex degree estimate $S_k \leq 2D_k$ gives better estimates for small or large energies, while the Brouwer estimate $S_k \leq B_k$ estimate is better in the middle of the spectrum. It is worth mentioning that the sequence list $k \to m + k(k+1)/2$ is concave up, while the lists of $k \to D_k = \sum_{j=1}^{k} d_j$ and $k \to S_k(G) = \sum_{j=1}^{k} \lambda_j$ are both concave down. If the cS_k for some $c \in (1,2)$ is tangent to B_k, then the estimate $S_k \leq B_k$ holds for all k!

9.6. If $m \to \infty$, keeping the loop numbers as equal as possible, the function $k \to m + k(k+1)/2$ is close to constant while $k \to \sum_{j=1}^{k} d_j + d_{j+1}$ is close to linear. We can so construct examples of quivers on n vertices for which the degree bound is smaller on half of the spectrum while the edge size bound is smaller on the other half of the spectrum.

9.7. For S_k we have now relations of the vertex degree sequences D_k and the Brouwer sequence $B_k = m + k(k+1)/2$ which when adapted to quivers with multiple connections becomes $B_k = m + r + k(k+1)/2$. We also adapt the Lew bound $H_k = m + r + k^2$. Let us summarize a few inequalities including the more elementary $D_k \leq B_{k-1}$.

Theorem 4. *Let G be an arbitrary quiver.*
a) $D_k \leq S_k$.
b) $S_k \leq 2D_k$.
c) $D_k \leq B_{k-1}$.
d) $S_k \leq H_k$.

Proof. a) $D_k \leq S_k$ is the Schur inequality (see e.g. [4, 3]).
b) Follows from $\lambda_j \leq d_j + d_{j+1}$ with the assumption $d_{n+1} = 0$ (see Theorem 1 in [19], a result for arbitrary quivers).
c) Assume first that G has no multiple connections. Take the sub-graph H of G which is the union of the k largest embedded star graphs in G. There are maximally $k(k-1)$ connections between the k centers of these star graphs so that H has at least $D_k - k(k-1)/2$ edges. As this is $\leq m$, we have $D_k \leq m + k(k-1)/2 = B_{k-1}$.
We finished the case without multiple connections. If any multiple connection is introduced, then right hand side increases by 2 while the left hand can only increase maximally by 2.
d) The result for graphs is shown in [22]. This upgrades to quivers. For each $1 \leq k \leq n$, adding a loop increases the left hand side by 1 or less while the right hand side increases by 1. When adding a multiple connection, the left hand side increases by maximally 2, while the right hand side increases by 2. $\qquad\square$

Appendix: Quiver gradient

9.8. We recall some **quiver calculus** from [19]. Quivers are a class of geometric objects containing finite simple graphs. They are natural because the set of quivers is a **pre-sheaf category** and so forms a **topos**. The nomenclature is not always uniform. Quiver-graphs are sometimes called "graphs" [2], "general graphs" [5], "pseudo graphs" [31], "loop multi-graphs" [30] or "réseaux" [8].

9.9. The calculus on graphs parallels the calculus on Riemannian manifolds. In particular, Hodge theory works, as maybe pointed out first by Eckmann [9] (in arbitrary dimensions). The linear algebra approach to cohomology is technically simpler: the k-**th cohomology** can be identified with the space of **k-harmonic forms**. They form the kernel of the k'th form Laplacian. Their dimension is the k'th Betti number b_k. For quivers, we only deal with one-dimensional structures, for which only the zero'th and first cohomology are relevant.

9.10. Given a quiver $G = (V, E)$, with $E \subset V \times V$. By enumerating the $n \geq 1$ vertices and $m \geq 1$ edges arbitrarily and by orienting the edges arbitrarily, we get a $m \times n$ matrix $F = d_0$ defined by $F_{e,v} = 1$ if $e = (w, v)$ and $F_{e,v} = -1$ if $e = (v, w)$ and $F_{e,v} = 0$ if $e = (v, v)$. This matrix is the **quiver gradient**. It maps **0-forms**, functions on vertices to **1-forms**, functions on oriented edges. The transpose matrix $d_1 = F^T$ is the **quiver divergence**. It maps 1-forms to 0-forms by telling a vertex v how much the total in-flux and out-flux coming from edges attached to v. The Kirchhoff matrix $K = d_1 d_0 F^T F$ is now a map on 0-forms, represented by a $n \times n$ matrix, while $K' = F^T F = d_0 d_1$ is a map on 1-form, represented by a $m \times m$ matrix. The matrix K does not depend on the orientation of the edges. The matrix K' does. But the spectrum of K' does not.

9.11. The nullity of K is the 0'th Betti number b_0. The nullity of K' is the first Betti number b_1. The Euler characteristic of the quiver is $\chi(G) = n - m$. Euler-Poincaré tells $\chi(G) = b_0 - b_1$. A modern point of view is to combine the spaces of 0-forms and 1-forms and write down d_0, d_1 as a nilpotent block matrices satisfying $d_0^2 = 0$ and $d_1^2 = 0$ and introduce the **Dirac matrix** $D = d_0 + d_1$ which then gives the **Hodge Laplacian** $L = D^2 = K \oplus K'$ consisting of the two blocks. If A is a general linear map on forms, (here a $n \times m$ matrix), one can define the super trace $\text{str}(A) = \sum_{1 \le k \le n} A_{kk} - \sum_{n+1 \le k \le n+m} A_{kk}$. As in Riemannian geometry, the super trace of any power of the Hodge Laplacian is zero $\text{str}(L^j) = 0, j \ge 1$ while the super trace of the identity matrix $1 = L^0$ is the Euler characteristic $\chi(1) = n - m$. The **McKean Singer equation** $\text{str}(e^{-tL}) = \chi(G)$ follows. Evaluating this at $t = 0$ gives $m - n$ while evaluating this at $t \to \infty$ gives $b_0 - b_1$. The **McKean-Singer symmetry** has as a consequence that K and K' are essentially isospectral. This is a fact exploited by Anderson-Morley [1].

APPENDIX: CLOVERS AND RIBBONS

9.12. A **clover** is a graph with $n = 1$ vertices with m self-loops. The quiver gradient F in that case is the $n \times 1$ matrix $F = \begin{bmatrix} 1 \\ \cdots \\ 1 \end{bmatrix}$ so that $K = F^T F = [m]$ and $K' = F F^T = \begin{bmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 1 \end{bmatrix}$. The Betti numbers are $b_0 = \dim\ker(K)) = 0$ and $b_1 = \dim\ker(K') = m - 1$, matching $\chi(G) = n - m = 1 - m = b_0 - b_1$.

9.13. A **ribbon** is a graph with $n = 2$ vertices and $m \ge 2$ multiple connections and no loops. The quiver gradient is a $m \times 2$ matrix and $K = \begin{bmatrix} m & -m \\ -m & m \end{bmatrix}$ with eigenvalues $\lambda_1 = 2m, \lambda_2 = 0$. The unmodified Brouwer estimate fails already for $k = 1$, we have $\lambda_1 = 2m > m + 1(1+1)/2 = m + 1$. But the modified estimate $\sum_{k=1}^{k} \lambda_j \le m + r + k(k+1)/2$ holds, where r is the number of redundant edges. In the case of the ribbon with m connections, we can take away $r = m - 1$ redundant edges. The ribbon has the estimate $\sum_{k=1}^{k} \lambda_j \le m + k(k+1)/2 + r$ sharp.
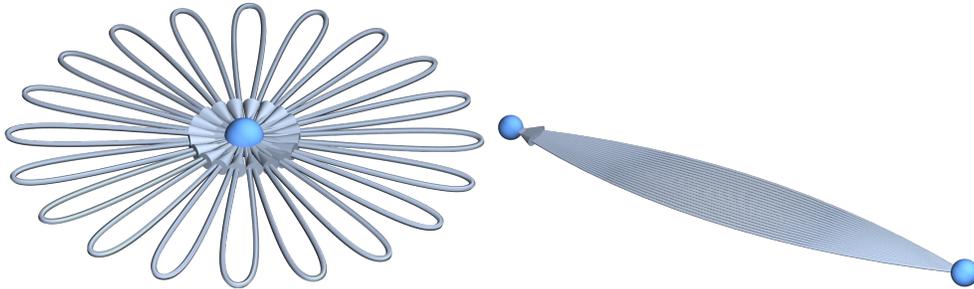

FIGURE 1. For a clover with m loops, the Brouwer is sharp $\lambda_1 = m$ For a ribbon, a graph $n = 2$ vertices and $m \ge 2$ connections, the original Brouwer estimate is false because $\lambda_1 = 2m$ and $m + 1(1+1)/2 = m + 1$. But the modified Brouwer estimate with $B_k = m + r + k(k+1)/2$ holds.

Appendix: Hadamard perturbation

9.14. The **first Hadamard perturbation formula** $\lambda'(t) = v^T(t)Ev$ describes using the eigenvector $v = v(t)$ how the eigenvalues $\lambda = \lambda(t)$ of a symmetric matrix L change if it is perturbed as $L(t) = L + tE$, where E is an other symmetric matrix. The eigenvector v of L to the eigenvalue λ is assumed to have length 1. Write $v(t)$ for the perturbed normalized eigenvector to the eigenvalue $\lambda(t)$. Take the eigenvalue equation $L(t)v(t) = \lambda(t)v(t)$ and multiply from the left with $v^T(t)$ to get $v^T(t)L(t)v(t) = \lambda(t)v^T(t)v(t) = \lambda(t)$. After differentiating using the product rule and using $L'(t) = E$ and $v \cdot v = 1$ and $Lv = \lambda v$, we end up with $\lambda'(t) = v^T(t)Ev$. This derivative is bounded above by the norm of E.

9.15. Since the first derivative stays uniformly bounded, also the higher derivatives do and $\lambda_j(t)$ stays smooth. It has been known since Rellich's work in the 1930ies that the eigenvalues of a smooth real symmetric matrix-valued function can be globally labeled so that each $\lambda_j(t)$ remains smooth. It is not totally obvious, as the second Hadamard perturbation formula involves terms of the form $\lambda_i - \lambda_j$ in the denominator which suggest troubles when eigenvalues collide. This becomes indeed a problem in the non-selfadjoint case. There is also a smooth parametrization of the orthonormal eigenvector frame, but at eigenvalue crossings, a switching of labels can occur. See Chapter II of [15].

9.16. In the case of the **rank**-1 perturbation $E = e_l \cdot e_l^T$ (which is a projection onto the l'th basis vector, because all matrix entries are $E_{ij} = 0$ except for $E_{ll} = 1$), this gives $\lambda'(t) = v_l(t)^2 \geq 0$. And because the normalization assumption was $\sum_j v_j^2 = 1$, we have $0 \leq \lambda'(t) \leq 1$. If a new loop is added to a quiver, then this means for the Kirchhoff matrix that it changees from K to $K + E$. All eigenvalues can then only increase. Because the sum of all eigenvalue changes is 1 the total change of the trace from $t = 0$ to $t = 1$. We therefore also have $\sum_{j=1}^{k} \lambda_j' \leq 1$. The eigenvalue sum change is between 0 and 1 for any k.

9.17. If a new redundant edge (a, b) is added, then again the eigenvalues can not decrease. Now, the eigenvalues and also each of the sums $\sum_{j=1}^{k} \lambda_j$ can increase maximally by 2. This is proven differently in [11] (Lemma 13.6.1). It can also be found in [28] (Corollary 6.2.2).
This follows again from the Hadamard formula but using $E = (e_a - e_b) \cdot (e_a - e_b)^T$ which is twice a projection matrix. Hadamard perturbation gives $\lambda'(t) = v^T(t)Ev(t) = (v_a - v_b)^2 \geq 0$. Since each eigenvalue can only increase and the sum of all eigenvalues changes by 2 when moving from K to $K + E$ we have the general case because the right hand side $m + r$ increases by 2.

Appendix: Illustration

9.18. The Brouwer estimate is especially good for graphs with a small diameter. This can be explained with being close to the complete graph. If G is the graph complement of a graph with vertex degree $4d^2 \leq m$ then the Brouwer estimates holds. But in that regime, the vertex degree estimate $\lambda_j \leq d_j + d_{j-1}$ gives much better results.
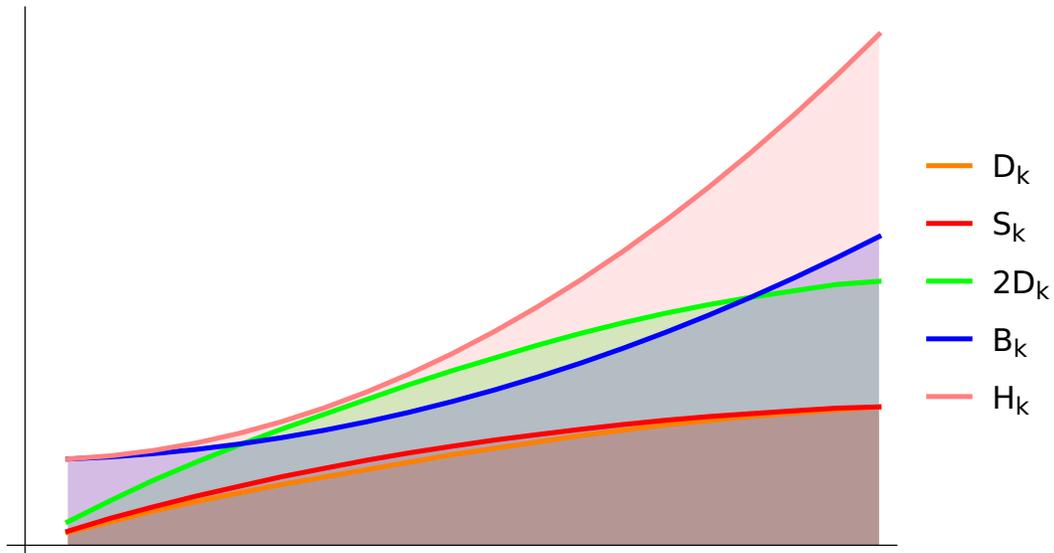
FIGURE 2. A random quiver without multiple connections. It has 20 vertices, 50 edges and 30 loops. To the right, we see the eigenvalue sum list $S_k = \sum_{j=1}^{k} \lambda_j$, the upper bound using $\lambda_j \leq d_j + d_{j-1}$, the lower bound both proven in [19] and the Brouwer upper bound. All except the Brouwer sequence $m + k(k+1)/2$ are concave down.
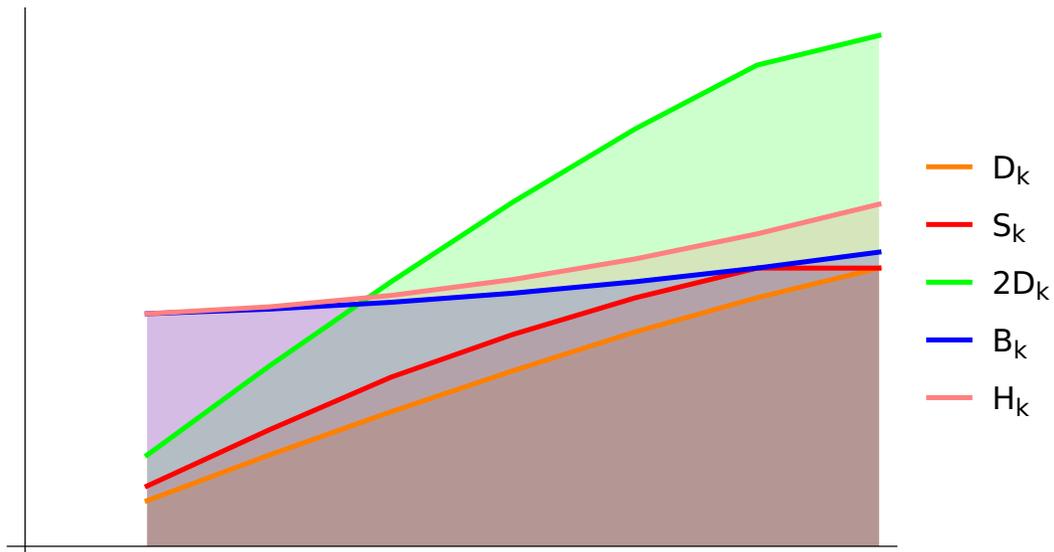

FIGURE 3. A complete graph K_7 with redundancy $r = 40$ and $m = 40 + 7*6/2 = 61$. Since $\lambda_n = 0$ for any graph without loops and for $k = n - 1$ we have $122 = \text{tr}(K) = \sum_{j=1}^{k} \lambda_j = m + k(k+1)/2 + r = 61 + 21 + 40$, the estimate is sharp here.

9.19. The Brouwer estimate is sharp also in the quiver case if we take a complete graph with multiple ribbon connections. The picture shows the case with

$$
K = \begin{bmatrix}
14 & -1 & -3 & -3 & -1 & -3 & -3 \\
-1 & 19 & -3 & -2 & -5 & -5 & -3 \\
-3 & -3 & 20 & -4 & -3 & -3 & -4 \\
-13 & -2 & -4 & 13 & -1 & -1 & -2 \\
-1 & -5 & -3 & -1 & 17 & -3 & -4 \\
-3 & -5 & -3 & -1 & -3 & 19 & -4 \\
-3 & -3 & -4 & -2 & -4 & -4 & 20
\end{bmatrix} .
$$

It illustrates also that the Brouwer bound is for graphs with large degrees better in general than the upper bounds in terms of the vertex degrees. The edge degree estimate is better if the graph is large and the degree remains small, like for discrete manifolds.

Appendix: Code

9.20. We conclude with an adaptation of the code given in [19] but showing also the Brouwer upper bound $B_k = m + k(k+1)/2$ as well as the Lew upper bound $H_k = m + k^2$. It illustrates the known inequalities $D_k \leq S_k \leq 2D_k$ and $D_k \leq B_k$ and $S_k \leq H_k$ [22]. The code was used to produce the graph in Figure 2:

```
QuiverGradient[s_]:=Module[{v=VertexList[s],e=EdgeList[s],n,m,F,a,b},
 n=Length[v];m=Length[e];      F=Table[0,{m},{n}];   Q[x_]:=Subscript[x,"k"];
 Do[{a,b}={e[[k,1]],e[[k,2]]};F[[k,a]]+=1;If[a!=b,F[[k,b]]+=-1],{k,m}];F];
RandomQuiver[{n_,m_,l_,c_}]:=Module[{G=RandomGraph[{n,m}],v,e,q={},R,A},
 v=VertexList[G];e=EdgeRules[G];R=RandomChoice;A=Append;U[u_]:=u[[1]]->u[[2]];
 Do[x=R[v];q=A[q,x->x],{1}];Do[q=A[q,U[R[e]]],{c}];Graph[v,Join[e,q]]];
n=20;r=0;G=RandomQuiver[{n,50,30,r}];F=QuiverGradient[G];K=Transpose[F].F;m=Length[F];
Eigen=Reverse[Sort[Eigenvalues[1.0*K]]];      d=Reverse[Sort[Table[K[[k,k]],{k,n}]]];
SignEigen=Reverse[Sort[Eigenvalues[1.0*Abs[K]]]];
DegreeUpper  = Table[d[[k]]+If[k<n,d[[k+1]],0],{k,n}];        (* which is <= 2d *)
s=Table[Sum[d[[j]],{j,k}],{k,n}];                  (* Schur            *)
f=Table[Sum[Eigen[[j]],{j,k}],{k,n}];              (* Eigenvalues      *)
c=Table[Sum[SignEigen[[j]],{j,k}],{k,n}];          (* Signed           *)
g=Table[Sum[DegreeUpper[[j]],{j,k}],{k,n}];        (* Knill [4]        *)
b=Table[m+k(k+1)/2 + r,{k,n}];                     (* Brouwer          *)
u=Table[m+k^2+r,{k,n}];                            (* Lew 2025         *)
ListPlot[{s,f,g,b,u,c},Filling->Bottom,PlotStyle->{Orange,Red,Green,Blue,Pink,Yellow},
   PlotLegends->{Q["D"],Q["S"],Q["2D"],Q["B"],Q["H"],Q["A"]},Joined->True,Ticks->None]
```

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